THE PERIODIC PROBLEM FOR THE KORTEWEG–DE VRIES EQUATION

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Introduction

The Korteweg–de Vries equation (KV) arose in the nineteenth century in connection with the theory of waves in shallow water. It is now known (cf., for example, [15]) that this equation also describes the propagation of waves with weak dispersion in various nonlinear media. In reduced form it is written

$$u_t = 6uu_x - u_{xxx}.$$

In 1967 in the well-known work of Gardner, Green, Kruskal, and Miura a remarkable procedure for integrating the Cauchy problem for this equation was discovered for functions $u(X)$ which are rapidly decreasing for $x \to \pm\infty$. This procedure consists in the following: we consider the Schrödinger (Sturm–Liouville) operator $L = -(d^2/dx^2) + u$; let $f(x,k)$ and $g(x,k)$ be such that $Lf(x,k) = k^2 f$, $Lg(x,k) = k^2 g$, whereby $f(x,k) \to e^{-ikx}$ for $x \to -\infty$ and $g(x,y)e^{-ikx}$ for $x \to +\infty$. We then have two pairs of linearly independent solutions $f, \bar{f}$ and $g, \bar{g}$ and a transition matrix

$$f(x,k) = a(k)g(x,k) + b(k)\bar{g}(x,k),$$
$$\bar{f}(x,k) = \bar{b}(k)g(x,k) + \bar{a}(k)\bar{g}(x,k).$$

If the potential u changes in time according to the KV equation, then the coefficients a and b vary according to the law $\dot{a} = 0$, $\dot{b} = 8ik^3 b$, and for the eigenfunction $f(x,k)$ there is the equation $\dot{f} = Af + \lambda f$, where $A = 4\frac{d^3}{dx^3} - 3\left(u\frac{d}{dx} + \frac{d}{dx}u\right)$, $\lambda = 4ik^3$. If $\lambda_n = +k_n^2 = (i\kappa_n)^2$ is a point of the discrete spectrum of the potential $u(x)$, then $\dot{\kappa}_n = 0$ and $\dot{c}_n = +8\kappa^3 c_n$, where c_n is the natural normalization of the eigenfunction. These formulas make it possible to reduce the Cauchy problem for the KV equation with rapidly decreasing functions to an inverse problem of scattering theory and to use the results of I. M. Gel'fand, B. I. Levitan, V. A. Marchenko, and L. D. Faddeev (cf. [2, 3, 4]). Subsequently, P. Lax [7] discovered that the KV equation is identical to the operator equation $\dot{L} = [A, L]$, where $L = -\frac{d^2}{dx^2} + u$, $A = 4\frac{d^3}{dx^3} - 3\left(u\frac{d}{dx} + \frac{d}{dx}u\right)$, since $\dot{L}$ is the operator of multiplication by $\dot{u}$, while $[A, L]$ is the operator of multiplication by the function $6uu' - u'''$. In particular, it follows from this that the spectrum of the operator L is an integral of the KV equation (this is also true for the periodic problem). This fact reveals the algebraic meaning of the procedure in [5] and is extremely useful for the application of these ideas to other problems. Further, L. D. Faddeev and V. E. Zakharov [8] have shown that the KV equation is a completely integrable Hamiltonian system, where the canonical variables are $\frac{2k}{\pi}\ln|a(k)|$, $\arg b(k)$, κ_n^2, $\ln(b_n)$, $b_n = ic_n\left(\frac{da}{dk}\right)_{k=i\kappa_n}$. In particular, the eigenvalues $\lambda_n = k_n^2$ are commuting integrals of the KV equation also in the periodic problem [and not only in the case of rapidly decreasing functions $u(x)$]. A number of other equations have subsequently been found which admit the "Lax representation" $\dot{L} = [A, L]$ for a certain pair of operators A, L. P. Lax [7] and Gardner [6] have shown that the known polynomial integrals $I_n(u) = \int_{-\infty}^{\infty} P_n(u, \dots, u^{(n)})\, dx$ of the KV equation (the I_n are expressed in terms of the spectrum of the operator L) all determine equations

$$\dot{u} = \frac{d}{dx} \frac{\delta I_n}{\delta u(x)}, \tag{1}$$

admitting the Lax representation $\dot{L} = [A_n, L]$, where $L = (d^2/dx^2) + u$ and the A_n are certain skew-symmetric operators of order $2n + 1$,

$$\begin{aligned} I_0 &= \int u^2\, dx, \quad I_1 = \int \left(\frac{u'^2}{2} + u^3 \right) dx, \\ I_2 &= \int \left(\frac{u''^2}{2} - \frac{5}{2} u^2 u'' + \frac{5}{2} u^4 \right) dx, \\ A_0 &= \frac{d}{dx}, \quad A_1 = 4 \frac{d^3}{dx^3} - 3 \left(u \frac{d}{dx} + \frac{d}{dx} u \right), \\ A_2 &= \frac{d^5}{dx^5} - \frac{5}{2} u \frac{d^3}{dx^3} - \frac{15}{4} u' \frac{d^2}{dx^2} + \frac{15u^2 - 25u''}{8} \frac{d}{dx} + \frac{15}{8} \left(uu' - \frac{u'''}{2} \right). \end{aligned} \tag{2}$$

We shall call these equations "higher KV equations." Further, beginning with the papers [9, 10], a number of other important equations were found which admit the Lax representation $\dot{L} = [A, L]$, where L is no longer a Schrödinger operator (and is not always symmetric). In the papers of L. D. Faddeev, V. E. Zakharov, and A. B. Shabat ([10, 11, 12]) the needed generalization of scattering theory was carried out for the new operators which has made it possible to solve the direct and inverse problems and to carry through the "Kruskal" integration of the Cauchy problem for rapidly decreasing (as $x \to \pm\infty$) initial data. A considerable literature has recently been devoted to discovering such new equations and carrying over the Kruskal mechanism to them. However, even for the original KV equation the periodic problem has not moved forward. The only new result in the periodic case, which was obtained by the method of Gardner, Green, Kruskal, and Miura, is the theorem of Faddeev and Zakharov to the effect that the eigenvalues of the operator L are commuting integrals of the KV equation as a Hamiltonian system (we remark that the integrals themselves can be expressed in a one-to-one manner in terms of the previously known polynomial integrals I_n which are thus also involutive). This result has not been used in an essential way until the present work, but here it plays an important role (cf. §2).

The interaction of simple waves [solutions of the type $u(x - ct)$, usually called "solitons"] are of major interest in the theory of the KV equation. This interaction is described by means of so-called "multisoliton solutions" where $b(k) \equiv 0$ for real k.[1] These solutions decay into solitons for $x, t \to \pm\infty$ and describe their interaction for finite t. For this case the Gel'fand–Levitan equations are completely solvable (cf., for example [8, 9]). Another method of obtaining multisoliton solutions has been developed in [13]. However, all these results refer to the case of rapidly decreasing functions $u(x)$. In the periodic case the solitons $u(x - ct)$ of the KV equation are of a more complicated structure; there are many more of them and their

[1] I. M. Gel'fand has informed the author that such potentials $u(x)$ were first considered by Bargmann.

interaction has not been studied at all. In the present paper we propose a method of studying certain analogs of the "multisoliton" solutions of the KV equation which, generally speaking, are found to be not only periodic, but also conditionally periodic functions $u(x)$ describing the interaction of periodic solitons. Our work is based on certain simple but fundamental algebraic properties of equations admitting the Lax representation which are strongly degenerate in the problem with rapidly decreasing functions (for $x \to \pm\infty$), and have therefore not been noted. Finally, it is essential to note the nonlinear "superposition law for waves" for the KV equation which in the periodic case has an interesting algebraic-geometric interpretation. The superposition law will be discussed in the second part of the work.

In conclusion, we call the attention of the reader to the following circumstance: in classical mechanics and mathematics the appearance of integrals in conservative systems (conservation laws) is almost always related to a Lie symmetry group of the problem in question. Other fundamental algebraic mechanisms of integrability were previously unknown. However, there were several exceptions: for example, the Jacobi case (geodesics on a triaxial ellipsoid) or the case of Kovalevskaya (the problem of the motion of a solid body with a fixed point in a gravitational field). Other exceptional examples are now known. There is not the slightest doubt that all these cases are the manifestation of a Kruskal-type algebraic mechanism based on the possibility of a Lax-type representation for these dynamical systems.

1. The Schrödinger (Sturm–Liouville) Equations with Periodic Coefficients. The Monodromy Matrix

We shall first list systematically simple facts which we shall need.

Let $u(x)$ be a smooth function where $u(x+T) = u(x)$, and let $L = (d^2/dx^2) + u$. We consider on the line the equation $L\psi_k = \lambda\psi_k$, where $\lambda = k^2$ is a real number. We consider the pair of linearly independent solutions $\psi_k(x, x_0), \bar{\psi}_k(x, x_0)$, where $\psi_k(x_0, x_0) = 1$, $\psi'_k(x_0, x_0) = ik$, or the pair $\phi_k, \bar{\phi}_k$, where $\phi_k(x_0, x_0) = 1$, $\bar{\phi}_k(x_0, x_0) = i$. (The pair $\psi_k, \bar{\psi}_k$ is more convenient but is meaningful only for $k^2 > 0$.) We can define the "monodromy matrix"

$$T(k, x_0) = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad a = a(k, x_0), \quad b = b(k, x_0),$$

where

$$\psi_k(x+T, x_0) = a\psi_k(x, x_0) + b\bar{\psi}_k(x, x_0),$$
$$\bar{\psi}_k(x+T, x_0) = \bar{b}\psi_k(x, x_0) + \bar{a}\bar{\psi}_k(x, x_0),$$

or the analogous matrix in another basis. In the basis $\phi_k, \bar{\phi}_k$, for example, the monodromy matrix is an entire function of $\lambda = k^2$. The trace of the matrix $\operatorname{Sp} T = a + \bar{a} = 2a_R$ is real, while the determinant is equal to one, $\det T = |a|^2 - |b|^2 = 1$ for all real k, since the Wronskian determinant is conserved.

The eigenvalues of the matrix $T(k, x_0)$ do not depend on the point x_0 and have the form: $\mu_\pm = a_R \pm \sqrt{a_R^2 - 1}$. In particular, we have two cases: $\mu_\pm = e^{\pm ip}$, $|a_R| \leq 1$; $\mu_\pm = e^{\pm p}$, $|a_R| \geq 1$, where p is a real number ($a_R = \cos p$ for $|a_R| \leq 1$). The Bloch eigenfunctions of the operator L are those solutions of the equation $Lf = k^2 f$ such that $f(x+T) = e^{\pm ip} f(x)$, where the number p is called the "quasi-momentum." We have periodic eigenfunctions for $e^{ip} = 1$ or $a_R = 1$, and antiperiodic eigenfunctions $f(x+T) = -f(x)$, where $a_R = -1$. The permitted zones are the regions on the axis $\lambda = k^2$, where $|a_R| \leq 1$, and the forbidden zones are the regions on the axis $\lambda = k^2$, where $|a_R| \geq 1$. The boundaries of the permitted and forbidden zones are the points $|a_R| = 1$, where the numbers $\lambda_n = k_n^2$ are the eigenvalues of the periodic or antiperiodic problem. These eigenvalues may be nondegenerate or doubly degenerate. Since $a = a_R + ia_I$, $b = b_R + ib_I$, and $|a|^2 - |b|^2 = 1$, it follows that at points of both spectra, where $|a_R| = 1$, we have $|a_I| = |b|$, $\lambda = \lambda_n = k_n^2$.

In the nondegenerate case $|a_I| = |b| \neq 0$, $\lambda = \lambda_n$. In the degenerate case $|a_I| = |b| = 0$, $\lambda = \lambda_n$ and therefore the forbidden zone in the degenerate case contracts to zero. Thus, the degenerate points of both spectra where $b = 0$ for $\lambda = \lambda_n$ are not boundaries of permitted zones. In the sequel we shall call "essential" only those nondegenerate points of both spectra which are boundaries of Bloch zones and which completely determine the zone structure on the axis $\lambda = k^2$. There is the following simple fact: multiplying the period T by an integer $T \to nT$ does not change the zone structure. Indeed, multiplication of the period $T \to nT$ raises the monodromy matrix to a power $T(k, x_0) \to T^n(k, x_0)$. Thus, the forbidden zones remain forbidden zones $e^{\pm p} \to e^{\pm np}$ and the permitted zones remain permitted zones $e^{\pm ip} \to e^{\pm inp}$, although in the interior of the permitted zones there appear new degenerate ("inessential") levels where $b \equiv 0$ which for increasing n fill out the entire zone in a dense manner.

This fact makes it possible to carry over the definition of the zone structure of a potential to almost periodic functions by approximating them by periodic functions with increasing period. However, in the literature there are no investigations of the convergence of such a process (apparently it has never been studied). It is possible that for linear combinations of periodic functions even with two periods T_1, T_2, the result may depend on the arithmetic of the number T_1/T_2.

We note that the function $\operatorname{Sp} T = 2a_R$ is an entire function of λ in the entire complex plane and can be defined up to a constant multiple as an infinite product (this was indicated to the author by L. D. Faddeev):

$$\frac{1}{2} \operatorname{Sp} T(\lambda) = 1 + \text{const} \prod_j (\lambda - \lambda_j),$$

where the λ_j are the points of the spectrum of the periodic problem.

The points of the spectrum of the antiperiodic problem $f(x + T) = -f(x)$ are thus determined in principle by the equation $a_R = -1$ starting from the purely periodic spectrum. However, for this it is necessary to know all the points of the spectrum of the periodic problem, while we do not wish, for example, to regenerate points of the spectrum. Therefore, we shall continue to work with both spectra.

We defined the monodromy matrix $T(k, x_0)$ by choosing an initial point x_0, although its eigenvalues not depend on the initial point. Therefore, the matrices $T(k, x_0)$ for fixed k but various x_0 are conjugate. From this it follow immediately that with regard to the dependence of this matrix on the parameter x_0, which we now denote by x, a differential equation of the form

$$\frac{dT}{dx} = [Q, T] \tag{3}$$

is satisfied, where the matrix Q is easily computed as the transition matrix $(1+Q\,dx)$ from the basis $[\psi_k(x, x_0 + dx), \bar{\psi}_k(x, x_0 + dx)]$ to the basis $[\psi_k(x, x_0), \bar{\psi}_k(x, x_0)]$. Therefore $(1 + Q\,dx)T(1 - Q\,dx) = T(k, x_0 + dx)$ or $T' = [Q, T]$.

In our bases the matrix Q has the form

$$\begin{aligned} Q &= -ik\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{iu}{2k}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad (\text{basis } \psi_k, \bar{\psi}_k), \\ Q &= -\frac{i}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} - \frac{i}{2}(u-k^2)\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad (\text{basis } \phi_k, \bar{\phi}_k). \end{aligned} \tag{4}$$

Thus, the monodromy matrix T can be sought as a periodic solution of Eq. (3) which satisfies the condition $\det T = |a|^2 - |b|^2 = 1$. At the nondegenerate points of both spectra $|a_R| = 1$, where $b(k_n) \neq 0$ and $|b| = \pm a_I$ the following equation is obtained from Eq. (3) by direct substitution:

$$\phi' \pm \frac{u}{k}\sin\phi = -2k + \frac{u}{k}, \quad k = k_n, \tag{3$'$}$$

where $\phi = \arg b(k,x)$ for $k = k_n$, $a_I = \mp|b| \neq 0$. In principle, the nondegenerate points of the spectrum k_n^2 are determined from the requirement that the latter Eq. (3$'$) should have a periodic solution. In general, this condition may also include "extraneous roots" k_n; however, for the zero potential $u \equiv 0$ we see that $\Delta\phi = 2\pi n = 2kT$ gives the points of the spectrum exactly (analogously for the case $u = \text{const}$). Formally this equation is applicable only in the nondegenerate case, but in view of the stability of the properties of this equation under small (smooth) variations of the potential u, use can be made of the fact that all the levels become nondegenerate after almost any small perturbation. From this there follows:

Proposition 1.1. 1) *For potentials close to zero* (*or to a constant*) *the condition of the existence of periodic solutions of Eq.* (3') *includes all the nondegenerate points of the spectrum k_n and is not solvable for k_n which are not spectral points*;

2) *If* a) *the function $\phi(k,x)$ is known as a function of u for a given potential u for all k, x, where it is meaningful* (*i.e., on the entire line with the exception of degenerate points of both spectra $a_I = b = 0$*), *and* b) *the function $(u/k)\sin\phi$ extends as a smooth function of the variables x, k to all x, k including the degenerate points of the spectrum $k = k_n$, then each spectral points satisfies the transcendental equation*

$$2\pi n = \int_{x_0}^{x_0+T} \left(2k - \frac{u}{k} \pm \frac{u}{k}\sin\phi\right) dx. \tag{5}$$

Assertion 1) of Proposition 1.1 was proved above, since for a constant potential the condition on the spectral points is exact. For the proof of assertion 2) we remark that the function $\sin\phi$ at all points x, k where it is well defined (i.e., $b \neq 0$) depends smoothly on the potential $u(x)$. If after an arbitrarily small perturbation δu of the potential u which makes all levels nondegenerate Eq. (5) is not satisfied for given k, then this point k is not a spectral point for the potential u. This implies assertion 2) of Proposition 1.1.

2. Potentials with a Finite Number of Zones and Multisoliton Solutions of the KV Equation

We recall that for an infinite period $T = \infty$ where $u(x) \to 0$ for $x \to \pm\infty$; the monodromy matrix is defined from the transition from $-\infty$ to $+\infty$. The multisoliton solutions $u(x,t)$ of the KV equation are determined from the condition $b \equiv 0$ for real k for these potentials u at any t. In this case a is defined by its zeros (the spectral points) $k_n = i\kappa_n$ in the upper half plane $x_n > 0$ by the formula $a = \prod_j \frac{k-i\kappa_j}{k+i\kappa_j}$, whereby to a single soliton $u(x-ct)$ there corresponds a single spectral point $k_1 = i\kappa_1$, where $\kappa_1^2 = +c/4$ (cf. [5]). What is the right analog of multisoliton solutions in the periodic problem? We suppose that the periodic potential u is such that it has only a finite number of zones; this means that starting from some number $n > n_0$ all points of both spectra $a_R = \pm 1$ are doubly degenerate and that the entire half line $k^2 \geq k_{n_0}^2$ forms a single zone. The degeneracy condition for a spectral point is the condition $b(k_n, x) \equiv 0$ for $k_n^2 \geq k_{n_0}^2$. On multiplying the period T by an integer $T \to mT$ the entire zone is filled out by degenerate levels $b \equiv 0$ as $m \to \infty$. On the other hand, with correct passage to the period $T \to \infty$, whereby the potential $u_T(x)$ with period T tends to a rapidly decreasing potential for $T \to \infty$, the zones of finite size contract to isolated points of the discrete spectrum. All this indicates that it is natural to suppose that the right analog of multisoliton solutions are the finite-zone potentials. The property that a potential be a finite-zone potential is conserved in time by virtue of the KV equation. How should one seek such solutions of the KV equation? What are finite-zone potentials like?

We consider the "higher KV equations" of order n

$$\dot{u} = \frac{d}{dx}\left(\frac{\delta I_n}{\delta u(x)} + c_1 \frac{\delta I_{n-1}}{\delta u(x)} + \cdots + c_n \frac{\delta I_0}{\delta u(x)}\right), \tag{6}$$

where $I_n = \int P_n(u, u', \ldots, u^{(n)})\, dx$ is a polynomial integral of the KV equation and $c_1, \ldots, c_n$ are arbitrary constants. Equation (6) has order $2n+1$ and by the theorem of Lax and Gardner [6, 7] it admits the Lax representation

$$\dot{L} = [L, A_n + c_1 A_{n-1} + \cdots + c_n A_0], \tag{7}$$

where $L = -(d^2/dx^2) + u$ and the operators A_0, A_1, A_2 are indicated in the introduction, $A_0 = (d/dx)$. We shall now indicate the corollary of the Zakharov–Faddeev theorem mentioned in the introduction which, while not noted by them, is extremely important for our subsequent purposes.

Proposition 2.1. *The set of all fixed points (stationary solutions) of any one of the higher KV equations is an invariant manifold also for any other of the higher KV equations (in particular, for the original KV equation) considered as a dynamical system in function space.*

Proof. All the higher KV equations are Hamiltonian systems; the Poisson brackets of the integrals are equal to zero $[I_n, I_m] = 0$ for all n, m. Therefore, all the higher KV equations commute as dynamical systems in function space. Therefore, the set of fixed points for one of them is invariant with respect to all the remaining ones. This proves Proposition 2.1. □

We have the following basic theorem:

Theorem 2.1. 1) *All the periodic stationary solutions of the higher KV equations*

$$\frac{d}{dx}\left(\sum_{i=0}^{n} c_i \frac{\delta I_{n-i}}{\delta u(x)}\right) = 0 \tag{8}$$

are potentials $u(x)$, the number of zones of which does not exceed n.

2) *The equation*

$$\frac{\delta I_n}{\delta u(x)} + \sum_{i=1}^{n} c_i \frac{\delta I_{n-i}}{\delta u(x)} = d \tag{8$'$}$$

is a completely integrable Hamiltonian system with n degrees of freedom depending on $(n+1)$ parameters $(c_1, \ldots, c_n, d)$, whereby the collection of n commuting integrals of this system and all the parameters $(c_1, \ldots, c_n, d)$ are expressed in terms of $2n+1$ nondegenerate eigenvalues of both spectra of the potentials $u(x)$ which form the boundaries of the zones.

Proof. As is known, in the case of rapidly decreasing functions u(x) from the Lax representation $\dot{L} = [A, L]$ it is easy to derive an equation for the eigenfunctions ψ_k of the operator L: $\psi_k = A\psi_k + \lambda\psi_k$. In the periodic case the analogous derivation gives

$$\psi_k = A\psi_k + \lambda\psi_k + \mu\bar{\psi}_k, \quad \dot{\bar{\psi}}_k = A\bar{\psi}_k + \bar{\mu}\psi_k + \bar{\lambda}\bar{\psi}_k, \tag{9}$$

where $\lambda + \bar{\lambda} = 0$.

Indeed, $(L - k^2)\psi_k = 0$. Therefore,

$$0 = \dot{L}\psi_k + (L - k^2)\psi_k = (AL - LA)\psi_k + (L - k^2)\psi_k = (L - k^2)(\psi_k - A\psi_k).$$

Since $(L - k^2)\psi_k = (L - k^2)\bar{\psi}_k = 0$, we obtain the desired result with unknown coefficients $\lambda(x_0, t), \mu(x_0, t)$. To determine the coefficients we make use of the fact that $\psi_k(x_0, x_0) = \psi'_k(x_0, x_0) = 0$. From this for $x = x_0$ we have

$$\begin{gathered} (A\psi_k)_{x=x_0} + \lambda + \mu = 0, \quad (\dot{\psi}_k)_{x=x_0} = 0, \\ \left(\frac{d}{dx}A\psi_k\right)_{x=x_0} + ik(\lambda - \mu) = 0, \quad (\dot{\psi}'_k)_{x=x_0} = 0 \end{gathered} \tag{10}$$

in the basis $\psi_k, \bar{\psi}_k$. (In the basis $\phi_k, \bar{\phi}_k$ it is necessary to let $k \mapsto 1$ in the lower equation.)

We consider the matrix $\Lambda = \begin{pmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{pmatrix}$, where $\lambda + \bar{\lambda} = 0$. This is a matrix of the Lie algebra of the group $SU_{1,1}$ to which the monodromy matrix $T(k, x_0)$ belongs. The matrix Λ, as is evident from Eq. (10), depends on $u(x_0, t), u'(x_0, t), \ldots, u^{(2n)}(x_0, t), k$.

In order to study the time dependence of the monodromy matrix $T(k, x_0)$ by Eq. (6) it is necessary to compute ψ_k and ψ' at the point $x = x_0 + T$, where T is the period. Having done this we obtain $(x = x_0)$

$$\begin{aligned} \dot{a} + \dot{b} &= A(a\psi_k + b\bar{\psi}_k) + \lambda(a\psi_k + b\bar{\psi}_k) + \mu(\bar{b}\psi_k + \bar{a}\bar{\psi}_k), \\ ik(\dot{a} - \dot{b}) &= \frac{d}{dx}A(a\psi_k + b\bar{\psi}_k) + \lambda(a\psi'_k + b\bar{\psi}'_k) + \mu(\bar{b}\psi'_k + \bar{a}\bar{\psi}'_k). \end{aligned} \tag{11}$$

Substituting relation (10) into (11) and performing simple computations, we obtain

$$\dot{a} = \mu\bar{b} - b\bar{\mu}, \quad \dot{a} + \dot{\bar{a}} = 2\dot{a}_R = 0, \quad \dot{b} = (\lambda - \bar{\lambda})b + (a - \bar{a})\mu = 2\lambda b + 2ia_I\mu. \tag{11$'$}$$

Proposition 2.2. *On the basis of the higher KV equation the dependence of the monodromy matrix on time is described by the equation*

$$\dot{T} = [\Lambda, T], \tag{12}$$

where the matrix Λ is found from formula (10) *starting from the time dependence of the eigenfunction basis $(\psi_k, \bar{\psi}_k)$:*

$$(\dot{\psi}_k, \dot{\bar{\psi}}_k) = A(\psi_k, \bar{\psi}_k) + \Lambda(\psi_k, \bar{\psi}_k).$$

This proposition has been proved above. It implies:

Corollary 2.3. *The equation*

$$\Lambda' - \dot{Q} = [Q, \Lambda], \tag{13}$$

holds and is the compatibility condition for the two-parameter family of matrices $T(k, x, t)$ satisfying the equations $\dot{T} = [\Lambda, T]$, $T' = [Q, T]$, where the matrix Q is that of formula (4).

Proof. Since $\dot{T} = \dot{T}'$, it follows that $[\Lambda', T] + [\Lambda, T'] = [\dot{Q}, T] + [Q, \dot{T}]$. Using the Jacobi identity, we obtain $[\Lambda' - \dot{Q} - (\Lambda, Q), T] = 0$.

This proves the corollary (it now follows easily from elementary properties of the Lie algebra of the group $SU_{1,1}$). □

We now proceed to the basic Theorem 2.1.

If $u(x)$ is a stationary solution of Eq. (6), then the corresponding monodromy matrix $T(k, x)$ is also stationary $\dot{T} \equiv 0$ and $\dot{Q} \equiv 0$. From Corollary 2.3 we obtain the equation

$$\Lambda' = [Q, \Lambda], \tag{14}$$

where $\operatorname{Sp} \Lambda = \lambda + \bar{\lambda} = 0$. The eigenvalues of the matrix Λ are integrals of the system (8). Since $\operatorname{Sp} \Lambda = 0$ for the eigenvalues of the matrix Λ we have $\alpha_{\pm}(k) = \pm\sqrt{\det \Lambda}$, where $\det \Lambda = |\lambda|^2 - |\mu|^2$.

Further, $\det \Lambda$ is a polynomial in k^2 of degree $2n+1$, as is easily verified from the form of the matrix Λ, whereby the leading coefficient is a nonzero constant. The roots of the polynomial $\det \Lambda = 0$ are numerical integrals of the system $\Lambda' = [Q, \Lambda]$; formally the coefficients of the polynomial $\det \Lambda$ in the variable k^2 are written in the form of polynomial expressions in $u(x)$ and its derivatives with constant coefficients. We remark that the equation $\Lambda' = [Q, \Lambda]$ is equivalent to the original equation $\frac{d}{dx}\left(\sum c_i \frac{\delta I_{n-i}}{\delta u(x)}\right) = 0$ for the function $u(x)$. Thus, we have $2n + 1$ integrals of this system (the roots of the equation $\det \Lambda = 0$ or the coefficients of the polynomial $\det \Lambda$), the first $n + 1$ of which are formed from the constants $(c_1, \ldots, c_n, d)$.

We shall prove that these integrals are commutative and define all the zone boundaries of the potential $u(x)$. Using Eq. (12) $[\Lambda, T] = 0$, we can obtain the following relations:

$$1)\ (a - \bar{a})\mu = 2\lambda b, \quad 2)\ \bar{b}\mu = \bar{\mu} b. \tag{15}$$

From this we immediately obtain

$$e^{2i\phi} = b/\bar{b} = \mu/\bar{\mu}, \quad \text{where} \quad \phi = \arg b, \quad ia_I/b = \lambda/\mu. \tag{15$'$}$$

At the (nondegenerate) points of the spectrum $k^2 = k_n$

$$\left|\frac{ia_I}{b}\right| = \left|\frac{\lambda}{\mu}\right| = 1, \quad k = k_n, \tag{16}$$

or $|\lambda|^2 - |\mu|^2 = \det \Lambda = 0$. Thus, the zone boundaries are the zeros of the polynomial $\det \Lambda = 0$. Passage to the limit of infinite period whereby u tends to a rapidly decreasing function shows that the last n coefficients of the polynomial $\det \Lambda$ are algebraically independent integrals which are polynomials in u and its derivatives for almost any values of the first $n + 1$ constants $(c_1, \ldots, c_n, d)$. We thus have n independent integrals of the Hamiltonian system (8$'$) with n degrees of freedom. From the previously mentioned theorem of Zakharov and Faddeev it can be seen also that these integrals commute. Indeed, the manifold of functions [the stationary points of Eq. (6)] lies in a complete function space where all $2n+1$ integrals are commutative. On the finite-dimensional manifold of functions in question the symplectic form is degenerate. The integrals $(c_1, \dots, c_n, d)$ after restriction to the manifold have zero Poisson brackets with all functions on this manifold. It is easy to verify that the symplectic form on this manifold is in fact obtained by restriction of a form from the entire function space. This implies the complete integrability of Eq. (8′) and all the assertions of the basic Theorem 2.1. □

Remark 1. It is evident from Eqs. (15′) that $\phi = \arg\mu + m\pi$, $\phi = \arg b$, and thus $\pm\sin\phi$ is expressed in terms of u and its derivatives. Combining this fact with Proposition 1.1, we find that also the degenerate points of finite-zone potentials $u(x)$ can be obtained from the transcendental equation(5):

$$2\pi n = \int_0^T \left(2k - \frac{u}{k} \pm \frac{u}{k}\sin\phi\right) dx$$

where $\pm\sin\phi = \sin(\arg\mu)$.

Remark 2. Periodic and conditionally periodic solutions are obtained when the level surface of all the commuting integrals found of Eq. (8′) for whatever values of the constants $(c_1, \dots, c_n, d)$ is compact (i.e., is an n-dimensional torus). In general, for randomly chosen eigenvalues or zone boundaries (or, what is the same thing, values of the constants and integrals), periodic functions are obtained with n incommensurable periods. It is thus more natural to solve the inverse scattering problem in almost-periodic functions rather than in periodic functions only.

Remark 3. The matrix Q itself has the form of the matrix Λ for the "zero-order KV equation" $\{\dot{u} = u'\}$, where the operator $A_0 = d/dx$, $[A_0, L] = u'$. The stationary solutions in this case are constants constituting 0-zone potentials. Further, for any n the transformation $u \to u + c$ takes an n-zone potential into an n-zone potential, and by means of this transformation it can be achieved that the function $u = v + \text{const}$ satisfies the equation

$$\frac{\delta I_n}{\delta u(x)} + c_2 \frac{\delta I_{n-2}}{\delta u(x)} + \cdots + c_n \frac{\delta I_0}{\delta u(x)} = d,$$

where $c_1 = 0$ (it is assumed that $c_0 = 1$). By studying the form of the matrix $\Lambda = \Lambda_R + i\Lambda_I$ starting from the the formulas (10), it is possible to show that the matrix Λ_R has the form $(n \geq 1)$

$$\Lambda_R = \left(\sum_{q=0}^{n-1} \gamma_q \left(\frac{d}{dx}\frac{\delta I_q}{\delta u(x)}\right) k^{2(n-1-q)}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where the constants γ_q are expressed in terms of the constants $c_1, \dots, c_n$.

Remark 4. The stationary solutions of the KV equation of order q are contained among the stationary solutions of the KV equation of order $n > q$ as degenerate tori of Eq. (8′), where the $n+1$ constants $(c_1, \dots, c_n, d)$ and the values of the n integrals $I_1, \dots, I_n$ are chosen such that these integrals depend on the entire level surface. We shall now consider stationary solutions of the KV equations for $n = 1$ and $n = 2$.

1) *The Case* $n = 1$. In this case we have the equation

$$cu' + 6uu' - u''' = 0,$$

$$u'' = 3u^2 + cu + d, \quad u'^2 = 2\left(u^3 + \frac{cu^2}{2} + du + E\right).$$

We obtain the elliptic function

$$x = \int \frac{du}{\sqrt{2u^3 + cu^2 + 2du + 2E}},$$

where $u(x - ct)$ is a solution of the KV equation of the type of a simple wave. According to Theorem 2.1 $u(x)$ is a 1-zone potential. This fact was first proved by E. Ince in 1940 in another language and by another method [14]; the Sturm–Liouville equation with an elliptic potential is a special case of the Lamé equation arising from the Laplace operator on an ellipsoid where $1/2\, n(n+1)$-fold elliptic functions also occur in the potential. (As shown in [14], they are n-zone potentials which are a degenerate case of the general n-zone potentials given by Theorem 2.1).

In the basis $\psi_k, \bar{\psi}_k$ the matrix Λ has the form

$$\begin{aligned} \lambda &= \frac{ik}{2k^2}(u'' - 2u^2 + 8k^2) - ick + \frac{icu}{2k}, \\ \mu &= -u' + \frac{ik}{2k^2}(-u'' + 2u^2 + 4k^2 u) - \frac{icu}{2k}, \end{aligned} \tag{17}$$

and the characteristic polynomial has the form

$$\det \Lambda = 4k^6 + 2ck^4 + \frac{c^2 + 4d}{4}k^2 + \frac{cd - 2E}{4}. \tag{17'}$$

We see that $k_1^2 + k_2^2 + k_3^2 = -c/2$. If the period $T \to \infty$ and $u(x)$ tends to a rapidly decreasing function, then $E \to 0$, $d \to 0$. Therefore, $k_3 \to 0$, $k_1^2 \to -\kappa^2$, $k_2^2 \to -\kappa^2$, and the zone contracts to the eigenvalue $k^2 = -\kappa^2$ where $\kappa^2 = c/4$.

We note that the transcendental equation for all the degenerate points of the spectrum follows from Remark 1.

2) *The Case* $n = 2$. On the basis of Remark 3 we consider only an equation of the form

$$\frac{d}{dx}\left(\frac{\delta I_n}{\delta u(x)} + 8c\frac{\delta I_0}{\delta u(x)}\right) = 0, \tag{8''}$$

where $8I_0 = \int (8u^2)\, dx$, $I_2 = \int \left(\frac{u''^2}{2} - \frac{5}{2}u^2 u'' + \frac{5}{2}u^4\right) dx$ [we obtain all 2-zone potentials by adding a constant to the solutions of Eq. (8″)]. The Lagrangian of the dynamical system (8″) has the form

$$L = L_2 + 8cL_0 - du = L(u, u''), \quad 8L_0 = 8u^2, \quad L_2 = \frac{u''^2}{2} - \frac{5}{2}u^2 u'' + \frac{5}{2}u^4.$$

We denote by q the quantity $q = (\partial L/\partial u'')$. The energy of a system in which the Lagrangian depends on two derivatives has the form

$$E = L - u''q + u'q'.$$

We denote u' by p_q and q' by p_u. Then $E = H(u, q, p_u, p_q) = V(u, q) + p_u p_q$, and Eqs. (8″) assumes the Hamiltonian form

$$p_u' = -\frac{\partial H}{\partial u}, \quad p_q' = -\frac{\partial H}{\partial q}, \quad u' = \frac{\partial H}{\partial p}, \quad q' = \frac{\partial H}{\partial p_q},$$

where $V = L - u''q = -\frac{q^2}{2} - \frac{5}{2}qu^2 - \frac{5}{8}u^4 + 8cu^2 - du$. However, the study of this Hamiltonian system with two degrees of freedom is not so simple. We therefore compute its remaining integral by using the Lax representation $\Lambda' = [Q, \Lambda]$ of this dynamical system. The operator A has here the form

$$A = A_2 + 16cA_0 = \\ = \frac{d^5}{dx^5} - \frac{5}{2}u\frac{d^3}{dx^3} - \frac{15}{4}u'\frac{d^2}{dx^2} + \frac{15u^2 - 25u''}{8}\frac{d}{dx} + \frac{15}{8}\left(uu' - \frac{u'''}{2}\right) + 16c\frac{d}{dx}.$$

Computing the matrix Λ by Eqs. (10), we obtain (in the basis $\psi_k, \bar{\psi}_k$)

$$\begin{gathered} -16\lambda = \frac{ik}{k^2}\left(-\frac{u^{(4)}}{2} + 4uu'' + 3u'^2 - 3u^3 + 2k^2u^2 + 16k^4\right) + 16ick - 8\frac{icu}{k}, \\ 16\mu = (6uu' - u''' + 4k^2u') + \\ + \frac{ik}{k^2}\left(-\frac{u^{(4)}}{2} + 4uu'' + 3u'^2 - 3u^3 + 2k^2u'' - 4k^2u^2 - 8k^4u\right) - \frac{8icu}{k}, \\ \det\Lambda = k^{10} + 2ck^6 - \frac{d}{16}k^4 + k^2\left(c^2 + \frac{E}{32}\right) + \frac{I + 16cd}{16^2}, \end{gathered} \tag{18}$$

where

$$\begin{gathered} I = p_u^2 - 2up_up_q + (2q - 16c)p_q^2 + D, \\ D = u^5 + 16cu^3 - 4uq^2 + 32cuq - 2dq. \end{gathered} \tag{19}$$

The integrals I and $E = H$ are commutative, and the Hamiltonian system (8$''$) is thus completely integrable. Let $p_u^2 = \alpha_1^2$,

$$(2q - 16c)p_q^2 = \begin{cases} \alpha_2^2 & \text{for } q \geq 8c, \\ -\alpha_2^2 & \text{for } q < 8c. \end{cases}$$

Then

$$\begin{gathered} \alpha_1^2 \pm \alpha_2^2 = I - D(u,q) + u(E - V), \quad \pm\alpha_1^2\alpha_2^2 = (2q - 16)(E - V)^2, \\ \pm 2\alpha_j^2 = A \pm \sqrt{a^2 - 4B}, \quad j = 1, 2, \end{gathered} \tag{19$'$}$$

where $A = I - D + 2u(E - V)$, $B = (2q - 16c)(E - V)^2$. We find that for given values of the constants c, d, E, I we must seek regions of the (q, u)-plane where

$$A \geq 0, \quad A^2 - 4B \geq 0, \quad q \geq 8c. \tag{20}$$

Compact regions satisfying inequality (20) give tori in the phase space (u, q, p_u, p_q). It is evident from the equations that compact regions are possible only in the half space $q \geq 8c$. Indeed, for $q < 8c$ the expression $A \pm \sqrt{A^2 - 4B}$ always has roots of different signs which coalesce only at the isolated points $A = 0$, $B = 0$.

For these potentials (periodic and conditionally periodic with two periods) the relation $\sum_j k_j^2 = 0$ is satisfied.

As noted in Remark 3, the remaining 2-zone potentials found in Theorem 2.1 are obtained by adding a constant $u \to u + \text{const}$, where this constant is the sum of the eigenvalues — the zone boundaries — of the new potential $v(x)$. The equations giving the singular points of the Hamiltonian system, as is easily seen, reduce to the conditions

$$p_u = p_q = 0, \quad q = -\frac{5}{2}u^2, \quad 10u^3 - 16cu + d = 0$$

or

$$u = u(c,d), \quad q = q(c,d), \quad E = V(c,d), \quad I = D(c,d).$$

In parameter space this is a two-dimensional surface $E = E(c,d)$, $I = I(c,d)$. In general, under these conditions we obtain in phase space compact separated level surfaces $E = \text{const}$, $I = \text{const}$ of torus type with one degenerate cycle. The trajectories of the KV equation on these surfaces describe the interaction of a periodic simple wave with a rapidly decreasing soliton. The interaction of two rapidly decreasing solitons (a two-soliton solution of the problem with rapidly decreasing initial data) is obtained [up to an additive constant $U(c,d)$] if still another condition on the parameters is satisfied under which both cycles on the torus degenerate to a point. This relation has the form of a condition on the parameters under which the polynomial $\det\Lambda$ has three distinct roots, two of which are double roots (the two zones contract to points).

With the exception of this special case, the evolution in time of two-zone potentials according to the Korteweg–de Vries equation is characterized by two constants Δ_1, Δ_2 such that

$$u(x+\Delta_1, t+\Delta_2) = u(x,t).$$

Calculation of these constants Δ_1, Δ_2 in terms of the zone boundaries will be given in the second part of the work.

In the second part of the work we shall study n-soliton solutions in more detail. Does there exist a superposition law synthesizing them from single-soliton solutions — an algebraic function of pairs of solitons (elliptic functions) which contains doubly valued points (roots) and therefore, in general, leads beyond the field of elliptic functions? In terms of the characteristic polynomials this appears as follows: there are two solitons $u_1(x, c_1, d_1, E_1)$, $u_2(x, c_2, d_2, E_2)$ with matrices $\det\Lambda^{(1)} = (k^2-\lambda_1)(k^2-\lambda_2)(k^2-\lambda_3)$, $\det\Lambda^{(2)} = (k^2-\mu_1)(k^2-\mu_2)(k^2-\mu_3)$.

Suppose that the roots λ_j, μ_j are nonmultiple and $\lambda_1 = \mu_1$ (a condition for the possibility of composition); the remaining 4 roots $\lambda_2, \lambda_3, \mu_2, \mu_3$ are all distinct. The superposition law of solitons is such that the 2-soliton potential $v = F(u_1, u_2)$ has a characteristic polynomial $\det\Lambda$ in the form of the least common multiple of the initial polynomials $\det\Lambda = (k^2-\lambda_2)(k^2-\lambda_3)(k^2-\mu_2)(k^2-\mu_3)(k^2-\lambda_1)$, where $\lambda_1 = \mu_1$. The correct analog of the amplitude a_k for rapidly decreasing functions is here the quantity $\det\Lambda$. However, this definition of superposition is ineffective. Completely different representations of the superposition law are possible; we shall discuss these in Part II.

Remark 5. V. B. Matveev and L. D. Faddeev have informed the author that in 1961 N. I. Akhiezer essentially formulated and began the solution of the problem of constructing examples of finite-zone potentials starting from the results of [2, 3] on the inverse scattering problem on the half line. In this work [1] N. I. Akhiezer developed an interesting approach to the construction of finite-zone potentials using facts from the theory of hyperelliptic Riemann surfaces. His construction, however, gives for a prescribed zone structure only a finite number of potentials which satisfy specific parity conditions in x. Theorem 2.1 of the present work gives many more periodic and almost-periodic n-zone potentials — they depend on n continuous parameters for a prescribed zone structure. Since the KV equation $\{\dot u = 6uu' - u''\}$ is not invariant under the transformation $x \to -x$, it follows that during evolution in time other potentials will be obtained from those of Akhiezer which are not contained in his construction of n-zone potentials. Judging from the work [1],

N. I. Akhiezer was not familiar with the work of E. Ince [14] which actually proved (in another language) that an elliptic function is a 1-zone potential. It is curious to note that the three proofs of this particular fact which follow from the works of E. Ince [14], N. I. Akhiezer [1], and the present work are, in principle, all different.

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