GROUND STATES OF A TWO-DIMENSIONAL ELECTRON IN A PERIODIC MAGNETIC FIELD

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Abstract. The two-dimensional Pauli Hamiltonian for an electron with spin 1/2 in a transverse magnetic field has the following property: addition of any doubly periodic (but not small) increment to the homogeneous field leaves the ground state, i.e., the lower Landau level, fully degenerate, despite the loss of symmetry, and is separated from the next levels by a finite gap (all the remaining levels spread out to form a continuous spectrum). For an integral or rational flux

$$NM^{-1} = (2\pi)^{-1} \int_0^{T_1} \int_0^{T_2} B \, dx \, dy$$

the aggregate of the magnetic Bloch functions of the ground state can be explicitly obtained in terms of elliptic functions, and forms an N-dimensional linear space if the quasimomenta are fixed. The dependence on the quasimomenta forms a topologically nontrivial object (vector bundle over a torus), from whose structure it is possible to determine the possible decays into magnetic bands under small potential (periodic) perturbations.

1. Introduction

The great difficulties encountered when attempts are made to describe the states of an electron in a periodic external field, when another magnetic field (also periodic or uniform) is present in addition to the periodic potential, are well known. In the presence of a magnetic field, the discrete symmetry group of the external field, consisting of shifts by the vectors of the main translations of the lattice, generates in general a non-commutative symmetry group of the Hamiltonian, since the translations are accompanied by a "gauge" winding of the ψ functions by a phase factor. The basis translations commute only when the magnetic-field flux through all the two-dimensional elementary cells has an integer number of quanta. In this case (or in a rational one that reduces to an whole-number enlargement of the lattice) it is possible to define a magnetic analog of the Bloch functions that are eigenfunctions for the basis magnetic translations. Even in this case, however (see below), the topology of the family of the Bloch functions turns out to be much more complicated than in a purely potential field. In the case of an irrational number of flux quanta there are many other difficulties, which will be dealt with in the text. In particular, it is difficult to define spectral characteristics that would vary continuously with the field.

This paper is based on the use of an analytic observation that arises in instanton theory, and some fragments of which date back to the paper of Atiyah and Singer.[2] We started from an interpretation of this observation, given by Aharonov and Casher[3] in the language of the ground state of an electron in a localized field

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on a plane. It was found possible to apply these premises to the important case of a periodic field, i.e., to find all the ground states for a two-dimensional Pauli operator in a periodic magnetic field directed along the z axis; the number of ground states turned out to be "the same" as in a uniform field with the same flux. This means that the degeneracy of the ground state is not lifted by the periodic magnetic field, despite the loss of translational symmetry. The curious topological structure of the family of Bloch functions (in the case of an integer or rational number of flux quanta) makes it possible to investigate the decay of the ground state into magnetic band when a periodic potential is turned on.

We consider the nonrelativistic Pauli Hamiltonian with spin 1/2 for a two-dimensional case, when the magnetic field, which depends only on the variables x and y, is doubly periodic and directed along the z axis:

$$\begin{gathered} B(x+T_1, y) = B(x, y+T_2) = B(x,y); \\ B = \partial_x A_y - \partial_y A_x, \quad \partial_x A_x + \partial_y A_y = 0. \end{gathered} \tag{1}$$

Let $A_x = -\partial_y \varphi$, $A_y = \partial_x \varphi$, where

$$(\partial_x^2 + \partial_y^2)\varphi = B(x,y). \tag{2}$$

The Pauli operator is $(c = \hbar = m = 1)$

$$H = -(\partial_x - ieA_x)^2 - (\partial_y - ieA_y)^2 - e\sigma_3 B(x,y) \tag{3}$$

and commutes with σ_3. This operator reduces to a pair of scalar operators H_x on functions of the type $\sigma_3\psi = x\psi$, where $x = \pm 1$.

The magnetic flux

$$\varphi = \int_0^{T_1} \int_0^{T_2} B(x,y)\, dx\, dy$$

is the principal "topological" characteristic. To be sure, a topological interpretation of the flux (as a characteristic class) is meaningful only for whole-number fluxes. There are no known reasonable characteristics of the spectrum of the operator H (i.e., characteristics that vary regularly with the value of the field B) if $e\varphi/2\pi$ is an irrational number.

To study the ground states of an electron in a periodic field B, we use the observations of Aharonov and Casher,[3] who considered the ground states for the Pauli operator in a finite field B concentrated in a finite region of the x, y plane. The operator H_x is of the form $H_x = A_x A_x^*$, where

$$A_x^* = -i(\partial_x + exA_y) + x(\partial_y - exA_x).$$

It turns out that the ground states are solutions of the equation $H\psi = 0$, such that $\sigma_3\psi = x\psi$. In fact, since $A_x A_x^* = H_x$, it follows that $\langle H_x\psi, \psi\rangle \geq 0$, therefore $\varepsilon \geq 0$. By obtaining the solutions of the equation $H_x\psi = 0$ (if they exist) we obtain in fact the ground states. From the condition $\langle H_x\psi, \psi\rangle = 0$ it follows that $\langle A_x^*\psi, A_x^*\psi\rangle = 0$, or $A_x^*\psi = 0$. The order of the equation is lowered in analogy with the situation in instanton theory.

Next, a state with $A_x^*\psi = 0$ is possible only under the condition $xe\varphi > 0$. Let $e\varphi > 0$ and $x = 1$. We make the substitution:

$$\psi(x,y) = \exp\{-e\varphi(x,y)\} f(x,y), \tag{4}$$

where φ is the solution of Eq. (2). We see that $f(x,y)$ is an analytic (and entire) function (!) if the initial eigenfunction ψ is quadratically integrable:

$$\partial f(x,y)/\partial\bar{z} = (\partial_x + i\partial_y) f(x,y) = 0. \tag{5}$$

From the condition of quadratic integrability of ψ, together with the easily calculated asymptotic form of φ for a finite field B, it follows that $f(z)$ is a polynomial of degree $\leq [e\varphi/2\pi] - 1$, $z = x + iy$. The fractional character of the flux $e\varphi/2\pi$ does not play any role in the derivation.[3]

It is curious to note that quadratic integrability is easily established for all polynomials of degree $\leq N-1$ for an essentially fractional flux $e\varphi/2\pi = N + \delta$, where $0 < \delta < 1$. This is in fact assumed in the paper of Aharonov and Casher.[3] For a whole-number flux $N = e\varphi/2\pi$ the quadratic integrability is not quite clear for a highest order polynomial of degree $N-1$ (incidentally, only in the whole-number case it is possible in principle to make any comparison whatever with the Atiyah–Singer index formula for operators on compact manifolds,[2] and even that in a quite indirect manner, by going over to a sphere instead of a plane. This transition alters the entire spectrum, and preserves only the exact solutions for the ground states–the zero modes[1].

It is of interest to ascertain whether the spectrum of the operator H on a plane (for a finite field B) has a gap that separates the ground states from the remaining ones. Of course, this question can have, even for the whole-number flux, different answers for a plane R^2 and for a sphere S^2, where the spectrum is always discrete.[2]

2. Ground States in a Periodic Field

We recall first the elliptic functions we need. Let for simplicity the lattice be rectangular and let the lattice points be given by $z_{m,n} = mT_1 + inT_2$. The so-called σ function is given in the form of the infinite product

$$\sigma(z) = z \prod_{(m,n)\neq(0,0)} \left(1 - \frac{z}{z_{m,n}}\right) \exp\left\{\frac{z}{z_{m,n}} + \frac{1}{2}\frac{z^2}{z_{m,n}^2}\right\}$$

and has the translational properties

$$\begin{gathered} \sigma(z+T_1) = -\sigma(z)\exp\{2\eta_1(2+T_1/2)\}, \\ \sigma(z+iT_2) = -\sigma(z)\exp\{2\eta_2(2+iT_2/2)\}; \\ \eta_1 = \zeta(T_1/2), \quad \eta_2 = i\tilde{\eta}_2 = \zeta(iT_2/2), \quad \zeta(z) = [\ln\sigma(z)]_z. \end{gathered} \tag{6}$$

We choose the solution of Eq. (2) in the form

$$\varphi = (2\pi)^{-1} \iint_K \ln|\sigma(z-\zeta)| B(x',y')\, dx'\, dy', \quad \zeta = x' + iy', \tag{7}$$

where K is the unit cell and its area is $|K| = T_1T_2$. The ground states $H_x\psi = 0$, as will be explained later, must be sought for any integer value of the magnetic flux $(2\pi)^{-1}e\varphi = N$ in the form

$$\psi_A = e^{-e\varphi}\sigma(z-a_1)\sigma(z-a_2)\ldots\sigma(z-a_N)e^{az}, \tag{8}$$

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[1]The reader must be warned that the paper by Aharonov and Casher[3] contains wrong references to the known work of Atiyah and Singer. Fortunately, these references have no bearing on the matter.

[2]The quadrature non-integrability of one of the eigenfunctions in Ref. 3 points to the absence of a gap in this case.

where the conditions on the constants $A = (a, a_1, \dots, a_N)$ will be indicated later [see (14) below]. The need for choosing the analytic function $f(z)$ in the form of a product of σ functions and an exponential will also be made clear later.

When shifted by the basis periods of the lattice T_1 and T_2

$$(x, y) \to (x + T_1, y), \quad (x, y) \to (x, y + T_2)$$

the function φ acquires an increment

$$\begin{aligned}
\Delta_1\varphi &= \varphi(x + T_1, y) - \varphi(x, y) \\
&= \pi^{-1}\varphi\eta_1 x + \eta_1 \left[T_1\varphi/2\pi - \pi^{-1} \operatorname{Re} \iint \zeta B(\zeta)\, dx'\, dy' \right], \\
\Delta_2\varphi &= \varphi(x, y + T_2) - \varphi(x, y) \\
&= -\pi^{-1}\tilde\eta_2\varphi y - \tilde\eta_2 \left[T_2\varphi/2\pi - \pi^{-1} \operatorname{Im} \iint \zeta B(\zeta)\, dx'\, dy' \right].
\end{aligned} \tag{9}$$

Since $A_1 = -\partial_2\varphi$, $A_2 = \partial_1\varphi$, the action of the group of magnetic translation is defined on any function in the following manner (see Refs. 1, 4, and 5) in the case of T_1^*:

$$\psi(x, y) \to \psi(x + T_1, y) \exp\{-ie\eta_1\varphi y/\pi\}; \tag{10a}$$

in the case T_2^*

$$\psi(x, y) \to \psi(x, y + T_2) \exp\{-ie\tilde\eta_2\varphi x/\pi\}, \tag{10b}$$

and the commutator depends on the magnetic flux

$$T_2^* T_1^* \exp\{ie\tilde\eta_2 T_1\varphi/\pi\} = T_1^* T_2^* \exp\{ie\eta_1 T_2\varphi/\pi\}. \tag{11}$$

We note that $\eta_2 T_1 - \tilde\eta_2 T_1 = \pi$ (see Ref. 6).

At integer $N = (2\pi)^{-1}e\varphi$ the complete basis of the ground states $H\psi = 0$ must be sought in the form of “ magnetic-Bloch” functions—the eigenvectors of the magnetic translations (10) with eigenvectors of unity modulus:

$$T_1^*\psi = \exp(ip_1T_1)\psi, \quad T_2^*\psi = \exp(ip_2T_2)\psi. \tag{12}$$

Starting from the postulate (8), we obtain by simple calculation

$$\begin{aligned}
T_1^*\psi_A &= \psi_A \exp\Bigg[-ie\varphi\eta_1 y\pi^{-1} - e\Delta_1\varphi \\
&\qquad + 2\eta_1 \left(Nz + NT_1/2 - \sum_{j=1}^{N} a_j \right) + iN\pi + aT_1 \Bigg], \\
T_2^*\psi_A &= \psi_A \exp\Bigg[-ie\varphi\tilde\eta_2 x\pi^{-1} - e\Delta_2\varphi \\
&\qquad + 2i\tilde\eta_2 \left(Nz + iNT_2/2 - \sum_{j=1}^{N} a_j \right) + iN\pi + iaT_2 \Bigg], \\
z &= x + iy, \quad A = (a, a_1, \dots, a_N).
\end{aligned} \tag{13}$$

From the condition $(2\pi)^{-1}e\varphi = N$ it follows by virtue of (13) that all the functions (8) are eigenfunctions for both magnetic translations T_1^* and T_2^*. The requirement that the eigenvalues of T_1^* and T_2^* have a unity modulus imposes the following conditions on the constants:

$$\begin{aligned} \operatorname{Re} a &= \operatorname{Re}\left\{ \eta_1 T_1^{-1} \left[2\sum_{j=1}^{N} a_j - \frac{e}{\pi} \iint_K \zeta B(\zeta)\, d^2\zeta \right] \right\}, \\ \operatorname{Im} a &= \operatorname{Im}\left\{ \tilde{\eta}_2 T_2^{-1} \left[2\sum_{j=1}^{N} a_j - \frac{e}{\pi} \iint_K \zeta B(\zeta)\, d^2\zeta \right] \right\}. \end{aligned} \tag{14}$$

It follows therefore that when the conditions (14) are satisfied, the functions (13) are, by virtue of (8), Bloch functions with quasimomentum values (the magnetic field enters only in the constant):

$$\begin{aligned} p_1 + ip_2 &= -2\pi i T_1^{-1} T_2^{-1} \sum_{j=1}^{N} a_j + \text{const}, \\ p_1 + N\pi T_1^{-1} &= \operatorname{Im}\left(a - 2\eta_1 T_1^{-1} \sum_{j=1}^{N} a_j \right), \\ p_2 + N\pi T_2^{-1} &= \operatorname{Im}\left(a - 2\tilde{\eta}_2 T_2^{-1} \sum_{j=1}^{N} a_j \right). \end{aligned} \tag{15}$$

At fixed p_1 and p_2 we obtain an N-dimensional family of magnetic-Bloch eigenfunctions $\sum a_u = \text{const}$, where $N = (2\pi)^{-1} e\varphi$. This family is obtained from one function ψ_A with parameters $A = (a, a_1, \ldots, a_N)$ in the following manner: we take any elliptic (doubly periodic meromorphic) function $\chi(z)$ with poles in part of the points $a_1, \ldots, a_{nN}$ such that the product $\psi_A \chi$ has no poles. Then the product $\psi_A \chi$ is again a magnetic-Bloch function of the type (8), accurate to a constant factor, with the same quasimomenta p_1 and p_2. These functions χ form an N-dimensional linear complex space. We obtain by this procedure all the Bloch functions from one of them.

The family of Bloch functions $\lambda\psi_A$ at arbitrary λ and fixed p_1 and p_2 is thus a linear N-dimensional complex space designated $C^N(p_1, p_2)$. Simple mathematical considerations show that under the conditions (14) the family (8) is the complete basis of the solution $H\psi = 0$ at integer $N = (2\pi)^{-1} e\varphi$.

3. Some Conclusions

1. If the magnetic flux is rational, $(2\pi)^{-1} e\varphi = NM^{-1}$, where N and M are integer mutually prime numbers, then enlargement of the lattice permits reduction of the problem to an integer flux, by increasing the dimension of the unit cell by M times, $K \to MK$ (nothing is further dependent on further enlargement). For the cell MK, the flux is equal to $2\pi N/e$. By the same token we obtain the complete basis of ground states (8) in the space of functions on the x, y plane. We take large integers $\bar{L}_1$ and $\bar{L}_2$ (such that the product $\bar{L}_1 \bar{L}_2$ is divisible by M) and impose on the state periodic boundary conditions with periods $\bar{L}_1 T_1$ in x and $\bar{L}_2 T_2$ in y [the periodicity is understood to be relative to the magnetic translations (10)]. We obtain a certain number D of states, proportional to $e\bar{L}_1 \bar{L}_2 \varphi$, in analogy with the case of constant B (see Ref. 7). Dividing this number by the area of the large cell $\bar{L}_1 \bar{L}_2 |K|$, we conclude that the number of ground states per unit area is proportional to the magnetic flux and is equal to $(2\pi|K|)^{-1}e\varphi$; where $|K|$ is the area of the unit cell of the initial lattice with periods T_1 and T_2. We see that this number (albeit defined only for a rational flux ratio) varies continuously with the field and runs through all real values of the flux.

2. Comparison with the case of a uniform field $B = B_0$ shows that in an alternating field there are at the zero level "just as many" [see (17) and (18) below] eigenfunctions $\lambda\psi_A$ as in the uniform field. This circumstance is quite unexpected. The point is that the group of magnetic translations (10) in a uniform magnetic field B_0 is a continuous group (isomorphous to the known nilpotent Heisenberg–Weyl group); each Landau level realizes an irreducible infinite representation of this continuous group.

The spectrum is therefore in fact "discrete": the Heisenberg–Weyl group has only one infinite-dimensional irreducible representation that does not reduce to a commutative group if the representation of the center is specified as multiplication by a nonzero number fixed by the value of the magnetic flux through a unit area.

On going to a periodic field $B = B_0 + \Delta B$, where $\varphi_0 = \varphi$, we lose the continuous Heisenberg–Weyl symmetry. We are left with only the discrete subgroup Γ of the magnetic translations (10), which conserves the Hamiltonian H. The restriction of the irreducible representation with a continuous group to the subgroup Γ already ceases to be irreducible. Therefore even a small periodic perturbation should broaden the Landau level into a magnetic band in accord with some expansion of the Landau level in the irreducible representations of the group Γ; next, in the case of an irrational flux through the unit cell, certain difficulties would arise, since the expansion of the Landau level in irreducible representations of the discrete group Γ is not unique, and these expansions are difficult to classify; in the rational case it is impossible to find good spectral characteristics that vary regularly together with the field B.

In our case, in the absence of an electric field, despite the loss of continuous symmetry, no broadening of the level occurs. We are left with energy degeneracy that is not connected with any irreducible representations of the group Γ. It is clear that there is no such degeneracy at the higher energy levels.

3. The Hilbert space $\mathcal{H}_0$ spans the states (8), in analogy with the "discrete" spectrum; in the complete Hilbert space $\mathcal{H}$ all the quadratically integrable functions $\psi(x, y)$, the subspace $\mathcal{H}_0$ is singled out by the direct term $\mathcal{H} = \mathcal{H}_0\mathcal{H}_1$, in analogy with the expansion in the direct sum of the subspaces of the Landau levels for the case of a uniform field $B = B_0$. Accurately speaking this means, in particular, that in the space $\mathcal{H}_0$ of the ground states it is possible to choose not a Bloch basis but a quadratically integrable one (the "Wannier basis"; the integral is taken over the unit cell of the reciprocal lattice, and all the paired differences $a_k - a_j$ are fixed):

$$W(x, y) = \iint \psi_A(x, y)\, dp_1\, dp_2. \tag{16}$$

where ψ_A is given by (8), and

$$p_1 + ip_2 = -2\pi i T_1^{-1} T_2^{-1} \sum_{j=1}^{N} a_j + \text{const.}$$

Here $W(x,y)$ is an N-dimensional space. All the possible magnetic translations of the space of the functions $W(x,y)$ yield a complete basis at the zero level, where the shifted spaces $W_{m,n}$ are pairwise orthogonal:

$$W_{m,n}(x,y) = (T_1^*)^m (T_2^*)^n W(x,y),$$
$$\langle W_{m,n}, W_{m'n'} \rangle = 0 \quad \text{if } (m,n) \neq (m',n').$$

For finite periodic perturbations ΔB of the uniform magnetic field B_0, where $\Delta B \ll B_0$, we can rigorously prove that the nonzero level remains separated from the next state by a finite barrier. This result has been established for any rational flux. It is therefore valid also for an irrational one. The degeneracy follows from the results of Sec. 1. We have found by direct calculations the finite perturbation of the basis set of functions of the Landau ground state when a periodic magnetic field is added to the unform one (without change of the flux); the perturbed functions have the same energy. We have so far not proved rigorously that for sufficiently large perturbations ΔB of the uniform field B_0 the next level cannot drop to the zeroth level and touch it from above; it appears, however, that in a certain sense this is "almost" always the case.

4. As follows obviously from the conclusion, the Pauli operator can be replaced by a zero-spin Schrödinger operator, where the potential $v(x,y)$ is proportional to the magnetic field: $v = -eB$ on the subspace $\sigma_3\psi = \psi$. All the results are then preserved. If we add a purely potential (albeit small) periodic increment $u(x,y)$, disturbing the connection between u and B, the degeneracy is lifted: for an integer flux $(2\pi)^{-1}e\varphi = N$ (or a rational one NM^{-1} that reduces to a whole-number enlargement of the lattice) the space of the ground states spreads to form a single magnetic band or several magnetic bands (for details see Sec. 4 below). The result is a dispersion law $\varepsilon = \varepsilon_i(p_1,p_2)$, where p_1 and p_2 are the quasimomenta (for the enlarged lattice in the rational case) and $i = 1,2,\ldots,N$.

The set of Bloch functions (8), (14), forms a complex "vector bundle" (i.e., a linear space that depends on the parameters: see Ref. 8) with layer $C^N(p_1,p_2)$, where the base (i.e., the set of parameters) is a two-dimensional torus defined as the factor of the p plane over the reciprocal lattice. This bundle is topologically nontrivial.

5. The results given in the preceding subsections for the spectrum are valid also for an irrational magnetic flux $(2\pi)^{-1}e\varphi$, since they depend continuously on φ in our case. It is possible to identify directly the eigenfunctions of the ground state in this case, too. To be sure, the Bloch functions (8) no longer have any meaning, since no enlargement of the lattice will reduce the problem to an integer flux. It is possible, however, to propose the following prescription: we consider a uniform field B_0 and obtain for it some eigenfunction basis (for any, possibly irrational, value of the flux) in the form $\psi_\alpha = \exp(-e\varphi_0)f_\alpha(z)$, where φ_0 takes the form (7) and $f_\alpha(z)$ is analytic, with α possibly a continuous index. Having obtained this basis, we perform the following operations: 1) we replace the function φ_0 by φ in accord with (7), replacing B_0 by B where the fluxes φ and φ_0 coincide, and 2) we replace in the quantity a defined by (14) the magnetic field B_0 by B.

This pair of operations is performed, strictly speaking, on the functions (8) that are meaningless in the irrational case. The answer, however, is written in the following form: the entire space of the solutions of the equation $H_n\psi = 0$ in the field B is obtained from the analogous space of solutions in the uniform field B_0, with the same flux $\varphi_0 = \varphi$, by multiplying by a single general function

$$\psi \to g(z, \bar{z})\psi, \quad B_0 \to B; \tag{17}$$

$$\begin{aligned} g(z,\bar{z}) = \exp\Bigg\{ & \frac{e}{2\pi} \iint_K \ln|\sigma(z-\zeta)|(B(\zeta) - B_0)\, dx'\, dy' \\ & - \frac{e}{\pi}\Bigg[\operatorname{Re}\left(\eta_1 T_1^{-1} \iint_K (B - B_0)\zeta\, dx'\, dy'\right) \\ & \qquad + i \operatorname{Im}\left(\tilde{\eta}_2 T_2^{-1} \iint_K (B - B_0)\zeta\, dx'\, dy'\right)\Bigg] z \Bigg\} \end{aligned} \tag{18}$$

$$\zeta = x' + iy'.$$

It follows from the results of this research that the case of a periodic field B is more natural than the class of finite localized fields, considered by Aharonov and Casher, among which there is no uniform field. The complexity of the phenomena in the periodic case and, in particular, the appearance of a nontrivial bundling in the description of the magnetic-Bloch functions (see Sec. 3, item 5) takes place already in a uniform field if it is correctly considered.

4. Perturbation by a Weak Potential Field

We discuss now the question of the spreading of the ground state to form a magnetic band upon addition of a weak potential perturbation $u(x, y)$ (electric field) periodic with the same lattice. As already indicated, the operator H at a fixed value of the spin is scalar. For a rational flux $NM^{-1} = (2\pi)^{-1}e\varphi$, with N and M mutually prime, we change to a lattice MT_1 and MT_2. Then the quasimomenta p_1 and p_2 are defined accurate to the addition of integer multiples $2\pi M^{-1}T_1^{-1}$ and $2\pi M^{-1}T_2^{-1}$, respectively. The Bloch functions (8) of the lattice MT_1, MT_2 for an unperturbed operator H form a linear space of dimensionality MN at arbitrary fixed p_1 and p_2 from a unit cell K^* with area

$$|K^*| = 4\pi^2 M^{-2} T_1^{-1} T_2^{-1}.$$

The perturbation of the Hamiltonian $H = H + u(x, y)$, where

$$u(x + T_1, y) = u(x, y + T_2) = u(x, y),$$

defines a Hermitian form $\hat{\varepsilon}$ on the functions (8):

$$\hat{\varepsilon}(\lambda\psi_A, \mu\psi_B) = \bar{\lambda}\mu \iint_K \bar{\psi}_A u(x, y)\psi_B\, dx\, dy, \tag{19}$$

$$A = (a_1, \dots, a_N), \quad B = (b_1, \dots, b_N), \quad |K| = T_1 T_2.$$

From the periodicity of the perturbation $u(x, y)$ with periods T_1 and T_2 it follows that:

1) the form $\hat{\varepsilon}$ differs from zero only when the quasimomenta are equal, or

$$\sum_{i=1}^{MN} a_i = \sum_{i=1}^{MN} b_i$$

by virtue of (15);

2) it follows from the structure of the magnetic-translation group that the matrix of the form $\hat{\varepsilon}$ on the linear space of the functions (8) of dimensionality MN (at

constant p_1 and p_2) has in a natural basis a block-diagonal form consisting of $MN \times N$ blocks, on all of which these forms are equivalent (in particular, they have the same eigenvalues ε_i).

To prove the last fact it is convenient to start with the minimum necessary enlargement of the lattice L_1T_1 and L_2T_2, where $L_1L_2 = M$. Then the quasimomenta p_1 and p_2 are defined in modulo of the lattice $[2\pi(L_1T_1)^{-1}, 2\pi(L_2T_2)^{-1}]$. For the magnetic translations (10) we have

$$T_1^*T_2^* = \xi T_2^*T_1^*, \quad \xi = \exp(2\pi i L_1^{-1}L_2^{-1}), \quad \xi^M = 1. \tag{20}$$

We fix arbitrary values of the quasimomenta $\tilde{p}_1$ and $\tilde{p}_2$ and the vector $e_{1,1}$ which is the eigenvector for $(T_1^*)^{L_1}$ and $(T_2^*)^{L_2}$ with eigenvalues

$$\exp(i\tilde{p}_1T_1L_1), \quad \exp(i\tilde{p}_2T_2L_2)$$

We consider the vectors

$$e_{q,s} = (T_1^*)^q(T_2^*)^s e_{1,1}$$

where $q = 0, 1, \ldots, L_1 - 1$, $s = 0, 1, \ldots, L_2 - 1$. From (20) we get

$$\begin{aligned} (T_1^*)^{L_1}e_{q,s} &= \exp(i(\tilde{p}_1 + 2\pi s M^{-1}T_1^{-1})T_1L_1\}e_{q,s}, \\ (T_2^*)^{L_2}e_{q,s} &= \exp(i(\tilde{p}_2 - 2\pi q M^{-1}T_2^{-1})T_2L_2\}e_{q,s}. \end{aligned} \tag{21}$$

Since everything is periodic with periods T_1 and T_2, the operators T_1^* and T_2^* commute with the perturbed Hamiltonian. Consequently the quadratic form $\hat{\varepsilon}$ is the same (equivalent) for all the quasimomenta obtained by the transformation

$$(\tilde{p}_1, \tilde{p}_2) \to (\tilde{p}_1 + 2\pi s/MT_1, \tilde{p}_2 - 2\pi q/MT_2). \tag{22}$$

By virtue of (22) we can change to the reciprocal lattice $(2\pi M^{-1}T_1^{-1}, 2\pi M^{-1}T_2^{-1})$, i.e., each of the quasimomenta p_1 and p_2 from the unit cell K^* with area $|K^*| = 4\pi^2M^{-2}T_1^{-1}T_2^{-1}$ must be set in correspondence with a direct sum of all N-dimensional spaces with quadratic form $\hat{\varepsilon}(\tilde{p}_1, \tilde{p}_2)$, of which there are exactly M by virtue of (22), where $M = L_1L_2$. The block character of the form on the MN dimensional space is now obvious, and the preservation of the M-fold degeneracy for a potential perturbation periodic in T_1 and T_2 follows from obvious arguments.

The metric on the family of functions (8) is given by

$$\hat{g}(\lambda\psi_A, \mu\psi_B) = \bar{\lambda}\mu \iint_K \bar{\psi}_A\psi_B \, dx \, dy. \tag{23}$$

The dispersion law for a perturbed Hamiltonian is defined as the extrema of the quadratic function $\varepsilon(\lambda\psi_A, \lambda\psi_A)$ under the condition $\hat{g}(\lambda\psi_A, \lambda\psi_A) = \text{const}$:

$$\nabla_{A,\lambda}(\hat{\varepsilon}(\lambda\psi_A, \lambda\psi_A) - \varepsilon\hat{g}(\lambda\psi_A, \lambda\psi_A)) = 0 \tag{24}$$

[the gradient is with respect to the variables λ and $A = (a_1, \ldots, a_{MN})$], and at a fixed quasimomentum

$$\sum_{j=1}^{MN} a_j = \text{const}.$$

The dispersion law $\varepsilon_i(p_1, p_2)$, as indicated above, is M-fold degenerate. We have in fact, generally speaking, N different eigenvalues $\varepsilon_1(p_1, p_2), \ldots, \varepsilon_N(p_1, p_2)$, each of which is M-fold. This is the consequence of the rational character of the flux with denominator M. We recall that the quasimomenta p_1 and p_2 are defined here in modulo $2\pi M^{-1}T_1^{-1}$ and $2\pi M^{-1}T_2^{-1}$.

The simplest cases are:

1) $N = 1$, $M = 1$. The dispersion law is defined as a single-valued continuous function on a torus, $\varepsilon(p_1, p_2)$; there is no degeneracy at fixed p_1 and p_2. Perturbations lead, obviously, to only one magnetic band.

2) $N = 1$, $M > 1$. Here too, there is only one magnetic band; at fixed p_1 and p_2 we have M-fold degeneracy ("vector bundling" on a torus). Only one magnetic band is produced (this result was first obtained by Zak[4] for a uniform field B_0).

For $N > 1$ (and arbitrary M) there are produced M-fold degenerate eigenvalues $\varepsilon_1(p_1, p_2), \ldots, \varepsilon_N(p_1, p_2)$; it can be assumed that $\varepsilon_i \neq \varepsilon_j$ at $i \neq j$. The equality of two eigenvalues of complex Hermitian matrix $\hat{\varepsilon}$ is specified by three real conditions. Therefore for a two-parameter family of matrices $\hat{\varepsilon}(p, p)$ we can (in a "typical" case) assume all the eigenvalues to coincide at arbitrary p_1 and p_2. If even two branches coincide accidentally at the point p_1, p_2 they cannot be shifted on going around this point—the branches are separated by virtue of the general properties of Hermitian matrices.

We shall move by varying p_1 (under the condition $p_2 = \text{const}$), where $-\pi/MT_1 \leq p_1 \leq \pi/MT_1$, and track the eigendirection initially numbered j. On going through the entire period we arrive, generally speaking to another eigendirection γ_{ij}. We obtain a permutation of the eigendirections (monodromy) $\gamma_1 \colon 1 \to \gamma_{11}, \ldots, N \to \gamma_{iN}$.

Analogously, by varying p_2 (with p_1 constant) we obtain after the circling the permutation $\gamma_2 \colon j \to \gamma_{2j}$. The permutations γ_1 and γ_2 commute: $\gamma_1\gamma_2 = \gamma_2\gamma_1$. We can therefore obtain for both permutations γ_1 and γ_2 a common minimum breakdown of the set of eigendirections $(1, \ldots, N)$ into cycles of length $n_1, \ldots, n_k$, where

$$\sum_{q=1}^{k} n_q = N.$$

Within the limit of each cycle η_q, the branches $\varepsilon_i(p_1, p_2)$ cannot be subdivided, since they are shifted after circling over the periods. In this case there are produced exactly k magnetic bands, corresponding to the different cycles n_q, although the energy values may overlap. The possible types of decay of the zeroth level into magnetic bands, due to the small periodic potential $u(x, y)$, is described by breakdown of the number N into terms

$$N = \sum_q n_q.$$

A more detailed classification of these types is determined by the commutation group of the permutations γ_1 and γ_2.

The principal spectral characteristic in each magnetic band with number j is the density of the number of states per unit area in the interval dE, which we designate by $\mu_j(E)\,dE$, where

$$\int \Bigl(\sum_j \mu_j\Bigr) dE = \frac{e\varphi}{2\pi|K|} = \frac{1}{4\pi^2}|K^*|M.$$

Here M is the multiplicity of the degeneracy at fixed p_1 and p_2. Irrational numbers $(2\pi)^{-1}e\varphi$ can be approximated by rational ones $N_i M_i^{-1}$, where the numerator N_i and the denominator M_i increase if $N_i M_i^{-1} \to (2\pi)^{-1}e\varphi$, $i \to \infty$. It is easily seen that with increasing N_i and M_i the dispersion law breaks up into more and more magnetic bands, so that all the characteristics, with the exception of the total density of the number of states per unit area

$$\sum_j \mu_j \, dE,$$

become meaningless.

At large N it is possible to pose the natural problem of calculating the statistical weights of various numbers of magnetic bands produced in the decay, bearing in mind the percentage of the configurations γ_1 and γ_2 with the number of bands in the given interval $k \pm \Delta k$ (i.e., the common cycles of a pair of commuting permutations) relative to the total number of configurations, i.e., classes of conjugacy of the commuting permutations. This is a purely combinatorial problem.

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