THE RELATIONSHIP BETWEEN HAMILTONIAN FORMALISMS OF STATIONARY AND NONSTATIONARY PROBLEMS

O. I. BOGOYAVLENSKII AND S. P. NOVIKOV

According to Gardner [2], Hamiltonian systems (flows) on functional spaces $u(x)$ can be formally written as follows:

$$\frac{\partial u}{\partial t} = \dot{u} = \frac{d}{dx}\frac{\delta I}{\delta u(x)}.$$

where $I = \int L\left(u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^n u}{\partial x^n}\right) dx$, $\frac{\delta I}{\delta u(x)} = \frac{\partial L}{\partial u} + \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}\frac{\partial L}{\partial u^{(k)}}$. Indeed, these systems are Hamiltonian on any class of functions in which I makes sense, since the Poisson bracket has the form $[I, I_1] = \int \frac{\delta I}{\delta u}\frac{d}{dx}\frac{\delta I_1}{\delta u}\, dx$.

In [1], Novikov has solved on the basis of the Korteweg–de Vries equation (KdV) and its higher order counterparts $\dot{u} = \frac{d}{dx}\frac{\delta I_n}{\delta u(x)}$, the stationary problem $\frac{\delta I_n}{\delta u(x)} =$ const. The latter equations are also Hamiltonian. In [1] it is asserted that there exists a relationship between the Hamiltonian formalisms of the stationary and the nonstationary problem, though this was not precisely formulated. In this paper we shall formulate and prove a general theorem on such a relationship.

Let us consider two formally Hamiltonian flows in a class of smooth functions on the x axis:

$$\dot{u} = \frac{d}{dx}\frac{\delta I}{\delta u}(X), \quad I = \int L\, dx, \quad \dot{u} = \frac{d}{dx}\frac{\delta I_1}{\delta u}(X_1), \quad I_1 = \int L_1\, dx,$$

where the Lagrangians $L = L(u, u^{(1)}, \dots, u^{(k)})$, $L_1 = L_1(u, u^{(1)}, \dots, u^{(k)})$ do not explicitly depend on x; $u^{(s)} = \frac{\partial^s u}{\partial x^s}$. For simplicity we shall assume that $k < n$. Let $I_0 = \int \frac{u^2}{2}\, dx$, $L_0 = \frac{u^2}{2}$, and the corresponding flow which commutes with X and X_1 will be denoted by X_0. This is a shift with respect to x. It is well known that commutativity of flows is equivalent [2] to

$$(1) \qquad \frac{\delta I}{\delta u}\frac{d}{dx}\frac{\delta I_1}{\delta u} \equiv \frac{d}{dx}Q(u, u^{(1)}, \dots, u^{(2n-1)}), \quad [I, I_1] = 0.$$

It follows from the commutativity of the flows X and X_1 that the fixed points of the flow X, i.e., the solutions of the equation

$$(2) \qquad \frac{\delta I}{\delta u} = \frac{\partial L}{\partial u} + \sum_{k=1}^{n}(-1)^k \frac{d^k}{dx^k}\frac{\partial L}{\partial u^{(k)}} = -h, \quad h = \text{const},$$

form an invariant set for the flows X_0 and X_1. Equation (2) has the form $\frac{\delta I_h}{\delta u(x)} = 0$, $L_h = L + hu$. It follows from (1) that $Q_h = Q + h\frac{\delta I_1}{\delta u(x)}$ is an integral of Eq. (2), $\frac{\delta I_h}{\delta u(x)} = 0$.

Date: Original article submitted September 22, 1975.

Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 10, No. 1, pp. 9–13, January–March, 1976.

In KdV theory, this construction of integrals of the stationary problem for higher order KdV was proposed in [3] and [4]; it differs from the construction of integrals in [1].

Equation (2) specifies a Hamiltonian system in the corresponding $2n$-dimensional phase space T; each solution of Eq. (2) is determined by its initial point in this phase space for an $x = x_0$. Hence, a flow X_1 that acts on the solutions of Eq. (2) determines a one-parameter family of flows $\phi_h(X, X_1)$ dependent on h in a finite-dimensional phase space T. If $X_1 = X_0$, then the flow $\phi_h(X, X_0)$ will evidently be a shift with respect to x along the trajectory of the Hamiltonian flow (2), and the corresponding integral Q_h will be the energy. We have a general theorem.

Theorem 1. *The flow $\phi_h(X, X_1)$ in phase space T is Hamiltonian with a Hamiltonian $(-Q_h)$ [we assume, of course, that the transition to canonical variables (p_i, q_i) is one-to-one and that $\partial^2 H/\partial p_n^2 \neq 0$].*[1]

Remark 1. Let us note that all the solutions of the stationary Eq. (2) can be, for example, increasing functions. Therefore, it makes no sense to restrict the symplectic structure to the set of fixed points of the flow X, thus, excluding the special case of completely integrable systems (for example, higher order KdV), where the entire region of solutions of Eq. (2) is filled with periodic and almost periodic functions. Precisely such a situation is encountered in [1].

For the proof, let us describe the phase space T. In this space we introduce the following coordinates and momenta:

$$q_i = u^{(i-1)}, \quad p_i = \frac{\partial L_h}{\partial u^{(i)}} + \sum_{s=1}^{n-i} (-1)^s \frac{d^s}{dx^s} \frac{\partial L_h}{\partial u^{(i+s)}}, \quad 1 \leq i \leq n. \tag{3}$$

A canonical structure on T has standard form $\omega^2 = \sum_{i=1}^n dp_i \wedge dq_i$. The Hamiltonian flow specified by (2) coincides with $\phi_h(X, X_0)$ and has a Hamiltonian

$$H = \sum_{i=1}^n p_i u^{(i)} - L_h = \sum_{i=1}^{n-1} p_i q_{i+1} + p_n q_n' - L_h(q_1, \ldots, q_n, q_n') = \tag{4}$$

$$= \sum_{i=1}^{n-1} p_i q_{i+1} + \Phi(q_1, q_2, \ldots, q_n, p_n),$$

where according to (3) we have $p_n = \partial L_h / \partial q_n'$ (the prime denotes the derivative with respect to x). Let note that

$$-p_1' - \frac{\partial H}{\partial q_1} \equiv \frac{\delta I_h}{\delta u}. \tag{5}$$

All the equations $p_i' + \frac{\partial H}{\partial q_i} = 0$, $q_i' - \frac{\partial H}{\partial p_i} = 0$, apart from the equation $p_1' + \frac{\partial H}{\partial q_1} = 0$, are equivalent to the conditions (3) and make it possible to express the derivatives $u^{(1)}, \ldots, u^{(2n-1)}$ in terms of the phase coordinates p_i and q_i.

[1] Theorem 1 holds also for other canonical structures (Poisson brackets) in the space of functions $u(x)$, for example, if $\{I, I_1\} = \int \left(\frac{\delta L}{\delta u} \cdot A_{2n+1} \frac{\delta L_1}{\delta u} \right) dx$, where $A_{2n+1} = \sum_{i=1}^n c_i \frac{\partial^{2i+1}}{\partial x^{2i+1}}$, and also in the space of vector functions $(p(x), q(x))$ have the form

$$\{I, I_1\} = \int \left(\frac{\delta L}{\delta p} \frac{\delta L_1}{\delta q} - \frac{\delta L}{\delta q} \frac{\delta L_1}{\delta p} \right) dx.$$

By virtue of (5), the identity (1) will be expressed in phase coordinates as follows:

$$\left(-\frac{\partial H}{\partial q_1} - p_1'\right)\left(\frac{d}{dx}\frac{\delta I_1}{\delta u}\right) \equiv \frac{\partial Q_h}{\partial p_1}p_1' - \sum_{i=2}^{n}\frac{\partial Q_h}{\partial p_i}\frac{\partial H}{\partial q_i} + \sum_{i=1}^{n}\frac{\partial Q_h}{\partial q_i}\frac{\partial H}{\partial p_i}. \tag{6}$$

Let us note that by virtue of our condition the identity (1) is valid for any function $u(x)$; therefore p_1' in (6) can be any number irrespective of p_i and q_i. Hence

$$\frac{d}{dx}\frac{\delta I_1}{\delta u} \equiv -\frac{\partial Q_h}{\partial p_1}. \tag{7}$$

By substituting this expression into (6), we obtain

$$\frac{\partial H}{\partial q_1}\frac{\partial Q_h}{\partial p_1} + \sum_{i=2}^{n}\frac{\partial H}{\partial q_i}\frac{\partial Q_h}{\partial p_i} - \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\frac{\partial Q_h}{\partial q_i} = \{Q_h, H\} = 0. \tag{8}$$

Let us consider the flow $\phi_h(X, X_1)$ in phase space. The derivatives along this flow will be denoted by a point. From the definition of this flow $\dot{u} = \frac{d}{dx}\frac{\delta I_1}{\delta u(X)}$ and from (7) it follows that $\dot{q}_1 = \frac{\partial Q_h}{\partial p_1}$. By induction let us assume that the equation $\dot{q}_i = -\frac{\partial Q_h}{\partial p_i}$ $(1 \le i < n)$ is valid; it hence follows by virtue of Jacobi's identity for Poisson's bracket $\{\,,\}$ that

$$\dot{q}_{i+1} = \dot{q}_i' = -\frac{d}{dx}\frac{\partial Q_h}{\partial p_i} = -\{H, \{Q_h, q_i\}\} = \{Q_h, \{\dot{q}_i, H\}\} + \{q_i, \{H, Q_h\}\}.$$

By virtue of (4), $\{q_i, H\} = -\frac{\partial H}{\partial p_i} = -q_{i+1}$; by virtue of (8), $\{H, Q_h\} = 0$. Therefore, $\dot{q}_{i+1} = -\{Q_h, q_{i+1}\} = -\frac{\partial Q_h}{\partial p_{i+1}}$, Hence, for any i $(1 \le i \le n)$ we have the equations

$$\dot{q}_i = -\frac{\partial Q_h}{\partial p_i}.$$

Let us differentiate the equation $q_n' = \frac{\partial H}{\partial p_n}$ along the flow $\phi_h(X, X_1)$, and the equation $\dot{q}_n = -\partial Q_h/\partial p_n$ along the flow $\phi_h(X, X_0)$. We obtain

$$(q_n')^{\cdot} = \sum_{i=1}^{n}\frac{\partial^2 H}{\partial p_n\,\partial q_i}\dot{q}_i + \frac{\partial^2 H}{\partial p_n^2}\dot{p}_n = -\sum_{i=1}^{n}\frac{\partial^2 H}{\partial p_n\,\partial q_i}\frac{\partial Q_h}{\partial p_i} + \frac{\partial^2 H}{\partial p_n^2}\dot{p}_n,$$

$$(\dot{q}_n)' = -\left\{H, \frac{\partial Q_h}{\partial p_n}\right\} = -\{H, \{Q_h, q_n\}\} = \{Q_h, \{q_n, H\}\} + \{q_n, \{H, Q_h\}\} =$$

$$= -\left\{Q_h, \frac{\partial H}{\partial p_n}\right\} = -\sum_{i=1}^{n}\frac{\partial^2 H}{\partial p_n\,\partial q_i}\frac{\partial Q_h}{\partial p_i} + \frac{\partial^2 H}{\partial p_n^2}\frac{\partial Q_h}{\partial q_n}.$$

By virtue of commutativity of differentiations with respect to the flows $\phi_h(X, X_1)$ and $\phi_h(X, X_0)$, the two expressions will be equal. Hence follows that $\dot{p}_n = \partial Q_h/\partial q_n$. By induction let us assume that the equation $\dot{p}_i = \partial Q_h/\partial q_i$ is valid where $1 < i \le n$. Let us differentiate this equation along the flow $\phi_h(X, X_0)$, and the equation

$p_i' = -\partial H/\partial q_i$ along the flow $\phi_h(X, X_1)$. Hence

$$(\dot{p}_i)' = \left\{H, \frac{\partial Q_h}{\partial q_i}\right\} = \{H, \{p_i, Q_h\}\} = -\{p_i, \{Q_h, H\}\} - \{Q_h, \{H, p_i\}\} =$$

$$= \left\{Q_h, \frac{\partial H}{\partial q_i}\right\} = \sum_{j=1}^{n} \frac{\partial^2 H}{\partial q_i\, \partial q_j} \frac{\partial Q_h}{\partial p_j} - \frac{\partial^2 H}{\partial q_i\, \partial p_n} \frac{\partial Q_h}{\partial q_n} - \frac{\partial^2 H}{\partial q_i\, \partial p_{i-1}} \frac{\partial Q_h}{\partial q_{i-1}},$$

$$(p_i')^{\cdot} = \sum_{j=1}^{n} \frac{\partial^2 H}{\partial q_i\, \partial q_j} \frac{\partial Q_h}{\partial p_j} - \frac{\partial^2 H}{\partial q_i\, \partial p_n} \frac{\partial Q_h}{\partial q_n} - \frac{\partial^2 H}{\partial q_i\, \partial p_{i-1}} \dot{p}_{i-1}.$$

Since

$$\frac{\partial^2 H}{\partial q_i\, \partial p_{i-1}} = 1$$

[by virtue of (4)], it hence follows that $\dot{p}_{i-1} = \frac{\partial Q_h}{\partial q_{i-1}}$. For any i ($1 \le i \le n$), we hence obtain the equations $\dot{p}_i = \frac{\partial Q_h}{\partial q_i}$. This completes the proof of Theorem 1.

Remark 2. It is not difficult to formulate and prove the theorem also in the case $k \ge n$, by expressing the higher order derivatives from Eq. (2).

Corollary 1. *Any Lie algebra L of formally Hamiltonian flows that commute with X can be holomorphically mapped on a Lie algebra of Hamiltonian fields $\phi_h(X, L)$ in a finite-dimensional phase space of stationary points of the flow X.*

In the case of higher order KdV, where the flow $X = X_n$ is defined by Kruskal integrals I_n, we have a commutative algebra L specified by flows X_k of the form

$$\dot{u} = \frac{d}{dx} \frac{\delta I_k}{\delta u(x)}, \quad k = 0, 1, \dots, n-1$$

see [2] and [5]). It is easy to show [4] that the flows $\phi_h(X_n, X_k)$ are linearly independent. Hence follows Theorem 1 the commutativity of this complete set of integrals. Gel'fand has pointed out to the authors that this assertion had been proved right away by him and Dikii [6] by a concrete analysis of Kruskal integrals; however in fact, it is a consequence of the very general and simple Theorem 1, as assumed in [1].

Let us note that Novikov [1] has given an entirely different construction of integrals of the stationary problem

$$\sum_{i=0}^{n} c_i \frac{\delta I_{n-i}}{\delta u(x)} = d = \text{const}$$

for higher order KdV; by the integrals obtained in [1], it was possible to explicitly define the spectrum of a Hill (Schrödinger) operator with a potential $u(x)$. For two-zone potentials ($n = 2$, $c_0 = 1$, $c_1 = 0$), by comparing the Lax–Gel'fand–Dikii integrals with the Novikov integrals ([1], Example 2), we obtain

$$R(E) = E^5 + \frac{c_2}{8} E^3 - \frac{d}{16} E^2 + \left(\frac{c_2^2}{64} + \frac{Q}{32}\right) E + \frac{-2Q_1 + 2c_2 d}{16^2},$$

where $(-Q)$ and $(-Q_1)$ are the Hamiltonians of the flows $\phi_d(X, X_0)$ and $\phi_d(X, X_1)$. The roots of the equation $R(E) = 0$ are the edges of the zones (lacunas).

Note Added in Proof. O. I. Bogoyavlenskii has obtained a formula that linearly expresses Novikov integrals [1] in terms of Lax–Gel'fand–Dikii integrals [3, 6] for any $n \geq 2$ [O. I. Bogoyavlenskii, Funktsional'. Analiz i Ego Prilozhen., **10**, No. 2 (1976)].

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L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR