

**MAGNETIC BLOCH FUNCTIONS AND VECTOR BUNDLES.
TYPICAL DISPERSION LAWS AND THEIR QUANTUM
NUMBERS**

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I. In previous joint papers by the author and B. A. Dubrovin [1], [2] we computed completely the basic states of a two-dimensional, nonrelativistic electron with spin $1/2$ in an external doubly periodic (in x and y) magnetic field $B(x, y)$ (directed along the z axis) and a zero electric field. The Hamiltonian in this case is the Pauli operator

$$(1) \quad H_0 = -\frac{1}{2} \left(\frac{\partial}{\partial x} - ieA_1 \right)^2 - \frac{1}{2} \left(\frac{\partial}{\partial y} - ieA_2 \right)^2 + e\sigma_3 B;$$

here $\hbar = m = c = 1$, $B = \partial_2 A_1 - \partial_1 A_2$, and $H_0 \psi = \epsilon \psi$. Suppose that the lattice is rectangular, $z_{m,n} = mT_1 + inT_2$, and that the magnetic flux is integral and positive (generalization to a rational flux presents no difficulties):

$$(2) \quad \Phi = \iint_K B dx dy, \quad e\Phi = 2\pi N,$$

K is an elementary cell, $0 \leq x \leq T_1$, and $0 \leq y \leq T_2$.

Since $H_0 \sigma_3 = \sigma_3 H_0$, we have a decomposition of the Hilbert space of square-summable, vector-valued functions ψ on the plane into a direct sum of two scalar spaces:

$$(3) \quad \mathcal{L}_2 = \mathcal{L}_2^{(+)} \oplus \mathcal{L}_2^{(-)}, \quad \sigma_3 \psi = \pm \psi, \quad H_{\pm}: \mathcal{L}_2^{(\pm)} \rightarrow \mathcal{L}_2^{(\pm)}.$$

As was indicated in [3] for a localized field B , the basic states for $\Phi > 0$ are found in the periodic case only in the space $\mathcal{L}_2^{(+)}$ and have energy $\epsilon = 0$:

$$(4) \quad H_+ \psi = 0.$$

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The formulas for the basic states are as follows (see [1] and [2]; for the properties of σ see [4]):

$$\begin{aligned}
\psi &= \psi_A = \lambda \exp(-e\phi) \prod_{j=1}^N \sigma(z - a_j) \exp(az), \\
\phi &= \frac{1}{2\pi} \iint_K \ln |\sigma(z - z')| B(x', y') dx' dy', \quad z = x + iy, \\
\sigma(z) &= z \prod_{m^2+n^2 \neq 0} (1 - z/z_{m,n}) \exp \left\{ z/z_{m,n} - \frac{1}{2} z^2/z_{m,n}^2 \right\}, \\
\text{Re } a &= \text{Re} \left\{ \frac{\eta_1}{T_1} \left[2 \sum_{j=1}^N a_j - \frac{e}{\pi} \iint_k z B dx dy \right] \right\}, \\
\text{Im } a &= \text{Im} \left\{ \frac{\eta_2}{T_2} \left[2 \sum_{j=1}^N a_j - \frac{e}{\pi} \iint_k z B dx dy \right] \right\}, \\
A &= (a_1, a_2, \dots, a_N, \lambda),
\end{aligned} \tag{5}$$

where λ is any number, $\eta_1 = \xi(T_{1/2})$, $i\eta_2 = \xi(iT_{2/2})$ and $\xi(z) = \sigma'/\sigma$.

The states (5) are “magnetic Bloch” states, i.e., they are the eigenstates for the operators of “magnetic translations” T_1^* and T_2^* , which commute with the Hamiltonian and have unimodular eigenvalues (this is a projective representation of the discrete group of translations):

$$\begin{aligned}
T_1^* \psi_A &= \exp(ip_1 T_1) \psi_A = \psi_A(x + T_1, y) \exp\{-ie\eta_1 \Phi y / \pi\}, \\
T_2^* \psi_A &= \exp(ip_2 T_2) \psi_A = \psi_A(x, y + T_2) \exp\{-ie\eta_2 \Phi x / \pi\}, \\
p_1 + ip_2 &= \frac{2\pi i}{T_1 T_2} \sum a_j + \text{const}, \quad T_1^* T_2^* = T_2^* T_1^* \exp(-e\Phi).
\end{aligned} \tag{6}$$

The states (5) form a complete basis in \mathcal{L}_2 of solutions of the equation $H_0 \psi = \epsilon \psi$ for $\epsilon = 0$ and generate a subspace \mathcal{L}_2^0 in \mathcal{L}_2 which is distinguished by the direct sum $\mathcal{L}_2 = \mathcal{L}_2^0 \oplus \mathcal{L}_2^1$ in a manner similar to the case of a discrete level; according to [2], it is possible to choose a discrete basis of localized “Wannier states” in \mathcal{L}_2^0 in place of the continuous magnetic Bloch basis (5).

We note a useful supplement to a result of [1] and [2].

Theorem 1. *For any integral or rational flux $e\Phi = 2\pi N M^{-1}$ the basic states (5) are separated from the remaining energy levels (eigenvalues of $H_0 \psi = \epsilon \psi$) by a finite gap $\Delta\{B\}$.*

Conjecture. The gap $\Delta\{B\}$ varies continuously with the magnetic field $B(x, y)$ in the class of doubly periodic fields with arbitrary periods (which may vary and therefore pass through irrational fluxes).

The proof of Theorem 1 for integral fluxes follows easily from the following consideration: fixing the quasimomentum (p_1, p_2) , we obtain an elliptic, selfadjoint operator $H_0(p_1, p_2)$ in a bundle over a compact manifold—the torus T^2 , where the connectivity is defined by the field B . Therefore, the spectrum $\epsilon_j(p_1, p_2)$ is discrete, of finite multiplicity, and depends continuously on the parameters p_1, p_2 . Following [1], [2], we know that a) $\epsilon_0 = \epsilon_{\min}(p_1, p_2) = 0$ for all (p_1, p_2) ; and b) the dimension of this eigensubspace is equal to N , and it varies continuously together with the

quasimomenta (p_1, p_2) (without bifurcations) because of formulas (5). Further, the next eigenvalue $\epsilon_1(p_1, p_2)$ is positive and depends continuously on (p_1, p_2) . The equality $\epsilon_1(p_1^0, p_2^0) = 0$ is impossible because of the absence of bifurcation of the eigensubspace with level $\epsilon_0 = 0$. Therefore, $\Delta = \min_{(p_1, p_2)} \epsilon_1(p_1, p_2) > 0$.

For rational fluxes the proof also reduces to the proof for integral fluxes.

Remark. The two-dimensional Pauli operator (1) with a zero electric potential on the subspace $\mathcal{L}_2^{(+)}$ reduces to the scalar Schrödinger operator (with spin 0)

$$(7) \quad H = H_+ = -\frac{1}{2} \left(\frac{\partial}{\partial x} - ieA_1 \right)^2 - \frac{1}{2} \left(\frac{\partial}{\partial y} - ieA_2 \right)^2 + eV(x, y)$$

with a nonzero but special electric potential V :

$$(8) \quad \partial_1 A_2 - \partial_2 A_1 = V(x, y)$$

(in the system of units $c = \hbar = m = 1$). Under the condition (8) we denote the operator H by H_0 . Later we shall also consider the general Schrödinger operator (7) where the condition (8) is not satisfied.

For the Schrödinger operator (7) we have two integrable cases: a) $V \equiv 0$ and the field $B = \text{const}$ is homogeneous; b) condition (7) is satisfied, but the field B is arbitrary (only the lowest level $\epsilon = 0$ can be integrated). In both cases we denote the operator H by H_0 .

II. An important property of the basic states (5) (which also occurs for the Landau levels in the homogeneous field $B = \text{const}$) is that the magnetic Bloch functions (5) for, integral number of quanta of the flux $N \neq 0$ form a topologically nontrivial vector bundle over the torus T^2 . Under variation of any a_j over a lattice period $a_j \rightarrow a_j + T_1$ or $a_j \rightarrow a_j + iT_2$ the σ -function is multiplied by an exponential. This variation is compensated by the variation of the quantity $a(a_1, \dots, a_N)$ in (5):

$$(9) \quad a(\dots, a_j + T_1, \dots) = a + 2\eta_1, \quad a(\dots, a_j + iT_2, \dots) = a + 2i\eta_2.$$

We thus obtain a “gluing law” for the complete space E of the vector bundle ξ with is defined by (9); from this it follows that

$$(10) \quad \begin{aligned} (\lambda, a_1, a_2, \dots, a_N) &\simeq (\lambda, a_{i_1}, a_{i_2}, \dots, a_{i_N}), \\ (\lambda, a_1, a_2, \dots, a_N) &\simeq (\lambda', a_1, a_2, \dots, a_j + T_1, \dots, a_N), \\ (\lambda, a_1, a_2, \dots, a_N) &\simeq (\lambda'', a_1, a_2, \dots, a_j + iT_2, \dots, a_N), \\ \lambda' &= \lambda \exp\{2\eta_1 a_j + \eta_1 T_1 + i\pi\}, \quad \lambda'' = \lambda \exp\{2i\eta_2 a_j - \eta_2 T_2 + i\pi\}. \end{aligned}$$

As indicated in [1] and [2], for a fixed quasimomentum we have a vector space $C^N(p_1, p_2)$ of functions ψ_A : they are all obtained from ψ_{A_0} by multiplication by a meromorphic, doubly periodic elliptic function with the same lattice, i.e. $\psi_A = \psi_{A_0} \chi$. The function $\chi(z)$ must have poles at some of the points a_j , so that the product again has no poles.

Lemma 1. *The mapping of quasimomentum*

$$p = p_1 + ip_2: E \rightarrow \frac{2\pi i}{T_1 T_2} \sum_j a_j + \text{const}$$

transforms the manifold E of all magnetic Bloch functions (5) of the basic state ($\epsilon = 0$) into a vector bundle ξ with fiber C^N over the torus T^2 obtained from the

reciprocal lattice (T_1^{-1}, T_2^{-1}) . This bundle is topologically nontrivial for all $N > 0$ and has nonzero first Chern class $c_1(\xi) = 1 \neq 0$.

This lemma is derived from (10) in a topologically standard way, and we shall not prove it.

Remark. This lemma is also true for the magnetic Bloch functions of any Landau level in a homogeneous field $B = \text{const}$.¹

III. Of course, the very fact of the occurrence of a situation of rank N (i.e., a bundle ξ with an N -dimensional fiber) for the magnetic Bloch functions over the torus T^2 implies very strong degeneracy for $N \geq 2$. This degeneracy should vanish under small perturbations. We shall consider small perturbations of the Hamiltonian by an electric, doubly periodic potential $W(x, y)$ with the same periods

$$(11) \quad H = H_0 + eW(x, y),$$

where the operator H_0 is any of those studied in §§I and II. For $N = 1$ a small perturbation (and therefore also a perturbation which is not small) leads only to the formation of a “dispersion law” $\epsilon(p_1, p_2)$ and spreading of any Landau level (or basic state for the operator (7), (8)) in a single magnetic zone due to the connectedness of the torus T^2 . The topology of the family of Bloch functions itself—the “dispersion law”—does not change under small deformation of the operator for $N = 1$ and remains the same as in §II for the operators H_0 . Thus, consideration of the single-quantum case $N = 1$ may lead to the illusion that the topology of all dispersion laws, although it is not trivial, is nevertheless completely determined by the flux of the external magnetic field B through an elementary cell—by the single integer N (this is actually the case for any small perturbations of the field $B = \text{const}$ for $N = 1$).

We consider the Hermitian form $\hat{W}(\psi_A)$ on the fibers of the bundle ξ which is defined by a perturbation $W(x, y)$ with the same periods (p_1 and p_2 are fixed):

$$(12) \quad \hat{W}(\psi_A) = \iint_K \psi_A W \bar{\psi}_A dx dy.$$

Here there arise the real eigenvalues

$$\epsilon_1(p_1, p_2) \geq \epsilon_2(p_1, p_2) \geq \cdots \geq \epsilon_N(p_1, p_2)$$

of the form \hat{W} on the fibers $C^N(p_1, p_2)$.

Lemma 2. a) *In the class of doubly periodic, real functions $W(x, y)$ the condition of coalescence $\epsilon_i = \epsilon_j$ for fixed (p_1, p_2) is given by three independent conditions on the Fourier coefficients (this is also true in three-dimensional space). In particular, for functions in “general position” $W(x, y)$ the coalescence $\epsilon_i(p_1^0, p_2^0) = \epsilon_j(p_1^0, p_2^0)$ for at least one quasimomentum p_1^0, p_2^0 of the given dispersion law has codimension 1 in the function space (i.e., it is realized only at isolated points with respect to the parameter τ for “typical” one-parameter families of potentials $W_\tau(x, y)$).*

¹For a homogeneous field,

$$e\phi = \frac{e\Phi}{2\pi} \left[\frac{\eta_1}{T_1} x^2 - \frac{\eta_2}{T_2} y^2 - \eta_1 x + \eta_2 y \right].$$

The operator A^n takes the functions (5) into the Bloch functions of the n th Landau level, where

$$A = -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + \frac{eB}{2} \left[\bar{z} + z \left(\frac{T_1 \eta_2}{\pi} - \frac{1}{2} \right) \right] - \frac{e\Phi}{4\pi} (\eta_1 + i\eta_2).$$

b) In the three-dimensional space of the parameters (p_1, p_2, τ) there may be stable singular points (p_1^0, p_2^0, τ_0) such that $\epsilon_i = \epsilon_j$ (for only one pair i, j) and the restrictions ξ_i^δ and ξ_j^δ of the one-dimensional bundles ξ_i and ξ_j to a small sphere S_δ^2 of radius δ surrounding the singular point are nontrivial (although their sum is trivial), $\xi_1^\delta \oplus \xi_2^\delta \sim 0$ on S_δ^2 ,

$$(13) \quad q = c_1(\xi_i^\delta) = -c_1(\xi_j^\delta).$$

On passing through the value of the parameter $\tau = \tau_0$ the dispersion laws “collide” and are changed by the quantum number $q = \pm 1$:

$$(14) \quad \begin{aligned} c_1 &= (\xi_j)_{\tau_0-\delta} = c_1(\xi_j)_{\tau_0+\delta} + q, \\ c_1 &= (\xi_i)_{\tau_0-\delta} = c_1(\xi_i)_{\tau_0+\delta} - q. \end{aligned}$$

c) For the Schrödinger operator in three-dimensional space the quasimomentum $p_3 = \tau$ plays the role of the parameter τ ; therefore, the condition of coalescence for one quasimomentum (p_1^0, p_2^0, p_3^0) is stable, and the situation of part b) occurs.

The following result is established using Lemma 2.

Theorem 2. a) In the case of a small perturbing potential $W(x, y)$ in “general position” the eigenvalues of the form $\hat{W}(\epsilon_1(p_1, p_2) > \epsilon_2(p_1, p_2) > \dots > \epsilon_N(p_1, p_2))$ are distinct for any (p_1, p_2) and provide a decomposition of the family (bundle) of magnetic Bloch functions ξ of the unperturbed operator H_0 into a direct sum of one-dimensional (fiber C^1) complex bundles

$$(15) \quad \xi = \xi_1 \oplus \xi_2 \oplus \dots \oplus \xi_N$$

with the single condition on the first Chern class

$$(16) \quad c_1(\xi) = 1 = \sum_{j=1}^N c_1(\xi_j).$$

b) The “monodromy group” generated by permutations of the eigenvalues ϵ_j under basic circuits of the torus T^2 is, in general position, always trivial. Therefore, precisely N “decay” dispersion laws $\epsilon_j(p_1, p_2)$ are formed, with topological quantum numbers $c_1(\xi_j) = m_j$ which can be any integers (positive or negative) with the single relation (16). These dispersion laws have rank 1 (i.e., the fibers are one-dimensional) and are therefore stable under further deformation (which is not small).

c) In the three-dimensional case the potential $W(x, y, z)$ occasions the decay of the family of magnetic Bloch functions (the bundle ξ) into a sum of bundles ξ_1, \dots, ξ_k (which are not necessarily one-dimensional), where each of the ξ_j has fiber of dimension k_j and decomposes into a sum of one-dimensional bundles after removal of the singular points from the torus T^3 according to the dispersion laws $(\epsilon_{j,1}, \dots, \epsilon_{j,k_j})$:

$$(17) \quad \epsilon_j = \sum_{s=1}^{k_j} \xi_{j,s} \quad \text{on } T^2 \setminus (P_{j1} \cup \dots \cup P_{jm}),$$

where the branches $\epsilon_{js} = \epsilon_{jt}$ with topological invariants $q_{j\alpha}$ coalesce at the points $P_{j\alpha}$.

Thus, by performing further large perturbations, we arrive at the following

Conclusion. For a “general” two-dimensional Schrödinger operator (7) in a stationary magnetic field which is periodic in (x, y) with an integral flux $N \geq 2$ and an electric field with a periodic potential there are a countable number of dispersion laws $\epsilon_j(p_1, p_2)$ for the magnetic Bloch functions. These dispersion laws (i.e., Bloch functions) form one-dimensional (fiber C^1) bundles over the torus T^2 of the reciprocal lattice and have “quantum numbers” $c_1(\xi_j) = m_j$ in no way connected with one another or with the flux N of the external magnetic field in the energy range where the perturbations of different Landau levels are “mixed” and cannot be separated from one another.² In a homogeneous magnetic field for sufficiently high energy levels a doubly periodic electric potential $W(x, y)$ produces only a small perturbation of the levels of the homogeneous field. Therefore, the perturbed dispersion laws which arise from them do not overlap; condition (16) is satisfied for the dispersion laws arising from each Landau level individually. For the general three-dimensional Schrödinger operator the “typical” dispersion laws do not form only one-dimensional bundles over the torus T^3 , and the pairs of branches ϵ_j and ϵ_k coalesce for singular values of the quasimomentum.

Remark 1. Comparison with some results of the author and Kričever (see [5]) on integrable cases of rank greater than 1 shows that the conclusion regarding the occurrence of dispersion laws with completely random quantum numbers is probably also valid for $N = 0$ in periodic problems of dimension ≥ 2 . This is probably also true for $N = 1$ if the perturbing potential is not small. However, here there is not an “integrable case” of even one dispersion law or of rank > 1 that might provide a proof from consideration of small perturbations.

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²The corresponding Bloch function $\psi_j(x, x_0, p)$, where $\psi_j = 1$ for $x = x_0$, has an algebraic number of zeros, equal to N for fixed $p\{x_{jk}(p)\}$ and equal to m_j for fixed $x\{p_{jl}(x)\}$. The poles are located at points $p_{jl}(x_0)$.