ALGEBRAIC CONSTRUCTION AND PROPERTIES OF HERMITIAN ANALOGS OF K-THEORY OVER RINGS WITH INVOLUTION FROM THE VIEWPOINT OF HAMILTONIAN FORMALISM. APPLICATIONS TO DIFFERENTIAL TOPOLOGY AND THE THEORY OF CHARACTERISTIC CLASSES. I

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Abstract. The complicated and intricate algebraic material in smooth topology (the theory of surgery) does not fit into the already existing concepts of stable algebra. It turns out chat the systematization of this material is most naturally carried through from the point of view of an algebraic version of the hamiltonian formalism over rings with involution. The present article is devoted to this task. The first part contains a development of the algebraic concepts.

Introduction

The present article can be read independently of its topological analogs, although it arose from a consideration of the general algebraic significance of surgery obstructions in a non-simply connected topology, especially from the objects $L_{2k+1}(\pi)$ introduced by Wall (see [20]) for the group π. Although it is true that for even n ($n = 2k$) these objects had been introduced earlier (see [13, 14] and [19]) and had been conjectured almost trivially from topological arguments for groups π without 2-torsion, it was not until 1968 that for odd n ($n = 2k + 1$) a reasonable algebraic formulation of a surgery obstruction was first conjectured by Wall in an important but unpublished work [20]. This latter study compelled the present author to return to the problem of an effective algebraic calculation of the invariants of quadratic forms over $Z(\pi)$, which had been considered in [13] and [14]. In addition, a careful analysis shows that the objects $L_n(\pi)$ in [20] for various values of n are not unified in a single "homology theory"; moreover, the objects $L_n(\pi)$, in the form in which they were introduced by Wall, cannot be incorporated into a unified theory of homology type with connections between $L_n(\pi)$ and $L_n(\pi \times Z)$; consequently questions remained open concerning the algebraic construction and the investigation of the analogs, for example, of the projection operators of Bass connecting $L_{n+1}(\pi \times Z)$ and $L_n(\pi)$, which can be studied algebraically only in the framework of a single "homology theory", although geometrical arguments sometimes allowed one to prove (noneffectively) the existence of such projection operators and even to apply this noneffective "existence theory" to the important problem of homotopic tori (see [7, 13] and [21]; I remark that, although the arguments in [16] are not completely correct, they can be corrected, at least for the calculation of the ranks of the groups $L_n(Z \times \cdots \times Z)$).

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The present article is devoted to the unification of all the objects of this kind into specific homology theories (of which there prove to be several). It turns out that this unification can be carried out simply and naturally in the language of the hamiltonian formalism. The most difficult step is that of relating surgery obstructions on manifolds of $2k-1$ dimensions with the obstructions in $2k$ dimensions, where the hamiltonian formalism makes it algebraically obvious that, in the category of modules with a scalar product of one symmetry, the objects of the other symmetry of scalar products appear as higher "K^i" (for $i \geq 2$). Although, according to Wall [20], a simpler passage from $2k-2$ to $2k-1$ dimensions for $L_n(\pi)$ is analogous to the passage from K^0 to K^1 in the category of modules with a symmetric (or, in individual cases, skew-symmetric) scalar product, it is convenient for us to take a different definition of the objects of the "type K^1" $\to L_{2k-1}(\pi)$, not in the language of automorphisms, which is artificial for these problems since it generalizes the usual construction of K^1 in the category of modules, but in a language which is more appropriate to the subject of the problem and is necessary for the passage to $n = 2k$. The equivalence of these objects to the groups $L_{2k+1}(\pi)$ will be true only in the narrow situation of Wall for one of the "K-theories", which will be constructed later, and will not, of course, be periodic; in this connection the groups $V_i^0(\pi)$ and $V_i^2(\pi)$ which extend $L_{2k+1}(\pi)$ to a "homology theory" will not coincide with $L_{2k}(\pi)$ and $L_{2k+2}(\pi)$ respectively (here $i = 1, 2$, in dependence on the evenness of k).

By using the algebraic concepts introduced in the present article we shall give an algebraic construction of the analogs of the projection operators of Bass for each of the homology theories and shall prove their main properties.[1] The geometrical interpretation, the applications and the essential unsolved problems will be presented in the second part (see §§ 9–12).

In the opinion of the present author, the methods developed here also shed light, from a different point of view from [1], on the abstract algebraic nature of periodicity, on which of the theories it holds for and when, and on what its relation is to the classical Bott periodicity for the usual homotopic K-theories (see § 4, Examples 3 and 4).

Here it is essential to note that the position occupied by the theory of surgery obstructions (the passage from $2k+1$ to $2k+2$ dimensions, where the sign of symmetry of the category changes and where the formulation we develop is essentially necessary) corresponds to the place in classical K-theory where, in order to construct the projection operator $K^0(X \times S^1) \to K^1(X)$ of Bass, it is necessary to construct the Bott periodicity operator $K^0 \to K^2$, since the mapping $K^2(X \times S^1) \to K^1(X)$ is obvious and in differential topology the appropriate analog of K^2 $(= L_{2k+2}(\pi))$ is defined as K^0. In addition we note that, in our notation, $K^i(x) = K^0(E^i, x)$, since we consider all the forms of extraordinary homologies as a covariant functor of the ring $C(X)$ of functions.

Standard notation and terminology. 1) $Z[\pi]$ is the group ring, with an involution generated by the mapping $\sigma \to \sigma^{-1}$, $\sigma \in \pi$. For any ring A with an involution we denote by $\bar{a}$ the image of the element a under the involution; this includes the ring of matrices $(a_{ij}) \in \mathrm{GL}(A, N)$, where $(\overline{a_{ij}}) = (\bar{a}_{ji})$.

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[1] It is important to note that our main theorems are proved for theories that are tensorially multiplied by $Z[1/2]$, although the majority of the constructions are valid without this assumption.

In addition, in all the theorems in the first part we shall assume that the ring A contains 1/2 (the ring $Z[\pi]$ is multiplied by $Z[1/2]$).

2) H_n is a hamiltonian space (module), namely a free module over A with a basis $x_1, \dots, x_n$, $p_1, \dots, p_n$ and a scalar product $\langle g, h\rangle \in A$, where $\langle ag, h\rangle = a\langle g, h\rangle$, $\langle g, h\rangle = \pm\overline{\langle h, g\rangle}$ ($\langle x_i, p_j\rangle = \delta_{ij}$). Two cases are possible: the hermitian case, when $\langle g, h\rangle = \overline{\langle h, g\rangle}$, and the skew-hermitian, when $\langle g, h\rangle = -\overline{\langle h, g\rangle}$. Here $\langle x, x\rangle = 0$ and $\langle p, p\rangle = 0$.

3) $L \subset H_n$ is a lagrangian plane, namely a free (or projective) submodule of H_n such that $\langle L, L\rangle \equiv 0$ and that $h \in L$ if $\langle h, L\rangle \equiv 0$. We shall assume that the module $L \subset H_n$ is distinguished as a direct summand.

4) A quadratic form is a projective module P with a scalar product $\langle g, h\rangle = \pm\overline{\langle h, g\rangle}$, where $\langle h, g\rangle \in A$, $h, g \in P$ (there are two cases: the hermitian and the skew-hermitian). For any module M we let M^* denote the conjugate module; then the scalar product is a homomorphism $M \to M^*$.

5) The "hessian of the action" on the lagrangian plane $L \subset H_n$ in an hermitian (skew-hermitian) hamiltonian space H_n is a quadratic form on L with the other symmetry sign; for $g, h \in L \subset H$ it is denoted by (g, h). By definition, $(g, h) = \langle g, \pi(h)\rangle$, where π is the projection of H_n onto the free submodule $(x_1, \dots, x_n)$; it is assumed that $p_1 = 0, \dots, p_n = 0$. In the classical case it is, in fact, the hessian of the action function $S(x) = \int_{x_0}^{x} p\,dx$ on the lagrangian submanifolds of H_n, which does not depend on the path of integration (locally) in view of the lagrangian property (all the tangent planes are lagrangian).

6) $K^0(A)$, $K^1(A)$ are the usual objects of algebraic K-theory with the involution $*\colon K^i \to K^i$ generated by the conjugate. When $A = Z[\pi]$ they are denoted by $K^i(\pi)$, $i = 0, 1$ (see § 1).

7) The Laurent extension of the ring A is the ring $A[z, z^{-1}]$, where $z^{-1} = \bar{z}$ and z, z^{-1} commute with A. For groups we have $Z[\pi \times Z] = A[z, z^{-1}]$, where z is a generator.

8) The usual projection operators of Bass (see §§ 5 and 6) are

$$K^1(A[z, z^{-1}]) \xrightarrow{B} K^0(A)$$

and

$$K^0(A) \xrightarrow{\bar{B}} K^1(A[z, z^{-1}])$$

such that $B\bar{B} = 1$, the "augmentation" is

$$K^1(A[z, z^{-1}]) \xrightarrow{\epsilon} K^1(A)$$

and the embedding is

$$K^1(A) \xrightarrow{\bar{\epsilon}} K^1(A[z, z^{-1}]),$$

where $\bar{\epsilon}\epsilon = 1$. It is well known that $B* = -*B$. We shall always assume that for the objects under consideration we have $K^0(A[z, z^{-1}]) = K^0(A)$.

9) In the subsequent text we shall always assume that the "evenness condition" of the hermitian (skew-hermitian) matrices $\phi = \phi_1 \pm \bar{\phi}_1$, is fulfilled for all quadratic forms (including the hessians of the action).[2]

10) We shall assume as standard the concepts of the canonical transformations of a hamiltonian space (hermitian or skew-hermitian) which preserve the form,

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[2] It is clear chat the evenness condition is important only in those constructions which remain meaningful over the integers (if there is no 1/2).

and also the hamiltonian equations which are obtained from the function H (the Hamiltonian), and the defining families of these transformations.

11) The scalar product $\langle\cdot,\cdot\rangle$ with values from $A[z,z^{-1}]$ generates a scalar product with values in A which is invariant under 'translation' by z, z^{-1}. We denote the latter scalar product by $\langle\cdot,\cdot\rangle_0 \in A$, and $\langle g,h\rangle = \sum_{-\infty}^{+\infty}\langle g, z^i h\rangle_0 z^i$.

§ 1. General remarks on the construction of various K-theories

It is usual to construct $K^1(A)$ in a category of A-modules, starting from the group of automorphisms of free or projective A-modules. An analogous idea was developed by Wall [20] in constructing $L_{2k+1}(\pi)$ in a category of modules with an hermitian (skew-hermitian) scalar product, since $L_{2k}(\pi)$ resembles K^0 in this category. Bass recently attempted to generalize this approach in order to define the higher K^i (in the usual sense). Steinberg and Milnor indicated another approach to the definition of K^2. They considered the group $E(A)$ (the commutant $[\mathrm{GL}(A,\infty), \mathrm{GL}(A,\infty)]$) which consists of elementary matrices $\alpha_{ij}(a)$ and their products, $i \neq j$. Let us recall that $\alpha_{ij}(a)$ is a matrix with units along the diagonal and $a \in A$ in the (i,j)-position. These authors introduced the group $\mathrm{St}(A)$ which is generated by the $\alpha_{ij}(a)$ and is factorized according to "obvious" and universal relations on these matrices; the generators are always valid where the elements $a \in A$ occur only as parameters (in a real ring $E(A)$ with A fixed, in general, more relations). There is an obvious homomorphism $\mathrm{St}(A) \to E(A)$, the kernel of which is called $K^2(A)$ (see [10]).

The idea developed in the present article consists of the following. If we wish to construct K^0 and K^1 in some category with direct sums and with a zero, then we must have:

a) a class of objects containing the zero and closed under summation;

b) a concept of equivalence between two objects of this class (then under this equivalence the objects of the chosen class form K^0);

c) a concept of a process which realizes this equivalence between two objects. Moreover, the processes realizing this equivalence are often in the form of an iteration of certain "elementary equivalences" between which there are trivial relations.

Then to construct K^1 we must consider all the processes that reduce one (fixed) object to another (fixed) object; the distinguishing characteristic of two such processes is an element of K^1; clearly it must be indicated when these two processes are equivalent to each other.

Example 1. In homotopy theory K-theory is defined as follows: $\tilde{K}^0(X)$ is simply $\tilde{K}^0(A)$, where $A = C(X)$; $K^{-1}(X) = K^1(A)$ is, by definition, $K^0(EX)$, where E is a suspension; this reduces to the usual treatment of K^1 in terms of automorphisms. Here we must bear in mind that $A = C(X)$ is a contravariant functor of X and that $K(A)$ is covariant in A; hence $K^{-1}(X) = K^1(A)$ by definition. For the "processes of reduction" to the zero element we consider the trivialization of the fiberings over X in the definition of $K^1(A)$, $A = C(X)$.

Example 2. General extraordinary homology theories. Here we have an object Y and put $H_Y^{-i}(X) = [E^iX, Y] = [X, \Omega^i Y]$, where $\Omega(Y)$ is the operation of taking the loop space, and $[\cdot,\cdot]$ denotes the homotopy classes. However the loop space $\Omega(Y)$ is simply the space of the "processes of motion" from the point y_0 to the point y_1 along the space Y. If $\alpha \in \Omega$ denotes the parametrized paths $\alpha(\tau)$, $0 \leq \tau \leq 1$, then we can interpret any element ψ of the function space $X \to \Omega Y$

as a "process of deforming" the constant mapping $X \to y_0|_{\tau=0}$ into the mapping $X \to y_1|_{\tau=1}$ by putting $\psi_{\tau_0}(x) = \psi(x)|_{\tau=\tau_0}$.

Example 3. The subclass chosen in the category of modules for the definition of K^0 consists of projective modules. An elementary equivalence is the addition of a one-dimensional free module $P \to P \oplus F_1 = P'$ and the inverse operation $P \to P'$, where $P' \oplus F_1 = P$. Thus it turns out that in this case the process which reduces a module to the zero element is by definition the choice of a free basis in the module $P \oplus F$, provided $P \oplus F$ is free. The distinguishing characteristic of two processes (the representative of an element from K^1) is the automorphism which is the distinguishing characteristic of two bases. Equivalence in K^1 is derived from the requirement that the superposition of automorphisms is equivalent to their "Whitney sum" $\oplus$.

Example 4. We start from a basic category where the objects are automorphisms with the Whitney sum $\oplus$. In the definition of K^1 such an object is taken to be equivalent to zero if it belongs to $E(A)$ or can be split into a product of elementary matrices $\alpha_{ij}(a)$. The actual decomposition of $x \in E(A)$ into the product $x = \prod_s \alpha_{i_s,j_s}(a_s)$ is taken to be the process which reduces x to the zero. The natural concept of the equivalence of processes is their equality in the group $\mathrm{St}(A)$. It is obvious that our approach gives the Milnor–Steinberg definition of $K^2(A)$ for the classification of the processes reducing x to the zero.

From an algebraic point of view the naturalness of this construction of K^2 is justified, for example, by an obvious transfer of the construction of Bass's projection operators which connect $K^{i+1}(A[z, z^{-1}])$ with $K^i(A)$ for $i = 1$ (see § 6). From the topological point of view this approach is particularly natural in combinatorial and smooth topology, where, for example, the projective modules $P \in K^0(\pi)$ for the fundamental group π are composed of elements of homotopy groups, and to solve problems it is necessary to annihilate them by the operations of pasting on a cell (or a handle) $P \to P \oplus F_1$, or of annihilating a cell (or handle) $P \to P'$, $P' \oplus F_1 = P$. The classification of these processes leads to the Whitehead torsion.

Example 5. The operations of altering a manifold by Morse surgery or of changing a mapping of manifolds of degree 1 play a significant role in differential topology. The main aim is to reduce a mapping of manifolds, for example, to a homotopy equivalence by a sequence of these surgery operations. The process of reducing a morphism of an n-dimensional manifold M_1 into M_2 to a morphism $M_1' \to M_2$ by surgery in the geometrical sense is a morphism of an $(n+1)$-dimensional manifold W into M_2 (or into $M_2 \times I$ modulo the boundaries and of degree 1), where $\partial W = M_1 \cup M_2'$, including the original transformations $M_1 \to M_2$ and $M_1' \to M_2$ on the edges. Here W is constructed in a canonical way from the handles corresponding to the surgery processes. More precisely, in the study of the mappings of manifolds (of degree 1) $f\colon M_1^n \to M_2^n$, where there is an element $\xi \in \mathrm{KO}(M_2^n)$ such that $f^*\xi$ is a normal bundle to M_1^n, attempts to reduce f by the elementary operations of Morse surgery to a homotopy (or simple) equivalence, encounter the obstructions denoted by $L_n(\pi)$, $\pi = \pi_1(M_2^n)$. The obstruction for $n = 4k$ is a projective module with a nonsingular hermitian scalar product, and for $n = 4k+2$ it is skew-hermitian. The scalar product is generated by the index of the intersection on a universal covering and has a value in $Z[\pi]$. If M_2^n is a finite Poincaré complex, the module is stably free (there are small additional structures of Arf-invariant type which we neglect and consider for $A = Z[\pi]$ only at the end of this article). For $n = 4k+1$ ($n = 4k+3$) the obstruction $\alpha(f) \in L_n(\pi)$ was recently reduced to an object similar to K^1 in a suitable hermitian (skew-hermitian) category of modules; it is described by Wall in the language of automorphisms of the hamiltonian module H_n. For even n (the quadratic forms) the trivialness of the obstruction $\alpha(f) \in L_{2k}(\pi)$ is almost obviously equivalent to the reducibility, in a stable sense, to the hamiltonian module. However, there is some difficultly in unifying these objects into a single "hermitian K-theory" for all n, especially in the passage from $L_{2k+1}(\pi)$ to $L_{2k+2}(\pi)$, where there is a change in the sign of symmetry of the scalar product in the category of modules; also it is desirable to regard L_{2k+2} as following "K^{2k+2}" in this category.

To realize our approach it is clearly necessary to start from one of these categories (hermitian or skew-hermitian); we then define K^0 over it by specifying a class of objects (modules with a scalar product), a concept of the equivalence of pairs of objects, a concept of the zero and then the algebraic concept of a process which realizes an equivalence to the zero element. Then it is necessary to classify the processes which send a fixed object into the zero element, to introduce the concept of equivalent processes and to define K^1. Next, starting from K^1, we must similarly construct K^2 and, moreover, connect K^2 in an hermitian category with K^0 in a skew-hermitian category, where K^2 arises naturally in a topological problem and is similar to $L_{4k+2}(\pi)$ if K^0 is $L_{2k}(\pi)$. If possible we must extend this construction further.

The detailed algebraic definitions are given in the following section and the geometric interpretation at the end of the article. The idea consists of the following: it is obvious that for the surgery of a $4k$-dimensional manifold $f\colon M_1^{2k} \to M_2^{4k}$ with a homotopic kernel which is nontrivial only in $2k$ dimensions, it is necessary mat this kernel (a projective module with an hermitian scalar product) reduces in a stable sense to the hamiltonian module H_m over $Z[\pi]$ (this is widely known). However, note that the geometrical process of reducing this kernel to a pure zero is a set of Morse surgeries over the cycles $p_1, \dots, p_m \in H_m$, realized by spheres without self-intersections and pairwise intersections. This means that $\langle p_i, p_j\rangle = 0$ for all i and j. In addition, it is easy to verify that the kernel will be zero after surgery if the (p_q) generate a lagrangian plane. It is important to note here that, for Morse surgery, the choice of a complete basis (x, p) in H_m is not necessary; it is sufficient to take only the "semibasis" $p = (p_1, \dots, p_m)$, which is a lagrangian plane (this is obvious for surfaces). Of course the semibasis can be augmented, but this procedure is not unique. Thus only (p) is chosen, and $(x) = x_1, \dots, x_m$ are defined uniquely in the factor-module $H_m/(p) \cong x$. More precisely, the X'-space $x'_1, \dots, x'_n$ is equivalent to $x = x_1, \dots, x_m$ if $\langle x'_i, p_j\rangle = \delta_{ij}$ and $\langle x'_i, x'_j\rangle = 0$; that is, if (X') is projected isomorphically onto X along (p).

Thus a process of reduction to the zero consists here of choosing a lagrangian plane L in H_m. For the initial-plane-process we take $L_0 = (p)$; the invariant of another lagrangian plane as an element of "K^1" is measured relative to the initial plane. We also fix $(x_1, \dots, x_m)$. By definition $(x'_1, \dots, x'_m)$ is equivalent to $(x_1, \dots, x_m)$ if $x'_i = \sum \alpha_{ij} p_j + x_i$ and $\langle x'_i, x'_j\rangle = 0$. The equivalences of pairs of $L \subset H_m$, where L is a lagrangian plane, are defined naturally; they are elementary hamiltonian transformations of H_m. If two different processes of elementary hamiltonian transformations send one object L into another L', then their distinguishing characteristic turns out to be a nonsingular skew-hermitian form (an element of

"K^2") which is the "hessian of the action" of the process of transforming the lagrangian plane; this process is interpreted as a lagrangian plane in a hamiltonian space of higher dimensions. It is necessary to add, as superfluous elements of the basis, the "Hamiltonians" of the elementary transformations and the corresponding "times".

This is described more formally and in greater detail in the next section. We remark that the construction of the analogs of the projection operators of Bass has interesting hamiltonian interpretations (see §§ 4 and 8). Actually we must construct two different theories, each of which partly contains objects of the type $L_n(\pi)$; as we have already remarked, it is not possible in principle to unify these in a single theory.

In addition, in § 4 we clarify the topological analog (met in boundary problems of classical and quantum mechanics) of this procedure. This analog was at the back of my mind as an ideal source when I was working through Maslov's book [9]; which indicated the connection between our objects and the ordinary complex K-theory of rings of functions [6] and also explained the link with Bott periodicity (Examples 3 and 4).

§ 2. Algebraic definition of hermitian K-theories for dimension 0 and 1

Let S_1 (S_2) denote the category of A-modules with the hermitian (skew-hermitian) scalar product $\langle g, h\rangle \in A$; we distinguish the following subclasses of these categories:

$D_i \subset S_i$ is the subclass of projective modules with the even scalar product $\langle g, g\rangle = g_1 \pm \bar{g}_1$ ($i = 1, 2$), nonsingular in the sense that the scalar product determines an isomorphism $P \to P^*$ of a projective module and the conjugate.

$D_i^0 \subset D_i$ is the subclass of invertible elements such that for any $P \in D_i^0$ we can find a $P' \in D_i^0$ with $P \oplus P' = H_m$. In fact, $D_i^0 = D_i$.

$D_i^F \subset D_i$ is the subclass of quadratic forms on free modules.

C_i is the class of "projective hamiltonian" forms such that for any $Q \in C_i \subset D_i^0$ we have the decomposition $Q = Q_1 + Q_2$, where $\langle Q_1, Q_1\rangle = \langle Q_2, Q_2\rangle = 0$.

Obviously $Q_1 = Q_2^*$. We remark that similar subclasses were chosen by the present author in [13] (Appendix 2). Wall considers only the class D_i^F (see [19] and [20]).

Definition of the objects of type K^0 for the category S_i. 1. Let $U_i^0(A)$ denote the group of type K^0 constructed from the class D_i, where the object equivalent to the zero element is stably isomorphic to the projective hamiltonian module $H = Q_1 + Q_1^*$, $Q_1 = (P)$.

2. Let $V_i^0(A)$ denote the group of type K^0 constructed from the class D_i^0 of invertible elements of D_i; the object equivalent to the zero element is stably isomorphic to the hamiltonian module H_m.

It is obvious that in both cases the sum is the ordinary direct "sum of Whitney".

We have the obvious homomorphisms $K^0(A) \xrightarrow{\lambda_0} V_i^0(A)$, where $\lambda_0(Q) = Q + Q^*$, and $\mu_0 \colon V_i^0 \to U_i^0$. By definition we have the exact sequence

$$K^0(A) \xrightarrow{\lambda_0} V_i^0(A) \xrightarrow{\mu_0} U_i^0(A).$$

When we turn to the construction of the objects U_i^1 and $V_i^1(A)$ of type K^1 in D_i and D_i^0, we must define a concept of a process that reduces an element from D_i or D_i^0 to the zero element.

Definition. a) By a process reducing the element $Q \in D_i$ to the zero element we mean a selected lagrangian plane $L \subset Q \oplus \tilde{H}$, which can be projective; here $\tilde{H}$ is a projective-hamiltonian module.

b) By a process reducing the element $Q \in D_i^0$ to the zero element we mean a selected lagrangian plane $L \subset Q \oplus H_m$, where L is a free module with a fixed basis.

The following simple lemma coordinates these two concepts of an equivalence to zero.

Lemma 2.1. *The object $Q \in D_i$ ($Q \in D_i^0$) admits a process reducing it to the zero element if and only if $Q \oplus \tilde{H}$ ($Q \in H_m$) is isomorphic to a projective-hamiltonian (hamiltonian) module.*

Proof. We first give a proof for the purely hamiltonian case $Q \in D_i^0$, where $L \subset Q \oplus H_m$, with the basis $p'_1, \dots, p'_N \in L$. We take elements $y_1, \dots, y_N$ such that $\langle y_i, p_j \rangle = \delta_{ij}$ (for the time being without the condition $\langle y_i, y_j \rangle = 0$). The matrix $\alpha_{ij} = \langle y_i, y_j \rangle$ is such that $\alpha = \beta \pm \bar{\beta}$ (evenness). Put $x'_i = y_i \pm \sum \beta_{ij} p'_j$; it is obvious that, under the sign appropriate to the number i, we have $\langle x'_i, x'_j \rangle = 0$. Note that our choice of the X'-plane and the expansion $\alpha = \beta \pm \bar{\beta}$ are not unique. Thus the lemma is proved for the class D_i^0. For the class $D_i \ni Q \oplus \tilde{H} \supset L$ we recall that the module L is projective, and consider $L' = -L$ together with the projective-hamiltonian module $\tilde{H}' = L' + L'^*$. Consider the stabilization $Q \oplus \tilde{H} \oplus \tilde{H}' \supset L \oplus L'$. In an obvious way, as an abstract module $Q \oplus H \oplus \tilde{H}'$ is stably free (and so can be assumed to be free) with the lagrangian plane $L \oplus L'$, which is also free. Hence the problem is reduced to the previous case, and so the lemma is proved. □

Corollary 2.2. *In the class D_i, for any object $Q \in D_i$ the object $\alpha(Q)$ (the module Q with the same scalar product in which only the sign is changed) is inverse to Q in $U_i^0(A)$. Here $D_i = D_i^0$.*

Proof. The lagrangian plane $P \subset Q \oplus \alpha(Q)$ is by definition $\{x \oplus x^*\} = P$, where $x^* \in \alpha(Q)$ is the same element and $\langle x^*, y^* \rangle = -\langle x, y \rangle$. Now use Lemma 2.1. □

Remark 2.3. In the case $Q \in D_i^F$, when the scalar product in Q is defined in the free basis $y_1, \dots, y_m$ by the matrix $\phi = (\phi_{ij})$ and $\phi = \phi_1 \pm \bar{\phi}_1$, the lagrangian plane $X = (x_1, \dots, x_m)$ complementary to $P = (y_j \oplus y_j^*)$ can be obtained by one of the standard formulas: if $\psi = \phi^{-1} = \psi_1 \pm \bar{\psi}_1$, then $X = (x_1, \dots, x_m) = \bar{\psi}_1(y) \oplus \pm \psi_1(y^*)$, where the sign is determined by the symmetry sign of the category of modules with a scalar product. We can do this also for D_i.

We now turn to the definition of the groups $V_i^1(A)$ and $U_i^1(A)$ which are analogous to K^1 in the appropriate category. In accordance with the program outlined in § 1, for this we need to define a concept of the equivalence of two processes reducing a fixed object $Q \in D_i$ or D_i^0 to the zero element. In the proof of Lemma 2.1 we pointed out that, in the class D_i, by stabilizing a module we can interpret these processes as lagrangian planes (in general, projective) in the ordinary hamiltonian space $H_n \in D_i^F$. We denote the initial process (that is, the plane $P = (p_1, \dots, p_n)$) by P; we select a basis and augment it to a basis $X = (x_1, \dots, x_n)$ tor the whole of H_n, where $\langle x_i, p_j\rangle = \delta_{ij}$ and $\langle x_i, x_j\rangle = 0$, although this is neither obligatory nor unique since X is defined a priori only as the factor-space $H_n/(p)$. However it is convenient. We will give first the definition of U_i^1 for D_i.

Another process $L \subset H_n$, the distinguishing characteristic of which we will measure with P, is the (projective) lagrangian plane L in the coordinate hamiltonian space. We will determine when $L_1 \subset H_n$ and $L_2 \subset H_n$ are equivalent. Obviously, if $L_1 = P$ then L_1 is assumed to be equal to the zero element. If $L_1 = X$, then L_1 is also taken to be trivial.

In the sense of the nonuniqueness of the choice of an X-plane, obviously L_1 will also be assumed to be trivial in the case when L_1 can be projected isomorphically onto X along P. Analogously for a projection onto P along X. A slight generalization of this is that $L_1 \subset H_n$ is trivial if $\pi(L_1) \subset X$ is a projective module in X which is distinguished as a direct summand. Then the projection onto P also has this property.

In addition we obviously allow a stabilization; that is, the transition from H_n to $H_{n+k} = H_n \oplus H_k$, where L_1 goes into

$$L_1 \oplus (p_{n+1}, \dots, p_{n+k}) = L_1' \subset H_{n+k}.$$

We allow also the operation of interchanging a basis element $x_1 \in X$ in some (free) basis of the X-plane and the element p_1 corresponding to it, where $\langle p_1, x_1\rangle = 1$ and $\langle p_1, x_j\rangle = 0$ for $j > 1$. Namely, $x_1 \to p_1$, $p_1 \to \pm x_1$, and $p_j \to p_j$, $x_j \to x_j$ for $j > 1$.

We remark that the already noted obvious operation of replacing X by X', being isomorphically projected onto X, decomposes into a superposition of more elementary operations. Namely, if $X' = X + \beta P$ (in matrix form) and if we always require that $\beta = \alpha \pm \bar{\alpha}$, then it is sufficient to confine ourselves to the elementary operations where $\alpha = (\alpha_{ij})$ with $\alpha_{ij} = 0$ for $i \neq i_0$, $j \neq j_0$ and $\alpha_{i_0,j_0} = \sigma \in A$ (σ is an element of some chosen additive basis in A). For $A = Z[\pi]$ we have that $\sigma \in \pi$.

Definition. By $U_i^1(A)$ we mean the stable equivalence classes of (possibly projective) lagrangian planes in H_n relative to the equivalences we have enumerated. The sum is the Whitney sum $\oplus$.

The following simple lemma holds.

Lemma 2.4. *For the element $(L, H_n) \in U_i^1(A)$, the canonical method of constructing an inverse element $(L', H_n) \in U_i^1(A)$ consists of the following: (L', H_n) is the lagrangian plane in H_n such that $L + L' = H_n$; in view of Lemma* 2.1 *it exists for sufficiently large n.*

Proof. We consider the sum $(L, H_n) \oplus (L', H_n)$ in the space $H_{2n} = H_n \oplus H_n'$ with the basis x, p, x', p', and choose a P''-plane in H_{2n} such that $P'' = (x - x', p - p')$. It is obvious that $L \oplus L' \subset H_{2n}$ is projected isomorphically when we impose the relations $P'' = 0$. Since we can pass easily to the basis (P'', X'') by the indicated elementary operations, the lemma is proved. □

Thus $U_i^1(A)$ is a group for $i = 1, 2$.

We now give an important hamiltonian operation that has an essential significance in what follows: we will regard it as an elementary operation although it can be obtained by a superposition of those already available.

1. A single operation. Let $L \subset H_n$ with the basis $x_1, \dots, x_n, p_1, \dots, p_n$; we go over to the sum $H_n \oplus H_1$, where the basis in H_1 is denoted by $H = p_{n+1}$, $t = x_{n+1}$

(energy-time) with a selected plane $H \subset H_1$. We obviously have $L' = L \oplus H \subset H_{n+1}$.

We make the transformation

$$H' = p'_{n+1} = H - \gamma x - \phi t = H - \sum \gamma_j x_j - \phi t,$$

where $\gamma_j \in A$, $\phi = \phi_1 \pm \bar{\phi}_1$, $p'_k = p_k \pm \bar{\gamma}_k(t)$, $x' = x$, $t' = t$, in dependence on the sign of symmetry of the category. (For a hermitian category we have $\phi = \phi_1 - \bar{\phi}_1$.) The new P'-plane is $P' = (p', H')$, and it determines a new direction of projection $\pi' \colon L \oplus H \to (x', t')$. This operation can obviously be regarded as an elementary hamiltonian transformation generated by the transition from time $t = 0$ to $t = 1$ in the solution of the hamiltonian equation with the Hamiltonian $H = \gamma x + \phi t$; namely, formally we have $\overline{\partial H}/\partial p_i = 0 = \dot{x}_i$ and so $x'_i = x_i(1) = x_i(0)$; next, $\pm\overline{\partial H}/\partial x_i = \bar{\gamma}_i = \dot{p}_i$, and therefore $p'_i = p_i(0) = p_i(0) \pm \bar{\gamma} t$, where t is the unit vector along the time axis, $t = x_{n+1}$.

Another definition of equivalence to the zero element in $U_i^1(A)$. The lagrangian plane $L \subset H_n$ is said to-be equivalent to the zero element in $U_i^1(A)$ if it can be reduced by an iteration of the above hamiltonian operations to a plane isomorphically projectible onto an X'-plane in the new projection [here one must take into account the step that $x' = (x'_j, t = x'_{n+1})$ and $p' = (p'_j, H' = p'_{n+1})$], or if it can be reduced to a plane which can be projected onto the projective direct submodule $\pi'(L') \subset X'$.

2. The iterated hamiltonian operation. It is easy to see that the iteration of this operation is written in the following matrix form: if $p = (p_1, \ldots, p_n)$, $x = (x_1, \ldots, x_n)$, $H = (H^{(1)}, \ldots, H^{(m)})$, and $t = (t_1, \ldots, t_m)$, put

$$H' = H - \gamma x - \phi t, \quad x' = x,$$
$$p' = p \pm \bar{\gamma} t, \quad t' = t,$$

where $\gamma = (\gamma_{ij})$ is an $m \times n$ matrix, $\bar{\gamma}$ is the $n \times m$ matrix adjoint to γ and $\phi = \phi_1 \pm \bar{\phi}_1$ is an $m \times m$ matrix where $\phi = \phi_1 - \bar{\phi}_1$ if we were to start from an hermitian category (ϕ is skew-hermitian) and vice versa.

The following simple lemma holds.

Lemma 2.5. *The single and iterated hamiltonian operations can be obtained by superposing those described earlier; conversely, these operations, together with the replacement of an X-plane by an X'-plane isomorphically projectible onto X along P, give all the operations we had earlier.*

Proof. If a hamiltonian operation is given, then (H', p') is projected isomorphically onto (P, H) along (X', t'). Since the interchange of X and P (with signs depending on $i = 1, 2$) is one of the operations we have already met, this part of the lemma is proved. The only part of the converse that is not obvious at once is the assertion that we can obtain the interchange of x_1 and p_1 from the hamiltonian operations. For this we put $H = x_1$. Then we have $H' = H - x_1$ and $p'_1 = p_1 \pm x_1$, and the remainder are unchanged. We see that (p_1, t) is projected isomorphically onto (x_1, t) in the new direction of projection; this allows us to interchange their places by changing the X'-plane into another X'-plane isomorphically projectible onto it. Thus the lemma is proved. □

Remark 2.6. The hamiltonian operations can be written in projective hamiltonian modules when (X) and (t) are projective. To do this we must take the pair of mappings $\gamma\colon (X) \to (H)$ and $\phi\colon (t) \to (H)$, where $(H) = (t)^*$ and ϕ is a skew-hermitian (hermitian) even quadratic form, and then define the transformation $Q \oplus Q' \to Q \oplus Q'$, where $Q = (X) + (P)$ and $Q' = (H) + (t)$ are projective hamiltonian. This transformation is defined, as before, by the formulas $(x, p, h, \tau) \to (x, p \pm \bar{\gamma}(\tau), h - \gamma x - \phi\tau, \tau)$ for $x \in (X)$, $p \in (P)$, $h \in (H)$ and $\tau \in (t)$. Here $\bar{\gamma}\colon (t) \to (P)$ is by definition conjugate to the mapping $\gamma\colon (X) \to (H)$, since $(X) = (P^*)$ and $(H) = (t)^*$. The skew-hermitian (hermitian) character of the form ϕ follows from the hamiltonian property of the transformation.

Remark 2.7. The iterated hamiltonian transformation of the module $H_n \oplus H_m$, with a projection along (P', H') chosen after the transformation and with the old lagrangian plane L, together determine a new lagrangian plane $L' = L \oplus (H)$ in the coordinates (P', H'), and $L' \subset H_n \oplus H_m$ can be imagined to be a "process" of transforming the plane $L \subset H$.

For our convenience it will be useful to have, formally speaking, a more general hamiltonian operation with a generating element $H = y + \phi t$, where $y \in H_n$, $y = \mu x + \gamma p$ and $\langle H', H' \rangle = \langle y, y \rangle \pm (\phi \pm \bar{\phi}) = 0$, $\phi \in A$; the sign depends on the symmetry of the category.

We naturally write the corresponding transformation as

$$H' = H - y - \phi t, \quad x' = x + \bar{\gamma} t, \quad p' = p \pm \bar{\mu} t, \quad t' = t$$

or as

$$t = t', \quad x = x' - \bar{\gamma} t', \quad p = p' \mp \bar{\mu} t', \quad H = H' + \mu x' + \gamma p' + [\phi - \langle y, y \rangle] t'.$$

The transformation we have obtained is equivalent, both in the formal algebraic sense and in a future geometric sense, to the following:

$$H' = H - \mu x - [\phi \pm \mu \bar{\gamma}] t, \quad x' = x, \quad p' = p \pm \bar{\mu} t, \quad t = t'.$$

This equivalence becomes obvious when one goes over the the basis

$$\tilde{x} = x \pm \bar{\gamma} t, \quad \tilde{t} = t, \quad \tilde{p} = p, \quad \tilde{H} = H - \gamma p.$$

This "general hamiltonian operation" is defined also over the projective lagrangian planes in a projective hamiltonian module as in Remark 2.6.

We give now the definition of the groups $V_i^1(A)$. (We can prove a purely algebraic theorem that these objects are equivalent to Wall's objects $L_{4k+1}(\pi)$ when $i = 1$ and $L_{4k+3}(\pi)$ when $i = 2$ for $A = Z[\pi]$, but this is not required.) Here we start from the class D_i^0 and must establish that it is a process reducing $Q \in D_i^0$ to the zero element.

We retain all the elementary operations occurring in the definition of $U_i^1(A)$, including the hamiltonian operations, but we recall that, both in the hamiltonian space and on the lagrangian plane $L \subset H_n$, there are defined and distinguished free bases $e_1, \ldots, e_n \in L$ and $x_i, p_j \in H_n$. The passage from one basis to another on L is permitted only by means of a unimodular transformation (in the sense of Whitehead and Dieudonné); that is, by iterations of the elementary matrices $E(A)$. Hence an object will be taken to be equivalent to the zero element if and only if $L \subset H_n$ can be reduced to a P-plane with a standard basis, where we allow only unimodular substitutions of this basis (it is clear that a stabilization is allowed). Then $V_i^1(A)$ is the group of equivalence classes in this sense (with regard for the basis on L and H_n). The fact that $V_i^1(A)$ is a group follows obviously from Lemma 2.4, in the proof of which all the transformations are unimodular and L' is a free module in H_n with a basis $e'_1, \dots, e'_n$ such that $\langle e'_i, e'_j \rangle = \delta_{ij}$.

Remark 2.8. It is clear that we can construct a (canonical) transformation $H_n \to H_n$ such that $P \to L$ and $X \to L'$, with regard for the bases, since L and L' are free. This is fundamental for the construction of $V^1_i = L_{4k\pm1}(\pi)$ for $A = Z[\pi]$ in the language of automorphisms starting from the analog of the usual K^1, although one will have to factorize with respect to more than a commutant because a reduction process (that is, a lagrangian plane $L \subset H_n$) is a pure object. This path (that is, the analog of the usual K^1) was the one taken by Wall (see [20], § 6). Of course it is not possible to describe the other groups U^1_i through automorphisms; but here, for the construction of higher V^j_i $(j = 2)$, it is convenient to start from the language of lagrangian planes, which is natural for given problems and which is directly appropriate to the geometrical sense in topology.

As in the 0-dimensional case, in the 1-dimensional case there are defined the homomorphisms $\lambda_1\colon K^1(A) \to V^1_i(A)$ and $\mu_1\colon V^1_i(U) \to U^1_i(A)$, where $\lambda_1(\alpha)$ is an X-plane with a basis which differs from the standard basis by a transformation with determinant α, and the construction of the homomorphism μ_1 is obvious.

To conclude § 2 we construct another object $W^0_i(A)$, which has a significant relation to topology. The elements are projective lagrangian planes $L \subset H_n$ in the hamiltonian module H_n, and the equivalence relation is the same as the the definition of $V^1_i(A)$, except that in $W^0_i(A)$ we take as equivalent to the zero element a plane L which is reduced by the same transformations to a plane isomorphically projectible onto an X-plane. There is an obvious mapping $W^0_i(A) \to U^1_i(A)$ and also a mapping $V^1_i(A) \to W^0_i(A)$. The sequence $K^1(A) \to V^1_i(A) \to W^0_i(A)$ is exact. There is a homomorphism $\tilde{K}^0(A) \to W^0_i(A)$ such that the sequence $\tilde{K}^0(A) \to W^0_i(A) \to W^1_i(A)$ is exact.

The construction of this homomorphism is of the following kind: for $\beta \in K^0(A)$ we consider the decomposition $X = \beta + (-\beta)$, $P = \beta^* + (-\beta^*)$, where β $(-\beta)$ is conjugate to β^* $(-\beta^*)$ relative to the scalar product in $H_n = (X, P)$, and $\langle \beta, -\beta^* \rangle = 0$. Put $L = \beta + (-\beta)^*$. The projective class of L, which is an invariant in $W^0_i(A)$, is here equal to $\beta - \beta^*$.

This construction is the exact analog of the mapping $K^0(A) \to V^0_i(A)$. It was considered by Golo [8] in connection with the realization of the obstructions of the present author, Siebenmann and Wall by means of manifolds; these are obstructions from K^0 (see [13] and [18]).

§ 3. Algebraic definition of hermitian K-theories in two dimensions and in some higher-dimensional cases. The interrelation between K^2 in an hermitian category and K^0 in a skew-hermitian category and vice versa

We now turn to the definition of the objects $U^2_i(A)$ and $V^2_i(A)$. We start from the definition of $U^2_i(A)$, the elements of which are, by definition, the distinguishing characteristics among the various processes which reduce a fixed lagrangian plane in a projective hamiltonian module to a plane isomorphically projectible along P, or to a plane projectible along P onto a direct projective submodule of X. We recall the following:

a) We can always start from the free hamiltonian module H_m (see the proof of Lemma 2.1).

b) For the processes one can take the iterated hamiltonian transformations (Lemma 2.5).

c) We can assume that the initial plane $L_0 \subset H$ coincides with $X = L_0$.

Since a process in $H_{n+m} = H_n \oplus H_m$ with the basis (x, t, p, H) is given by the lagrangian plane $L_0 \oplus (H)$ in the basis P', H', where $P' = P \pm \bar{\gamma} t$, $H' = H - \gamma x - \phi t$, $x' = x$, $t' = t$, $\phi = \phi_1 \pm \bar{\phi}_1$ and $L_0 = (x_1, \dots, x_n)$, in a completely obvious way we obtain that the plane $L \oplus (H)$ is projected isomorphically onto X', t' along P', H' if and only if the matrix ϕ^{-1} exists; in other words, a process sends X into an isomorphically projectible plane if and only if the matrix ϕ is nonsingular. Here we can assume that $\gamma = 0$, since the properties of this process for the initial $L_0 = X$ are determined only by the "hessian of the action" ϕ of this process: $\partial^2 S / \partial t_j \bar{\partial} t_j = \phi$ with respect to the time coordinates (here one must formally introduce $i = \sqrt{-1}$ if ϕ is skew-hermitian, since $\bar{i} = -i$; i is not introduced if ϕ is hermitian). In general the "hessian of the action" on $L_0 \oplus H$ is $\begin{pmatrix} 0 & 0 \\ 0 & \phi \end{pmatrix}$ independently of γ.

We now explain when $L \oplus H$ can be projected along (P', H') onto a direct submodule of (X', t').

Lemma 3.1. *$L \oplus H$ is projectible along (P', H') onto a direct submodule α if and only if we can find a direct decomposition $(t) = \alpha + \beta$ such that the restriction of the form ϕ to α is nonsingular and the restriction of the form ϕ to β is zero. Here ϕ is considered as a skew-hermitian (hermitian) form on (t).*

Proof. The module β is simply $\operatorname{Ker}\phi$. For $\gamma = 0$ the image of the projection coincides with the image of the transformation $(X, t) \to (X, t)$ with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix}$, and the lemma is obvious. For $\gamma \neq 0$ the matrix is $\begin{pmatrix} 1 & \gamma \\ 0 & \phi \end{pmatrix}$. It is obvious that the cokernel of the projection is isomorphic to α and that the kernel of the projection is isomorphic to $\operatorname{Ker}\phi$. Thus the lemma is proved. □

Thus each process sending $L = X$ into $L' = L \oplus H$ (in the new coordinates), which is projected onto X, t in a directed manner, determines in a canonical way a skew-hermitian (hermitian) form on a projective module; conversely, given any quadratic form on a projective module we can construct such a process (the evenness condition is always assumed to hold) and the form is the "hessian of the action" of this process.

Now the definition of $U_i^2(A)$ is obvious and coincides with the definition of $U_j^0(A)$, where $i \neq j$ and $i, j = 1, 2$. Note that we derived this result from our formalism, and did not take it from the beginning as a definition before analyzing the sequential process of constructing the higher analogs of K^0 and K^1. In what follows the process of constructing these objects also gives a simple construction for the analogs of the projection operators of Bass; this could be very difficult if we were to go to the definition of U_i^2 and V_i^2 tautologically.

It is now clear that the groups $V_i^2(A)$, which extend objects of the type of the odd-dimensional Wall groups $L_{2k+1}(\pi) = V_i^1$ $(i = 1, 2)$ to a higher dimension do not have the periodicity property. As earlier, for the definition of $V_i^2(A)$ we consider processes reducing an X-plane in H_n to a plane isomorphically projectible onto $(X', t') = (X, t)$ along $(P', H') = (P \pm \bar{\gamma} t, H + \gamma x + \phi t)$, where γ, ϕ are matrices, and for identical reasons we can assume that $\gamma = 0$ and ϕ^{-1} exists. However, in this case the reduction process does not end here; to complete it one must do the following.

(a) $L \oplus H$ must be combined with the plane (X', t') by elementary transformations. These transformations have the form $X'' = (X', t') + \lambda(P', H')$, where the λ are matrices acting on the basis of P', H' (λ is a square matrix and $\lambda = \mu + \bar{\mu}$ in view of the hamiltonian property). In a certain sense these transformations are trivial since, as we recalled, at the beginning there is a lack of difference between X'' and X', t' (we selected only P', H').

(b) When we have combined $L \oplus H$ with the (X', t')-plane, it is necessary by elementary (in the sense of Whitehead) transformations to combine the bases on $L \oplus H$ and on the (X', t')-plane; this can be done if and only if $\det \phi = 1$ (in the sense of Whitehead and Dieudonné). This means that we must expand ¡p as a product of elementary matrices: $\phi = \prod_s \alpha_{i_s, j_s}(a_s)$.

Thus, the unimodular matrix ϕ of an even skew-hermitian (hermitian) form on a free module is an invariant of the process; here ϕ is expanded as a product of elementary matrices: $\phi = \prod \alpha_{i_s, j_s}(a_s)$. For fixed ϕ we assume that two expansions as products of elementary matrices are equivalent if $gh^{-1} = 1$ in the group $\mathrm{St}(A)$, where g and h are these expansions regarded as elements of $\mathrm{St}(A)$. Thus we have the pairs $[\phi, g]$, $q \in \mathrm{St}(A)$. In addition, for ϕ we allow the unimodular change of variables $\phi \to \alpha\phi\bar{\alpha}$, where $\det \alpha = 1$ (note that the conjugation operator is also defined on $\mathrm{St}(A)$).

Let J_i denote the matrix of the simplest form

$$J_i = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}, \quad i = 1, 2;$$

the reduced forms have matrices $\alpha J_i \bar{\alpha}$. Note that J_i is lifted into $\mathrm{St}(A)$ for all A.

Definition. $V_i^2(A)$ consists of all pairs (ϕ, g), where $s(g) = \phi$ and $s\colon \mathrm{St}(A) \to E(A)$ is a canonical epimorphism, and where ϕ is a unimodular hermitian (skew-hermitian) even form on a free A-module. The pair $[\phi, g]$ is equivalent to the trivial pair if $g = \alpha J_i \bar{\alpha}$ in $\mathrm{St}(A)$. The sum is the Whitney sum $\oplus$. (For the form ϕ to be trivial it is necessary that it can be reduced to J_i by unimodular transformations, but this is sufficient only if $K^2(A) = 0$.)

The natural homomorphism $K^2(A) \to V_i^2(A)$ is defined by analogy with $j = 0, 1$; here the elements $g \in K^2(A)$ go into $J_i g$ and $s(J_i g) = J_i$.

There is a natural homomorphism of "change of the sign of symmetry of the category":

$$\kappa_i\colon V_i^2(A) \to V_j^0(A), \quad i \neq j, \quad i, j = 1, 2.$$

However, in the theory of V_i^* this is not, in general, an isomorphism; the mapping κ is an isomorphism only if $K^0(A) = K^1(A) = K^2(A) = 0$. In this case the theory of V_i^* coincides with that of U_i^*.

By analogy with $j = 0, 1$ we have the exact sequence

$$K^2(A) \to V_j^2(A) \to W_j^1(A).$$

It will be useful to give the outline of the definition of another "homology theory" of W_j^* by analogy with the theory of V_j^* (these objects are needed for geometrical applications).

1) **Definition of $W_j^0(A)$.** The elements $\alpha \in W_j^0(A)$ are represented by the projective lagrangian planes $L \subset H_n$ in a free hamiltonian module. A selected P- (or X-) plane in H_n is taken to be the zero element. The equivalence is realized by means of a stabilization, of a twisting of an X-plane onto an isomorphically projectible plane and by interchanging x_i and p_i in some basis x, p. We allow the "hamiltonian operations", where H and t are free modules. In this connection we only consider the "invertible elements" with the indicated properties, so that $W_j^0(A)$ is a group.

2) **Definition of $W_j^1(A)$.** The elements $\alpha \in W_j^1(A)$ are quadratic forms of the sign of symmetry other than j, but only on free modules (it is clear that they are nonsingular and even). The zero element is the form which is reducible to H_n by unimodular (in the sense of $K^1(A)$) transformations and with accuracy up to a stabilization. This definition is obtained from the "processes of reducing lagrangian planes", as before, where the desired form is the "hessian of the action" of the reduction process, which is interpreted as a lagrangian plane.

3) **Definition of $W_j^2(A)$.** An element $\alpha \in W_j^2(A)$ is an automorphism T of the hamiltonian space H_n with the sign of symmetry other than j, and with $\det T = 1$ (in the sense of K-theory); we assume that the lifting $h \in \mathrm{St}(A)$ is defined, where $s(h) = T$, $s\colon \mathrm{St}(A) \to E(A)$. The equivalence of the pairs $[T, h]$ is defined naturally by analogy with V_j^2.

As for V-theory we will not give here the detailed definitions and prove the elementary properties; U-theory will be studied algebraically later.

Note that for W-theory there is also the homomorphism of the "change of the sign of symmetry of the category" $W_i^2(A) \to W_j^0(A)$, $i \neq j$ and $i, j = 1, 2$; in general this is neither a monomorphism nor an epimorphism.

§ 4. The classical analogs of the algebraic procedure of constructing hermitian K-theories. The meaning of the objects U_i^* and V_i^* for rings of functions

We are going to consider here several questions that are indirectly related to the present study. Our aim in this section is to clarify the significance of the objects constructed in §§ 1–3, their advantages and defects, by means of special examples of basic rings and of simple (but well known in other domains) classical analogs.

Example 1. Let $A = R$. A quadratic form over R has one stable invariant, which is its signature. For skew-symmetric forms there are no invariants. In both cases each lagrangian plane in H_n can be reduced to one which is isomorphically projectible by elementary transformations. In the classical skew-symmetric hamiltonian space $H_n = (x_i, p_i)$ these elementary transformations are only the interchanges of x_1 and p_1 in some basis (with due regard for the sign).

Here there have been considered the (in general nonlinear) n-dimensional lagrangian submanifolds $L \subset H_n$ and the form of the action $\omega = p\,dx = \sum p_i dx_i$ on them, which is such that $d\omega|_L = 0$. In addition, the projection $\pi\colon H \to (X)$ along P and the cycle $W^{n-1} \subset L$ of the singularities of this projection play a particular role. For a Hamiltonian $H(x, p, t)$ the equations $[\dot p = \partial H/\partial x,\ -\dot x = \partial H/\partial p]$ define a family of trajectories which we can consider in the space H_{n+1} with the basis $(x, t = x_{n+1}, p, H = p_{n+1})$. By taking the initial data on L we obtain, in the usual way, a new lagrangian manifold L' of dimension $n+1$ in H_{n+1}, and also the "cycle of the singularities" $W^n \subset L'$ of the projection $\pi\colon L' \to (x, t)$ such that for $t = 0$ we have $W^n \cap L'_0 = W^{n-1} \subset L$, where $L = L'_0$. Each trajectory lies on L' whenever it originates on L at $t = 0$ and its finite segments at times $t \in [0, T]$ have an index of intersection with the cycle W^n; Maslov [9] has shown that this index of intersection is equal to the Morse index of the corresponding hamiltonian function $H(x, p, t)$ of a variational problem with $L \subset H_n$ as the manifold of initial data and with the functional

$$S(\gamma) = \int_\gamma \tilde{L}(x, \dot{x}, t)\, dt,$$

where

$$H(X, p, t) = \left(\dot{x}, \frac{\partial \tilde{L}}{\partial \dot{x}}\right) - \tilde{L}, \quad p = \frac{\partial \tilde{L}}{\partial \dot{x}}, \quad \gamma(0) \in L \subset H_n.$$

On the initial manifold $L \subset H_n$ the cycle $W^{n-1} \subset L$ has integral indices of intersection with the one-dimensional cycles, in contrast to the usual Stiefel class, where they are defined only modulo two; this is a simple consequence of the lagrangian property. However, Maslov [9] has given an interesting procedure for determining this index on L, which will itself be necessary for subsequent applications. Namely, we consider a curve $\gamma(\tau)$ on L that intersects W^{n-1} at the point $\gamma(\tau_0)$ and for $\epsilon > 0$ we suppose that, $\gamma(\tau_0 - \epsilon)$ and $\gamma(\tau_0 + \epsilon)$ do not lie on W^{n-1}. Let U^n be a (small) neighborhood of the point $\gamma(\tau_0) \in W^{n-1}$. The domain U^n is projected onto an X-plane with a degeneracy, for example, at $\gamma(\tau_0)$. We can find a (minimal) subspace $\bar{P} \subset P$ and a maximal $\bar{X} \subset X$, where $\bar{X} + \bar{P}$ is a lagrangian plane, such that U^n is projected isomorphically onto $\bar{X} + \bar{P}$. The dimension of $\bar{P}$ is equal to the corank of the projection π at $\gamma(0)$. For $\epsilon > 0$ the tangent planes at the points $\gamma(\tau_0 \pm \epsilon)$ are projected isomorphically onto $\bar{X} + \bar{P}$ and also onto X. We consider the (locally single valued) function of the action on $L \supset U^n$: $S(y) = \int_{y_0}^{y} p\, dx$, $y \in U^n$, and its hessian, in particular, at the points $\gamma(\tau_0 \pm \epsilon)$, $\epsilon > 0$. It has been proved mat the index of the intersection (locally) of the segment $[\gamma(\tau_0 - \epsilon), \gamma(\tau_0 + \epsilon)]$ of the curve with W^{n-1} is equal to the difference of the numbers of the negative squares (of the indices) of the hessians of the action at the points $\gamma(\tau_0 + \epsilon)$ and $\gamma(\tau_0 - \epsilon)$, and that this index of the hessian has a jump only at the points of intersection.

Note that in this situation, as is easily seen, a double jump of the index is a jump in the signature of the hessian of the action in going from $\gamma(\tau_0 - \epsilon)$ to $\gamma(\tau_0 + \epsilon)$.

Next, the hessians of the action were interpreted in [9] in terms of a sequence of elementary operations (the replacement of X by an isomorphically projectible X'' (X') and the interchange of X''_i (X'_i) and p_i) which send X into $\bar{X} + \bar{P}$, where X'' (X') is tangential to L at the point $\gamma(\tau_0 - \epsilon)$ ($\gamma(\tau_0 - \epsilon)$); this interpretation was necessary in the construction of a global analog of the WKB method (of a canonical operator), since each elementary operation of a rearrangement was followed by taking the Fourier transform, with respect to the appropriate coordinate, of functions which are finite in the image of the domain $\pi'(U^n)$ on $\bar{X} + \bar{P}$, and which revert to the coordinate representation. (Here the functions finite on U^n were first brought together on $\pi'(U^n)$ by means of the usual local operation, semiclassical in the $\bar{X} + \bar{P}$-representation, which is multiplication by $\exp[iS/h]/\sqrt{\det \pi'}$ and which requires the isomorphic projectiveness of the lagrangian manifold.)

Hence the answer essentially depended from which "side" of W^{n-1} one takes the initial point γ_0 in U^n where the hessian of the action determines the process, and it is necessary to take these centers "from the right side" of W^{n-1} simultaneously for all U^n_k covering L. The distinguishing characteristic of these processes (the hessians of the action) from the two sides led to the index of intersection which enters into the final result as the phase multiplier $e(i\pi/2)^{\gamma\circ W}$. Then the fact that the constructions were independent of the choice of coverings and of the initial point on L, as well as of the paths γ_k connecting this point with the center of U_k^n, led to the condition that for any closed path γ on L we have

$$e^{\left\{\frac{i}{h}\int_\gamma p\,dx + \frac{i\pi}{2}\gamma\circ W_{n-1}\right\}} = 1$$

which is the so called "Bohr–Sommerfield quantum condition" (in the one-dimensional case).

It is essential to note here that in all these procedures the definition of the Maslov index of $\gamma \circ W_{n-1}$ on L corresponds exactly to the formalism we used in §§ 1–3 to define the two-dimensional objects in the skew-symmetric category (for the case $U_j^2(R)$, $j = 2$), including the treatment of the class of a quadratic form (of the signature) in terms of processes of reducing objects of the earlier dimension (lagrangian planes) to isomorphically projectible objects. It is interesting to note that, although for $A = R$ and for the lagrangian submanifolds of H_n we can define, by the standard topological construction of a "tangent mapping", the higher characteristic classes from $H^n(U_n/O_n)$ (see [2], [17]), the one-dimensional class W^{n-1} and the formalism of defining it, which is retained in a more general and refined form in the construction of hermitian K-theories over rings with an involution, plays a particular role.

Example 2. Here we shall consider the rings $A_1 = C[\pi]$ ($\pi = Z \times \cdots \times Z$) and A_2, which is the ring of complex-valued (infinitely differentiable, or even analytic) functions on the torus T^n. In view of the Fourier series expansion of $\operatorname{Char} T^n = Z \times \cdots \times Z$, A_1 is embedded in A_2 as trigonometric polynomials in $e^{i\phi_1}, \dots, e^{i\phi_n}$ which are identified with the generators of the group π. For A_1 we have $\tilde{K}^0(A_1) = 0$ and, according to Bass, $K^1(A_1) = \pm\pi \subset A_1$. Also, $\tilde{K}^0(A_2) \otimes Q$ and $K^1(A_2)$ coincide with $\tilde{K}^0(T^n) \otimes Q = H^{\text{even}}(T^n, Q)$ and $K^1(T^n, Q) = H^{\text{odd}}(T^n, Q)$; that is, it is substantially larger, although A_1 was dense in A_2. Thus the fiberings over T^n, for $n \geq 2$, and the elements of $K^1(T^n)$ cannot be constructed from the polynomials alone but require convergent series that belong to A_2, although the speed of convergence can be arbitrary (even for the analytic case).

Next we consider the groups $U_j^0(A_1)$ $(= V_j^0(A_1))$ of hermitian (skew-hermitian) forms over $C[\pi]$ for $j = 1, 2$. Te note that, since $i = \sqrt{-1} \in A_1$, therefore $U_1^0 = U_2^0$, and so the hermitian and skew-hermitian cases are not distinct. Next, by taking the Fourier transform we can treat $U_j^0(A_1)$ as functions on T^n with a value in the nonsingular hermitian matrices over C defined by trigonometric polynomials.

It has been proved noneffectively from the topology (see, for example, [13] or [16]) that for $A_1 = C[\pi]$ the groups U_j^0 are nontrivial and are connected with $\Lambda^*(Z \times \cdots \times Z) = H^*(T^n)$. By regarding $\alpha \in U_j^0(A_1)$ as a function-valued form over T^n, we can see that the obstruction to its reduction to $V_j^0(A_1)$ is then interpreted in the sense of homotopic topology as an element of $\tilde{K}^0(T^n)$; the nontrivial homomorphism $U_j^0(A_1) \to \tilde{K}^0(T^n)$ arises automatically, and after we take the tensor product by Q it becomes an epimorphism.

Thus we see that, in contrast to the usual $K^0(A_1) = 0$, the groups $U_j^0(A_1)$ "catch" the vector bundles $\alpha \in \tilde{K}^0(T^n)$. This approach to the study of $V_j^0(A_1)$

is due to Gel'fand (who communicated it to the present author) and it was subsequently, developed jointly by Gel'fand and Miščenko [6]; it is important to note here that the rings $A_1' = Z[\pi_1]$, and more roughly $A_1'' = R[\pi_1]$, which appear in smooth topology (see [14]), lead to the study, after taking the Fourier transform, of the groups $KR(T'')$ (see [6]).[3]

Example 3. We consider the ring $A = C(X)$ of complex-valued functions on a finite complex X with a natural involution and norm. This example generalizes Example 2.

Here we mainly consider the objects $U^j(A)$ for all $j \geq 0$; we may omit writing the number of the sign of symmetry of the category because $i = \sqrt{-1} \in C(X)$.

Because U_1^* and U_2^* coincide, this theory is 2-periodic by definition (see § 3). Here the element $\beta \in U^0(A) = U^0(X)$ is a projective module (a vector fibering over X) with an hermitian form. By taking the trivial operations of adding the trivial fibering with a form whose signature is unity, we can always make the signature of the form on each fiber (equal to C_{2n}) equal to zero and can regard n as large. The process of reducing an element to the zero element consists of choosing a direct projective submodule with a zero form (a lagrangian plane). If the metric on the fiber is reduced to (he hamiltonian metric, the set of lagrangian planes in each fiber is a manifold coinciding with U_n as $n \to \infty$. Hence the obstruction to the construction of a cross-section or to the reduction is $[\beta] \in K^0(X)$. We have obtained a monomorphism $U^0(X) \to Z + \tilde{K}^0(X) = K^0(X)$ (a similar construction for $X = T^n$ was discussed in Example 2).

Conversely, it is easy to see that for $U^1(A) = U^1(X)$ a projective lagrangian plane in a free hamiltonian module defines an element $\alpha \in K^1(X)$, since the set of (lagrangian) planes in a fiber is U_n. This plane over A is projected onto a direct projective submodule in the X-plane if and only if $\alpha = 0$ in $K^1(X)$. It is easy to see that the monomorphism $U^0(X) \to K^0(X)$, which we constructed earlier, is an epimorphism. Thus we have arrived at a theorem which clarifies the significance of our constructions.

Theorem 4.1. *For the function ring $A = C(X)$, the theory of $U^*(A) = U^*(X)$ coincides with the usual complex K-theory.*

Remark 4.2. For $A = C(X)$, $V^0(A)$ coincides with $K^0(X)+K^0(X)$; the Gel'fand–Miščenko group V_F^0 lies in $K^0(X)+K^0(X)$ as $(x, -x)$; the group $U^0(A)$ is the factor group of $K^0(X) + K^0(X)$ with respect to the diagonal $(x, x) = 0$.

Remark 4.3. The most difficult point in the constructions in §§ 1–3, for which we developed the whole formalism (the passage from $U^1(A)$ to $U^2(A)$, where the sign of symmetry of the category changes), corresponds, when we make the passage to the rings $A = C(X)$, to the place where one must prove the Bott periodicity, namely the passage from K^1 to K^2, to a group which we must again regard as K^0, so that the difficulties which are met at this stage in smooth topology in constructing Bass type projection operators in an effective algebraic manner now become completely comprehensible; for example, one muse interpret $K^0(X)$ as $K^2(X)$ in the classical (Bott) case, where the projection operator of Bass is analogous to the mapping $K^2(X \times S') \to K^2(EX) \to K^1(X)$.

[3] In fact, Gel'fand and Miščenko [6] studied the subgroups $V_{j,F}^0(A)$ of $V_j^0(A)$ $(A = C(X))$ composed of forms on a free module, and proved that they coincide with $K_0(X)$.

Example 4. We consider the rings $A = R(X)$ with the trivial involution, where X is a finite complex. Here the hermitian category simply consists of the usual metrics and the skew-hermitian category is the skew-symmetric case.

Reasoning by analogy with Example 3, we easily obtain

$$U_1^0(A) \overset{C_2}{\leftrightarrow} Z + [X, BO] = KO^0(X),$$

$$U_1^1(A) \overset{C_2}{\leftrightarrow} [X, O] = KO^{-1}(X),$$

$$U_1^2(A) \overset{C_2}{\leftrightarrow} U_2^0(A) \leftrightarrow [X, B(U/O)] = KO^{-6}(X),$$

$$U_1^3(A) \overset{C_2}{\leftrightarrow} U_2^1(A) \leftrightarrow [X, U/O] = KO^{-7}(X),$$

where the subscript 1 denotes that we start from a symmetric category, BO is the classifying space, C_2 are finite groups of order 2^h and the cokernels of U_1^2 and U_1^3 in U_2^0 and U_2^1 also are of order 2^h. As in Example 3 we find that

$$U_1^*(A) \otimes Z[1/2] = KO^*(X) \otimes Z[1/2].$$

We see here that there is a substantial defect in our constructions which is connected with the pairing. One explanation of this defect may be as follows. In the construction of all homotopic homology theories we take, by definition, $K^{i+1}(EX) = K^i(X)$, where $EX = X \times S/X \vee S^1$. For function rings we have $C(X \times S^1) = A[z, z^{-1}]^\wedge$, where $\wedge$ denotes the completion, and the decomposition is $K^{i+1}(X \times S^1) = K^{i+1}(X) + \tilde{K}^i(X) + \tilde{K}^i(S^1)$, whereas, as in algebra, we take for the basic concept $A[z, z^{-1}]$ without the completion. In KO-theory we have $K^0(S^1) = Z_2$, which introduces a difference. Let us recall that we have always imposed the restriction $\tilde{K}^0(A[z, z^{-1}]) = \tilde{K}^0(A)$. Perhaps the reason is deeper. It would be useful to analyze the case of $KSC(X)$-theory (it is 4-periodic), the general KR-theories and also the "Sullivan theory" F/PL (or F/Top).

Let us note that in smooth topology our theory is applicable only to study of the objects $\otimes Z[1/2]$, since we have not considered Arf-invariants, so that the defects of this theory modulo 2 are natural.

In what follows we shall bear in mind that only $U_j^* \otimes Z[1/2]$ is a "homology theory", although the aim of the present article is only to construct the Bass projection operators, the existence of which is a weaker (although not much weaker) assertion than Stating that this is a homology theory.

§ 5. Construction of the analogs of the Bass projection operators and their inverses which connect U_j^0 and U_j^1, V_j^0 and V_j^1

We shall now define some analogs of the projection operators of Bass for the Laurent extension $A_z = A[z, z^{-1}]$ with the involution $\bar{z} = z^{-1}$ (for $A = Z[\pi]$ we have $A_z = Z[\pi \times Z]$), but only in one dimension.

1) **Definition of the projection operator** $B_U^0 \colon U_i^1(A_z) \to U_i^0(A)$**.** We consider an element $\alpha \in U_i^1(A_z)$ represented by a lagrangian plane $L \subset H_n$. We can assume that H_n is projective hamiltonian, but in view of the equality $K^0(A_z) = K^0(A)$, which is always assumed to hold, we assume that H_n is defined in the form $Q[z, z^{-1}]$, where Q is projective hamiltonian over A and the plane $L = L_0(z, z^{-1}) \subset H_n$, as an abstract module, is given in the same form.

We use the following notation: H^+ consists of elements of H_n which are nonnegative powers of z; H^- consists of elements with purely negative powers; similarly we define L^+ and L^- in the abstract module $L = L_0(z, z^{-1})$. H^+, H^-, L^+ and

L^- are (infinite-dimensional) A-modules. Let N be so large that $z^N L^+ \subset H^+$ and $E_N(L^+)$ consists of all $y \in H^+$ such that $\langle y, z^N L^+\rangle_0 = 0$ (recall that $\langle\cdot,\cdot\rangle \in A$ (see the beginning)). Since $E_N(L^+) \supset L \cap H^+$ we put

$$B^0_U(\alpha) = E_N(L^+)/z^N L^+ \in U^0_i(A).$$

Since a quadratic form is induced on the module $E_N(L^+)/z^N L^+$, it only remains to prove that this module is projective.

Lemma 5.1. *$E_N(L^+)/z^N L^+$ is a projective module; the natural form on it is nonsingular and even, and its class in $U^0_i(A)$ is independent of the arbitrariness in the choice of a representation of the element $\alpha \in U^1_i(A_z)$.*

Proof. We consider the module $E_{N_1}(L^-) \subset H^-$ which relative to $\langle\cdot,\cdot\rangle$ is orthogonal both to $z^{-N}L^- \subset H^-$ and to $E_N(L^+)+E_{N_1}(L^-)/(z^N L^+ + z^{-N_1}L^-)$. This module is isomorphic to a direct sum of L_0 taken several times with itself and with the module L_0^-, where H_n is decomposed into a sum of two lagrangian planes $L + L' = H_n$ and $L' = L'_0(z, z^{-1})$; this is at once clear in the coordinates of L, L' in H_n. Hence $E_N(L^+)/z^N L^+$ is projective over A. The evenness and nonsingularity are obvious, and it can be straightforwardly verified that the definition is invariant: it $L \subset H_n$ is projected onto a direct submodule, then, by stabilizing the problem, we can decompose L into the sum of submodules isomorphically projectible onto (X) and onto (P), which are equivalent. Next, if L is projected isomorphically onto X, then the projective hamiltonian decomposition of the module $E_N(L^+)/z^N L^+$ is indicated thus:

a) $z^i(P)$, $0 \le i \le N-1$,

b) $(z^i L_0)_+$, $0 \le i < \infty$ with $L = L_0(z, z^{-1})$, $L^+ = L_0(z)$, where the subscript $+$ denotes the projection $\pi_+ \colon H_n \to H^+$.

By noting that the operation B^0_U is invariant under the interchange of X and P we complete the proof of the lemma. □

2) **Definition of the projection operator $B^0_V \colon V^1_i(A_z) \to V^0_i(A)$.** Here the lagrangian planes have a selected basis and lie in the free hamiltonian module H_n. Again we put

$$B^0_V(\alpha) = E_N(L^+)/z^N L^+ \in V^0_i(A), \quad \alpha \in V^1_i(A).$$

It is obvious from the proof of Lemma 5.1 that an isomorphically projectible L leads to the projectible hamiltonian module $E_N(L^+)/z^N L^+$, with the chosen lagrangian plane in it, whose projective class belongs to $B(\alpha)$, where α lies in the image of $K^1(A_z) \to V^1_i(A_z)$ and $B\colon K^1(A_z) \to K^0(A)$, so that the invariance of this definition can be proved in a way analogous to Lemma 5.1.

3) **Construction of the embeddings $\bar{B}^0_U \colon U^0_i(A) \to U^1_i(A_z)$ and $\bar{B}^0_V \colon V^0_i(A) \to V^1_i(A)$.**

If $\alpha \in U^0_i(A)$ is represented by a projective module with the scalar product ϕ: $Q \to Q^*$, then $(-\alpha)$ is represented by the pair Q, $-\phi\colon Q \to Q^*$. We consider the infinite sum

$$Q_z = \cdots \oplus [(Q,\phi) \oplus (Q,-\phi)] \oplus [(Q,\phi) \oplus (Q,-\phi)] \oplus \ldots$$

with the natural scalar product and with the translation by z in period, $Q_z = \sum_j [(Q,\phi)\oplus(Q,-\phi)]z^j$. There is a natural lagrangian plane (P) which is generated by the elements $x \oplus x^*$, $x \in (Q,\phi)$, $x^* \in (Q,-\phi)$, $x = x^*$, since $\langle x\oplus x^*, y \oplus y^*\rangle = 0$. We note in Q_z another lagrangian plane L which is generated over $A_z = A[z, z^{-1}]$ by elements of the form $x \oplus zx^*$, $x \in (Q,\phi)$, $x^* \in (Q,-\phi)$, $x = x^*$. We take the plane $(P) = L_0$ as the initial plane and assume, by definition, that $\bar{B}^0_U(Q,\phi)$ is the class of the lagrangian plane $L \subset Q_z$, relative to $(P) \subset Q_z$, considered in $U^1_i(A)$. Similarly we define the operator $\bar{B}^0_V\colon V^0_i(A) \to V^1_i(A_z)$. Here the elements $(Q,\phi) \in D^0_i$ are invertible in the sense that we can find $(Q',\phi') \in D^0_i$ such that $(Q,\phi) \oplus (Q',\phi')$ is isomorphic to the hamiltonian module H_n with the basis $x_1,\dots,x_n,p_1,\dots,p_n$. We note that $p_i = y_i \oplus y^*_i$, where $y_i \in Q$, $y^*_i \in Q'$. Next we consider $Q_z = \sum_j H_n z^j$ similarly to the previous case and single out the lagrangian plane in Q_z with the basis $p_1,\dots,p_n \in H_n$. Let $p'_j = y_j \oplus zy^*_j$ generate the lagrangian plane $L \subset Q_z$ with a fixed basis.

Definition. $\bar{B}^0_U(Q,\phi)$ is the class of the plane L relative to $P \subset Q_z$ which is generated by $p = (p_1,\dots,p_n)$.

Lemma 5.2. *The operators $\bar{B}^0_U$ and $\bar{B}^0_V$ are properly defined on the groups $U^0_i(A)$ and $V^0_i(A)$, and their image in $U^1_i(A), V^1_i(A)$ belongs to the kernel of the natural mappings $U^1_i(A_z) \xrightarrow{\epsilon} U^1_i(A)$ and $V^1_i(A_z) \xrightarrow{\epsilon} V^1_i(A)$ generated by the homomorphism $A_z \xrightarrow{\epsilon} A$, where $z \xrightarrow{\epsilon} 1$. The operators B^0_U and B^0_V annihilate the images of the groups $\bar{\epsilon}U^1_i(A)$ and $\bar{\epsilon}V^1_i(A)$ respectively in $U^1_i(A_z)$ and $V^1_i(A_z)$.*

Proof. If the initial element $\alpha \in U^0_i(A)$ is represented by the projective module Q,ϕ and if we choose a lagrangian plane $P_1 \subset Q$, where $\phi/P_1 = 0$, then in the module $Q_z = \sum_j [(Q,\phi) \oplus (Q,-\phi)]z^j$ we can select, in a canonical manner, a new lagrangian plane $L' = \sum_j [(P_1,0) \oplus (P_1,0)]z_j$. It is easy to see that the distinguishing characteristics of both planes (the initial plane P with basis $y \oplus y^*$ and the other plane L with basis $(y \oplus zy^*)$) relative to L' are trivial, from which it follows that $\bar{B}^0_U$ is correctly defined. The proof for $\bar{B}^0_V$ is similar. Next, if we consider the operation of going from A_z to A on the module Q_z by putting $\epsilon(z) = 1$, the images of the lagrangian planes $P \subset Q_z$ and $L \subset Q_z$ will coincide (with due regard for the basis, wherever this remark is meaningful). It follows that this homomorphism annihilates the images of $\bar{B}^0_U$ and $\bar{B}^0_V$. Conversely, B^0_U and B^0_V annihilate the images of the embeddings $\bar{\epsilon}\colon U^1_i(A) \to U^1_i(A_z)$ and $\bar{\epsilon}\colon V^1_i(A) \to V^1_i(A_z)$, since on these images, by choosing in the construction the minimally possible $N = 0$, we obtain the result that $E_0(L^+) = L^+ = L \cap H^+$ and that $E_0(L^+)/L^+ = 0$. The lemma is proved. ☐

The following lemma is useful for topological interpretations.

Lemma 5.3. *If $\alpha \in V^0_i(A)$ is defined by the free module F with the basis $y_1,\dots,y_n$ and with a scalar product defined by an hermitian (skew-hermitian) nonsingular matrix $\phi = \phi_1 \pm \bar{\phi}_1$, where $\phi^{-1} = \psi = \psi_1 \pm \bar{\psi}_1$, then the image of the operator $\bar{B}^0_V(\alpha)$ can be naturally realized in the hamiltonian module H_n with the basis $x_1,\dots,x_n,p_1,\dots,p_n$, and with the lagrangian plane $L \subset H_n$, the basis of which is defined In matrix form by*

$$\psi L = (z^{-1}\psi_1 \pm \bar{\psi}_1)P + (z^{-1} - 1)X,$$

where the definition of the bases in the construction of $\bar{B}^0_V$ is the following:

$$P = y_i \oplus y^*_i, \quad L = y_i \oplus z^{-1}y^*_i,$$

$$X = \psi_1 y \oplus \mp\bar{\psi}_1 y, \quad y = y_1,\dots,y_n.$$

The element $\bar{B}^0_V(\alpha)$ is represented as an infinite sum $B^0_V(\alpha) = \sum_j (F_n \oplus F'_n)z^j$, the scalar products on F_n and F'_n are defined by the matrices ϕ and $-\phi$, and the *method for singling out the space* $X = \psi_1 y \oplus \pm\bar{\psi}_1 y$ *complementary to* $P = y \oplus y^*$ *was given in Remark* 2.3.

The proof consists in verifying the formula by a direct substitution.

Theorem 5.4. *The equalities* $B^0_U \bar{B}^0_U = \pm 1$ *and* $B^0_V \bar{B}^0_V = \pm 1$ *and the following direct decompositions hold*:

$$U^1_i(A[z, z^{-1}]) = \bar{\epsilon} U^1_i(A) + \bar{B}^0_U U^0_i(A),$$
$$V^1_i(A[z, z^{-1}]) = \bar{\epsilon} V^1_i(A) + \bar{B}^0_V V^0_i(A) + T_i(A),$$

also $B^0_V T_i(A) = 0$ *and* $0 = B^0_U \bar{\epsilon} = B^0_V \bar{\epsilon} = \epsilon \bar{B}^0_U = \epsilon \bar{B}^0_V = \epsilon T_i$.

Proof. We first prove that $B^0_U \bar{B}^0_U = \pm 1$ and that $B^0_V \bar{B}^0_V = \pm 1$. For the theory of U^*_i and for any $\alpha \in U^0_i(A)$ the construction of the element $\bar{B}^0_V(\alpha)$ includes the module $Q_z = \sum_j [(Q, \phi) \oplus (Q, -\phi)] z^j$ and the lagrangian planes $P = \sum_j (y \oplus y^*) z^j$ and $L = \sum_j (y \oplus z^{-1} y^*) z^j$; moreover, $Q^+_z = \sum_{j \geq 0} [(Q, \phi) \oplus (Q, -\phi)] z^j$ and L^+ consists of all elements of the form $\sum_{j \geq 0} (y \oplus y^* z^{-1}) z^j$. It is obvious that $zL^+ \subset H^+ = Q^+_z$ and that $E_1(L^+) = zL^+ + [(Q, \phi) \oplus 0]$. Hence $E_1(L^+) = (Q, \phi)$ and for U^* we have $B^0_U \bar{B}^0_U = 1$. The proof is similar for V-theory.

Thus we now have the direct decompositions

$$U^1_i(A_z) = U^1_i(A) + U^0_i(A) + S_i(A),$$
$$V^1_i(A_z) = V^1_i(A) + V^0_i(A) + T_i(A),$$

where $B^0_U S_i(A) = 0$ and $B^0_V T_i(A) = 0$.

We now turn to the algebraically more difficult part of the theorem; this is the investigation of the groups $S_i(A)$ and $T_i(A)$. For convenience we will talk in the language of bases as in the theory of $V^1_i(A)$, although all the arguments are true for U^1_i. Let $\alpha \in V^1_i(A_z)$ be represented by the lagrangian plane $L \subset H_n$ over the ring A_z with a basis $(e_1, \dots, e_n) \in L$ written in matrix form as $(e) = ax + bp$, where $a\bar{b} \pm b\bar{a} = 0$ and $b\bar{a}$ is the "hessian of the action" on L.

The simplest case a). $B^0_V(\alpha) = E_N(L^+)/z^N L^+ = 0$ for some N which we can take to be zero. Then $E_0(L^+) = L^+ \subset H^+$ and L^+ is a direct summand of H^+ as in an $A[z]$-module. In this case we can obviously choose another basis $e'_1, \dots, e'_n$ in L^+, perhaps stabilizing the pair $L^+ \subset H^+$, so that $(e') = a'x' + b'p'$, where the a' and b' are independent of z, so that in this case the theorem is obvious. The proof for U-theory is analogous.

Case b). $B^0_V(\alpha) = E_N(L^+)/z^N L^+ \neq 0$, but it contains a lagrangian plane $y_1, \dots, y_q \in B^0_V(\alpha)$ with $\langle y_i, y_j \rangle_0 = 0$, and we can find lifted elements $\tilde{y}_1, \dots, \tilde{y}_q \in E_N(L^+)$ such that $\langle \tilde{y}_i, \tilde{y}_j \rangle_0 = 0$ (this is obvious). For our investigation it will be convenient to describe the construction of the module $E_N(L^+)$ in greater detail. Namely, $E_{-N_1}(L^-) \subset H^-$, where $z^{-N_1} L^- \subset H^-$, defines in a canonical manner an inverse for $B^0_V(\alpha)$, and in the sum

$$E_{-N_1}(L^-) + E_N(L^+)/(z^{-N_1} L^- + z^N L^+)$$

there is distinguished in a canonical manner the lagrangian plane $(z^j e)$, where $-N_1 < j < N_1$ and (e) is the basis of the initial plane. For $j < N$ the elements $(z^j e)$ lie in $E_N(L^+)$. If L^* is complementary to L, $L + L^* = H_n$, and e^* is a basis in L^*, $\langle e^*_j, e^*_i \rangle = 0$, then it is obvious that for $j < N$ the elements $(z^j e^*)_+$ lie in $E_N(L^+)$. We have two A-submodules in $E_N(L^+)$ with the natural bases

$\{z^j e\}_+ = u_j$ and $\{z^j e^*\}_+ = u_j^*$, $j < N$, and any element from $E_N(L^+)$ mod $z^N L^+$ has the form $\gamma u + \mu u^*$. Of course, (u) and (u^*) are not free in general, and are not lagrangian in $E_N(L^+)/z^N L^+$. We can write the lagrangian plane (y) in the form $y = \gamma u + \mu u^* \in B_V^0(\alpha)$, although it may not be unique.

Lemma 5.5. *If the submodule $(u) = \{z^j e\}_+ \subset B_V^0(\alpha)$ is such that the scalar product $\langle \cdot, \cdot \rangle_0$ in it is identically zero, then the inclusion $(u) \subset L$ holds over $A[z, z^{-1}]$, and in the lagrangian plane L we can select another basis (e') not containing z^{-1} and such that the equality $E_0(L'^+)/z^0 L'^+ = 0$ holds in the new basis (e') (let us recall that, in U-theory, by a basis in L we mean a representation $L = Q[z, z^{-1}]$, where Q is projective).*

Proof. If $\langle u_i, u_j \rangle_0 = 0$, then, by noting that $\langle u, H^- \rangle_0 = 0$ and that by definition $\langle u, z^N L^+ \rangle_0 = 0$ we see that $\langle u, x \rangle_0 = 0$ for any $x \in L$, so that

$$\langle u, x \rangle = \sum \langle u, z^j x^+ \rangle_0 z^j = 0.$$

Hence $\langle u, L \rangle = 0$ and $u \in L$. Then, since $E_N(L^+)$ modulo $z^N L^+$ is represented as $(u) + (u^*)$, when $\langle u, u \rangle_0 = 0$ we have $(u) \cap (u^*) = 0$ and the module (u) is projective over A because $B_V^0(\alpha)$ is projective and the scalar product on it is nonsingular. Since in our case $(u) + z^N L^+ = L \cap H^+$, it follows that $z^N L^+$ is an A-free module and, as an $A[z]$-module, $L \cap H^+$ does not have z-torsion; therefore $L \cap H^+$ is a projective $A[z]$-module whose basis we will assume to be the positive part of L, denoting $L \cap H^+$ by L'^+, which is correct in U-theory (in V^*-theory we must have a free or stably free module $L \cap H^*$).

For U-theory Lemma 5.5 is a consequence; in V-theory an analogous argument shows that (u) $((u) + z^N L^+)$ is stably free over A $(A[z])$.

Thus the lemma is proved. □

The general case c). Our problem is now as follows: to reduce the initial lagrangian plane by a hamiltonian operation (iterated) to the conditions of Lemma 5.5 by using the lagrangian plane $(y) \in E_N(L^+)/z^N L^+$ over A which, being lifted into $E_N(L^+)$, is written as $\gamma u + \mu u^*$, where $u \in (z^j L)_+$ and $u^* = (z^j L^*)_+$, $-N_1 < j < N$. Let $\tilde{u} = z^j L^*$ for $-N_1 < j < N$, let $u = (\tilde{u})_+$ and $y = \gamma u + \mu u^*$, and let $\langle y, y \rangle_0 = \langle \gamma \bar{\mu} \pm \mu \bar{\mu} \rangle_0 = 0$. We pass to the new hamiltonian basis $x_L = L^*$, $p_L = L$ in H_n and let $\tilde{L}$ denote the (P)-plane written in the new basis x_L, p_L. We write the basis for the plane (y), lifted into $E_N(L^+)$, as $\gamma u + \mu u^* = \lambda x_L + \delta p_L$, and consider the (iterated) hamiltonian operation with the generating function $H = y + \phi t$, where ϕ is given precisely as a function of λ and δ (or of γ and μ):

$$H' = H - y - \phi t, \quad x'_L = x_L \pm \bar{\delta} t,$$
$$p'_L = p_L + \bar{\lambda} t, \quad t' = t,$$
$$\tilde{L}' = L' \oplus H = (P) \oplus H,$$
$$\phi = \langle y, y \rangle_+ = \overline{\langle y, y \rangle_-} = \sum_{i \geq 0} \langle y, z^i y \rangle_0 z^i, \quad \langle y, y \rangle_0 = 0.$$

If we revert to the basis x, p, H, t in place of the plane $L = p_L$ we will have the plane $L' = (p'_L, H')$. If we consider everything in a free basis (for V^*), then $L = aX + bP$, $L^* = cX + dP$, where $a\bar{b} \pm b\bar{a} = 0$, $c\bar{d} \pm d\bar{c} = 0$, $c\bar{b} + d\bar{a} = 1$ ($b\bar{a}$ and $d\bar{c}$ are the "hessians of the action" on L and L^*) and $\bar{X} = \bar{d} p_L + \bar{b} X_L$, $P = \bar{c} P_L + \bar{a} X_L$.

Thus in the basis (X, t, P, H)

$$L' = (aX + bP + \bar{\lambda}t,\ H - y - \phi t) = (P'_L, H'),$$

where y was defined above and ϕ (a function of λ, δ (or of γ, μ)) has been given. Next we carry out the following "splitting operation" (of the variables H, t).

We consider the space with the basis $(x, t_1, t_2, p, H_1, H_2)$ and elements $\tilde{y} \in E_N(L^+)$ such that $\langle \tilde{y}, y\rangle_0 = \delta_{ij}$, $\langle \tilde{y}, \tilde{y}\rangle = 0$ and $\tilde{y} = \gamma x_L + \delta p_L = \tilde{\lambda} u + \delta u^*$. We put

$$L'' = \begin{cases} p_L + \bar{\lambda}(t_1 + z^{-1}t_2) + \langle p_L, \tilde{y}\rangle (H_1 - z^{-1}H_2), \\ H'_1 = H_1 - y - \phi(t_1 + z^{-1}t_2) + \psi(H_1 - z^{-1}H_2), \\ z^{-1}H'_2 = z^{-1}H_2 - y - \phi(t_1 + z^{-1}t_2) - \psi(H_1 - z^{-1}H_2), \end{cases}$$

where

$$\psi = \langle y, \tilde{y}\rangle_1 = \sum_{j\geq 1} \langle y, z^j \tilde{y}\rangle_0 z^j, \quad \phi = \langle y, y\rangle_+ = \sum_{j\geq 1} \langle y, z^j y\rangle_0 z^j.$$

The following simple lemma holds.

Lemma 5.6. *As an element of* $U_i^1(A_z)$ *(or of* $V_i^1(A_z)$*) the lagrangian plane* $L'' \subset H_{n+2q}$ *is equivalent to the initial* $L \subset H_n$.

Proof. The initial lagrangian plane (p'_L, H') is obtained from L'' on the subspace $t = t_1 + z^{-1}t_2$, $H_1 - z^{-1}H_2 = 0$, and it is easy to see that its "stabilization" is L''. The assertion of the lemma becomes obvious in going over to the basis $\tilde{H} = H_1$, $\tilde{H}_2 = z^{-1}H_2 - H_1$, $\tilde{t} = t_1 + z^{-1}t_2$, $\tilde{x} = x$, $\tilde{p} = p$.

Thus the lemma is proved. □

Thus for L'' we have a situation satisfying Lemma 5.5, whence follows Theorem 5.4. In fact, we have

Lemma 5.7. *The following equality holds for all* $-\infty < j < \infty$ *and for a basis of* L'':

$$\langle (z^h L'')_+, (z^k L'')_+\rangle_0 = 0.$$

The proof follows from a direct calculation with the formulas for L''.

It is clear that the operations developed above have a meaning on projective lagrangian planes L, including the application of the hamiltonian operation (see Remark 2.6) and also the splitting operation and Lemma 5.7. Hence we have proved the fundamental Theorem 5.4 for U^*-theory. □

§ 6. The analogs of the projection operators of Bass which connect $U_i^2(A_z)$ and $U_i^1(A)$, $V_i^2(A_z)$ and $V_i^1(A)$

We will first consider U^*-theory and will construct the "Bass operators"

$$B_U^1 \colon U_i^2(A[z, z^{-1}]) \to U_i^1(A),$$
$$\bar{B}_U^1 \colon U_i^1(A) \to U_i^2(A[z, z^{-1}]).$$

To construct B_U^1 we consider the representative $\alpha \in U_i^2(A_z) = U_j^0(A_z)$, where $i \neq j$ and $i, j = 1, 2$; $Q = Q_0[z, z^{-1}]$. By definition this representative is the projective module $Q = Q_0[z, z^{-1}]$ with the "basis" Q_0 over A and with the scalar product $\phi \colon Q \to Q^*$ which is hermitian and nonsingular.[4]

[4]We will always assume that the decomposition of $Q^* = Q_0^*[z, z^{-1}]$ is compatible with $Q = Q_0[z, z^{-1}]$, where $(z^j Q_0^*, Q_0) = 0$ for $j = 0$ in the sense of the scalar product over A.

We next recall that, in deriving the objects from U_i^2 as processes over lagrangian planes, the form ϕ was interpreted naturally as a lagrangian plane in the projective hamiltonian module $\tilde{H} = Q + Q^*$, where $Q_0^* = (X)$ and $Q_0 = (P)$, and the basis of L has the form $\{P + \phi(P)\} = L'$. As a projective module L is isomorphic to Q. However, one must note that in the conjugate variables $\phi^{-1}\colon Q^* \to Q$ we pass to ϕ^{-1}, and so the basis $L'' = \{\phi^{-1}(X) + X\}$ is specified naturally in L. By definition L'_+ denotes the "nonnegative" part in L measured in the basis L': $L'_+ = \sum_{i\geq 0} Q_0 z^i$, and L''_- denotes the strictly negative part in L measured in the basis L'': $L''_- = \sum_{i<0} L'' z^i = \sum_{i<0} Q_0^* z^i$. Let $E_{N,N_1}(L) \subset \tilde{H}$ denote the submodule over A consisting of all elements $\gamma \in H$ such that $\langle \gamma, z^N L'_+\rangle_0 = \langle \gamma, z^{-N_1} L''_-\rangle = 0$, where N and N_1 are numbers for which $z^N L'_+ \subset \tilde{H}^+$ and $z^{-N_1} L''_- \subset \tilde{H}^-$ (it is obvious that $\langle \tilde{H}^+, \tilde{H}^-\rangle_0 = 0$).

Since $E_{N,N_1}(L) \supset L$, we take by definition

$$B_U^1(\alpha) = E_{N,N_1}(L)/(z^{-N_1} L''_- + z^N L'_+)$$

with the selected lagrangian plane

$$\tilde{L} = L/(z^{-N_1} L''_- + z^N L'_+) \subset B_U^1(\alpha)$$

and with the natural hamiltonian basis which we now give: if $Q_0 \subset Q$ is a basis, where $Q = \sum_{-\infty}^{+\infty} Q_0 z^i$, and $Q_0^* \subset Q*$ is a basis, where $Q^* = \sum_{-\infty}^{+\infty} Q_0^* z^i$, then $Q^+, Q^-, \tilde{H}^+, \tilde{H}^-$ are natural concepts and $z^j Q_0 \subset E_{N,N_1}$ for $0 \leq j < N$, $z^{-j} Q_0^* \subset E_{N,N_1}$ for $0 < j < N_1$, and also $(z^j L')_+$ for $0 \leq j < \infty$, $(z^{-j} L'')_-$ for $0 < j < \infty$, where the subscripts $\pm$ denote the projection operators $\tilde{H} \to \tilde{H}^{\pm}$. To obtain a hamiltonian basis in $B_U^1(\alpha)$ we must take $(\tilde{P}) = \{z^j Q_0, z^{-s} Q_0^*\}$ and $(\tilde{X}) = (z^\alpha L')_+$ for $\alpha < N$; $(\tilde{X}) = (z^{-\beta} L'')_-$ for $\beta < N_1$.

Thus we obtain a hamiltonian basis in $B_U^1(\alpha)$ and the lagrangian plane $\tilde{L}$. Hence the operator B_U^1 is defined.

Lemma 6.1. *The operator $B_U^1\colon U_i^2(A_z) \to U_i^1(A)$ is properly defined; to an allowable degree of accuracy the projective class $[\tilde{L}] \in \tilde{K}^0(A)$ coincides with the image $B(\det\phi)$ under the ordinary Bass projection operator $B\colon K^1(A_z) \to K^0(A)$, and $\det\phi \in K^1(A)$ is the determinant of the mapping $\phi\colon Q \to Q^*$, where $\det\phi$ is taken with the degree of accuracy with which it is defined.*

Proof. If the quadratic form ϕ on a module admits a "reduction to the zero" in the form of a lagrangian plane $V \subset Q$, $\langle \phi(V), V\rangle = 0$, then $Q = V + V^*$ (with accuracy up to a stabilization). Analogously, $Q^* = V^* + V$. We take a new hamiltonian basis in $H = Q + Q^*$, namely, $X' = V_Q + V_{Q^*}$, $P' = V_Q^* + V_{Q^*}^*$ where $V_Q = V \subset Q$, $V_{Q^*} = V \subset Q^*$ and similarly for V_Q^* and $V_{Q^*}^*$. The passage from the old basis to the new one is effected in $\tilde{H}$ by operations of permuting submodules in $X = Q$ and $P = Q^*$. Note that $\phi\colon Q \to Q^*$ has the form $\phi\colon V_Q \to V_{Q^*}$, $\phi\colon V_Q^* \to V_{Q^*}^*$. In the new basis X', P' in $\tilde{H}$ the lagrangian plane splits into a direct sum of modules lying in the X'- and P'-spaces. Hence it obviously follows that $B_U^1(0) = 0$, and the correctness of the definition is proved.

We prove the second part of the lemma. The projection $\pi\colon L \to X$ along $P = Q^*$, where $X = Q$, sends the basis $L' = P + \phi(P)$ into the basis ϕQ_0 on the X-space, and therefore $\pi(z^N L'_+) = (z^N \phi) Q_+$. Next, under the mapping π the basis $L'' = \{\phi^{-1}(X) + X\}$ goes into $Q_0 \subset X$. Hence $L/(z^{-N_1} L''_- + z^N L'_+)$ goes into $B(\phi)$.

Therefore the projective class of the module $\tilde{L}$ is $B(\det\phi) \in K^0(A)$ by definition of the projection operator $B\colon K^1(A_z) \to K^0(A)$, and the lemma is proved. □

Although we do not essentially need the projection operators $B^1_V\colon V^2_i(A_z) \to V^1_i(A)$ and $\bar{B}^1_V\colon V^1_i(A) \to V^2_i(A_z)$ later, we give a method of constructing them. We must indicate first of all the analog of the classical Bass projection operator $B^1\colon K^2(A_z) \to K^1(A)$ in the Milnor–Steinberg definition of K^2 (see § 1). To do this we recall once again the definition of $B\colon K^1(A_z) \to K^0(A)$. If $\alpha \in K^1(A)$ is represented by the matrix $\alpha\colon F \to F$ on a free module over $A_z = A[z, z^{-1}]$ and $z^N\alpha\colon F_+ \to F_+$, then we take $B(\alpha) = F_+/z^N\alpha F_+$. In his proof of the correctness of this definition Bass remarked that for $\alpha \in E(A_z) = [\mathrm{GL}[\infty, A_z], \mathrm{GL}[\infty, A_z]]$ we obtain stably free modules in the form $B(\alpha)$. Here we must note the following extract from his arguments: if $\alpha = \prod_s \alpha_{i_s,j_s}(a_s)$, $s \in A_z$, is represented as a superposition of elementary matrices (or "is lifted" into $\mathrm{St}(A_z) \xrightarrow{s} E(A_z)$), then there is distinguished, in a canonical manner, a free basis (in a stable sense) in the module $B(\alpha)$. Since by definition $K^2(A_z) = \operatorname{Ker} s$, to different liftings of the matrix α into $\mathrm{St}(A_z)$ there correspond different bases in $B(\alpha)$, and their distinguishing characteristic is an element from $K^1(A)$. This gives the projection operator $B\colon K^2(A_z) \to K^1(A)$.

The elements $\alpha \in V^2_i(A_z)$ are represented as pairs $[\phi, h]$, where ϕ is an even hermitian (skew-hermitian) form on the free module F with $\det\phi = 1$, and $h \in \mathrm{St}(A_z)$ and $s(h) = \phi$ (in the basis). To define the operator $B^1_i\colon V^2_i(A_z) \to V^1_i(A)$ we repeat exactly the construction of the definition of the operators B^1_U, but in addition note (on the basis of Lemma 6.1) that the projective class of the lagrangian plane $\tilde{L}$ in the tree hamiltonian module $E_{N_1,-N_1}(L)/(z^{-N_1}L''_- + z^N L'_+)$ will be trivial. Moreover, the lifting $h \in \mathrm{St}(A_z)$ allows us to distinguish a canonical basis on $\tilde{L}$ which gives us an element from $V^1_i(A)$. The proof of the correctness of this definition is very simple and is analogous to the previous one.

Thus we have constructed the operators B^1_U and B^1_V.

We turn to the construction of the embedding operator $\bar{B}^1_U\colon U^1_i(A) \to U^2_i(A_z)$. Suppose that the element $\alpha \in U^1_i(A)$ is represented by the lagrangian (projective) plane $L \subset H_n$ in the hamiltonian module H_n with the basis x, p. If L is free and has a basis $\tilde{e}$, then $\tilde{e} = ax + bp$ in matrix form, where $b\bar{a} \pm a\bar{b} = 0$ and $b\bar{a}$ is the "'hessian of the action" on L in matrix notation (a quadratic form on L, where $(\xi, \eta) = \langle \xi, \pi\eta \rangle$, $\xi, \eta \in L$, $\pi\colon H_n \to X$ and $\langle \cdot, \cdot \rangle$ is the usual scalar product on H_n).

We consider the module $E \ni e$, canonically isomorphic to L, and the free modules $(\tilde{X})$ and $(\tilde{P})$, isomorphic to X and P but extended freely to the ring $A_z \supset A$. We impose on the direct sum $E + (\tilde{P}) + (\tilde{X})$ all the relations of the form $(z-1)e = \tilde{e}$, where $e \in E$, $\tilde{e} \in L \subset (\tilde{X}, \tilde{P})$, and e and $\tilde{e}$ correspond to each other in view of the isomorphism between E and L. Let $\bar{B}^1(\alpha)$ denote the A_z-module we obtain. We introduce in $\bar{B}^1_U(\alpha)$ a scalar product $(\cdot, \cdot)$ with the other sign of symmetry to the one in H_n by putting

$$(\tilde{X}, \tilde{X}) = 0, \quad (\tilde{P}, \tilde{P}) = 0, \quad (\tilde{X}_i, \tilde{P}_j) = (z-1)\delta_{ij}.$$

It is easy to calculate that it follows from these formulas that the scalar product (E, E), bounded on E, coincides with the "hessian of the action" on L. By definition we can take for the element $\bar{B}^1_U(\alpha) \in U^2_i(A_z)$ the class of the quadratic form $(\cdot, \cdot)$ on the factor module of $\{E, \tilde{X}, \tilde{P}\}$ with respect to the relation $(z-1)e = \tilde{e}$. The following lemma holds.

Lemma 6.2. *The module $\bar{B}^1_U(\alpha)$ is free and the scalar product on it is even and nondegenerate. If the lagrangian plane $L \subset H_n$ representing α is free, then the matrix ϕ of the scalar product $(\ ,\ )$ on $\bar{B}^1_U(\alpha)$ is unimodular in the natural basis of the module $B^1_U(\alpha)$; that is, $\det\phi = 1 \in K^1(A_z)$ (in a multiplicative notation). The operation $\bar{B}^1_U\colon \alpha \to \bar{B}^1_U(\alpha)$ properly defines a homomorphism $\bar{B}^1_U\colon U^1_i \to U^2_i(A_z)$ such that $\epsilon\bar{B}^1_U = 0$, where $\epsilon\colon U^2_i(A_z) \to U^2_i(A)$.*

Proof. Consider the decomposition $H_n = L + L^*$ (with accuracy up to a stabilization) and, as before, let P_L denote L and X_L denote L^*. We choose the basis $E, \tilde{X}_L$ in the module $(E, \tilde{X}, \tilde{P})_{(z-1)\epsilon-\bar{\epsilon}}$, where $\tilde{X}_L = c\tilde{X} + d\tilde{P}$. From what has been proved earlier $L^* = X_L = cX + dP$ and $L = aX + bP$. It is easy to see that $\tilde{X}$ and $\tilde{P}$ are expressible in terms of E and $\tilde{X}_L$. Since $L + L^* = H_n$ is free, the module $\bar{B}^1_U(\alpha)$ is free over A_z. (Recall here that in U^1_i-theory we could put any lagrangian plane into an equivalence class in a free H_n.) Next we denote E by $\tilde{\tilde{P}}_L$. Note that $\tilde{\tilde{P}}_L, \tilde{X}_L$ are not lagrangian planes in $\bar{B}^1_U(\alpha)$. The nonsingularity of the matrix ϕ of the scalar product $(\ ,\ )$ on $\bar{B}^1_U(\alpha)$ follows from the fact that ϕ^{-1}, as a matrix, is obtained from ϕ after the following formal transformation:

$$z \to z^{-1}, \quad X_L \to P_L, \quad P_L \to \pm X_L,$$
$$\phi \to \phi^{-1}$$

and this is verified by a direct calculation when the lagrangian planes X_L and P_L are free. When X_L and P_L are projective, ϕ must be understood as the homomorphism of modules

$$\bar{B}^1_U(\alpha) \xrightarrow{\phi} \bar{B}^1_U(\alpha)^*;$$

the transformation ϕ^{-1} reverses the direction of the arrow and the inverse transformation ϕ^{-1} is formally conjugate to ϕ, considered as a homomorphism of A-modules (but not of A_z-modules), where in this "formal conjugate" multiplication by z is conjugate with itself and not with multiplication by z^{-1}.

Note that the transformation ϕ is written as $\phi = \phi(0) + \lambda\phi' \pm \bar{\lambda}\bar{\phi}'$, where $\lambda = z - 1$, $\bar{\lambda} = z^{-1} - 1$ and $\phi(0)$ is obtained from ϕ by the "augmentation" $A_z \xrightarrow{\epsilon} A$, where $\epsilon(z) = 1$; here $\phi(0), \phi'$ and $\bar{\phi}'$ are transformations over A, and $\phi(0)$, as a form on the A-module $\epsilon(\tilde{\tilde{P}}_L, \tilde{X}_L)$, has the form $(\epsilon\tilde{X}_L, \epsilon\tilde{X}_L) = 0$, $\epsilon\tilde{\tilde{P}}_L = (\epsilon\tilde{X}_L)^*$; the form $(\epsilon\tilde{\tilde{P}}_L, \epsilon\tilde{\tilde{P}}_L)$ on $\epsilon E = \epsilon\tilde{\tilde{P}}_L$ is the "hessian of the action" on L. In view of the evenness of the "hessian of the action", we now see that the form on $\bar{B}^1_U(\alpha)$ is even. In a basis on $L = ax + bp = e$, where $e^* = cx + dp$, if L, L^* are free we have $\bar{a}b \pm b\bar{a} = 0$, $c\bar{d} \pm d\bar{c} = 0$ and $a\bar{d} \pm b\bar{c} = 1$, and the matrices $\phi(0)$ and ϕ' take the form

$$\phi(0) = \begin{pmatrix} b\bar{a} & 1 \\ \pm 1 & 0 \end{pmatrix}, \quad \phi' \begin{pmatrix} 0 & 0 \\ c\bar{b} & \pm d\bar{c} \end{pmatrix},$$
$$\bar{\lambda} = z^{-1} - 1, \quad \lambda = z - 1, \quad \phi = \phi(0) + \lambda\phi' \pm \bar{\lambda}\bar{\phi}'$$

We now prove that the definition of the operator $\bar{B}^1_U$ is invariant under the elementary operations of a transformation on L.

1. If $L = (X) \subset H_n$, then it is entirely obvious that $\bar{B}^1_U(L) \cong 0$. Since $\bar{B}^1_U$ preserves the sum, the invariance relative to a stabilization is proved.

2. We can regard an interchange of (projective) submodules in the (X)- and (P)-spaces as the iteration: a) of a "stabilization" (that is, the addition of a module with the basis (H, t) and a selected H-plane); b) of the replacement of an extended

P-plane by a new P'-plane, isomorphically projectible onto P; c) of the interchange of the X'- and P'-planes as a whole.

Here our assertion is not obvious only in the case of the replacement of an X-plane by a new plane $X' = X + \delta P$, where $\delta = \mu \pm \bar{\mu}$ and X' is isomorphically projectible onto X along $P = P'$.

3. We now consider the replacement of X by $X' = X + \delta P$, $P' = P$, $\delta = \mu \pm \bar{\mu}$. To prove the equivalence of the forms $\bar{B}^1_U$ of the L-plane $L = P_L$ in the coordinates $(X, P) \subset H_n$, and of the L'-plane, which is the same L but in the coordinates $(X', P') \subset H_n$, we must consider the "hessian of the action" on L and L' and the bases $(E', \tilde{X}_{L'}) \subset \bar{B}^1_U(L')$ and $(E, \tilde{X}_L) \subset \bar{B}^1_U(L)$, where $\lambda E' = L'$, $\tilde{X}_{L'} = L'^*$ and $\lambda E = L$, $\tilde{X}_L = L^*$, $E = \tilde{P}_L$, $E' = \tilde{P}_{L'}$. The embedding of the form $\bar{B}^1_U(L')$ in $B^1_U(L)$ is done formally according to the following formulas:

$$\begin{gathered} E' \to E + a\mu P, \\ \tilde{X}_{L'} \to \tilde{X}_L + \lambda c\mu \tilde{P}, \\ \tilde{P}' \to \tilde{P}, \\ \tilde{X}' \to \tilde{X} + (z\mu \pm \bar{\mu})\tilde{P}, \\ \tilde{\lambda} = z - 1, \quad \mu \pm \bar{\mu} = \delta, \quad P = P', \quad X' = X + \delta P. \end{gathered}$$

A straightforward calculation shows that the scalar product is preserved under this embedding. It follows from this that the definition of $\bar{B}^1_U$ is invariant. We remark that in this proof we have essentially used the "evenness" of the "hessian of the action" $\delta = \mu \pm \bar{\mu}$ of the plane X' in the (X, P)-coordinates: without the evenness of δ we would not be able to define $\bar{B}^1_U$ correctly. In addition let us note that by putting $\epsilon\colon z \to 1$ we go over to $U^2_i(A)$ and the image $\epsilon\bar{B}^1_U \subset U^2_i(A)$ goes into 0, since the lagrangian plane $\epsilon\tilde{X}_L$ lies in the module $\epsilon\bar{B}^1_U(\alpha)$. Thus we have proved all the statements in Lemma 6.2 apart from those concerning the determinant $\det\phi \in K^1(A_z)$ in the case of free lagrangian planes $L \subset H_n$ with a selected basis $(e) = aX + bP = P_L$, which arise in the theory of $V^1_i(A)$. Actually we regard P_L as complemented to $(X_L, P_L) = (cx + dp, ax + bp)$, where $\pm\langle x_L, p_L\rangle = 1$, and we consider the matrix $\phi = \phi(0) + \lambda\phi' \pm \bar{\lambda}\bar{\phi}'$, where

$$\phi(0) = \begin{pmatrix} b\bar{a} & 1 \\ \pm 1 & 0 \end{pmatrix}, \quad \phi' \begin{pmatrix} 0 & 0 \\ c\bar{b} & \pm d\bar{c} \end{pmatrix},$$

Here it is necessary, by using the relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{b} \\ \pm\bar{c} & \pm\bar{a} \end{pmatrix} = 1$$

(the sign depends on the category), to show that $\det\phi = 1$ in $K^1(A_z)$ independently of the concrete ring A; thereby ϕ is lifted, by a universal method, into the group $\mathrm{St}(A_z)$, thus defining an element from $V^2_i(A_z)$. This means that we must give a "universal" factorization of ϕ as a product of elementary matrices. For this let us note that ϕ^{-1} is derived from ϕ by the simple unimodular change of variables $\tilde{\tilde{x}} \to \pm\tilde{\tilde{P}}_L$, $\tilde{P}_L \to \tilde{\tilde{x}}_L$, $z \to z^{-1}$. In view of this, $(\det\phi)(\det\phi^{-1}) = 1$, and hence $(\det\phi)^2 = 1$; this is a universal relation. Since for some rings with an involution $K^1(A_z)$ does not have 2-torsion, we see that the required universal relation $\det\phi = 1$ must hold. However, we have not completed the proof of this fact, which is required only in the theory of V^*_i. □

The construction of the operator $\bar{B}^1_V: V^1_i(A) \to V^2_i(A)$ follows from the "universal unimodular property" of the matrix ϕ; the correctness of the construction can be deduced from the corresponding arguments for $\bar{B}^1_U$, but we shall not dwell on V^*-theory.

The following theorems contain the main properties of the "Bass operators" $\bar{B}^k_U$ and B^k_U, where by $\bar{B}^k_U$ and B^k_U for $k \geq 2$ we mean the operators defined "by periodicity" starting from the isomorphism $U^k_j = U^{k-2}_i$, where $i \neq j$ and $j, i = 1, 2$ is the number of the category. In what follows we will omit the number k (the dimension of the domain of definition U^k_i) and write simply B_U and $\bar{B}_U$ understanding that these operators are defined for all k. When we consider the "multiple Laurent extension" $A[z_1, \dots, z_1^{-1}, \dots]$, where $i = 1, \dots, s$, we will write $B_U(z^*)$, where z^* is a nontrivial indivisible integral linear functional over a free abelian group with the generators $z_1, \dots, z_s$, bearing in mind that $B_U(z^*)$ "annihilates" the generator z (in the other basis) on which $(z^*, z) = 1$ and it goes into the subgroup Z^{s-1} on which $(z^*, Z^{s-1}) = 0$. We will write this as

$$B_U(z^*): U^*_j(A[Z^s]) \to U^*_j(A[Z^{s-1}_{(z^*)}]).$$

This follows essentially from previous arguments and theorems (in an invariant language).

If $Z^{s-1}_{(z^*_q)}$ is the subgroup spanned by all the elements of the selected basis $z_1, \dots, z_s$ apart from z_q we will write $B_U(z_q)$.

Theorem 6.3. [5] *The following holds*:

$$B_U \bar{B}_U = \pm 1.$$

The proof of this theorem is nontrivial for U^2_i, in contrast to the corresponding statement for U^1_i which was obvious (see the proof of Theorem 5.4).

Theorem 6.4. *The Bass operators for different variables anticommute*:

$$-B_U(z_1)B_U(z_2) = B_U(z_2)B_U(z_1),$$
$$-B_U(z_1)\bar{B}_U(z_2) = \bar{B}_U(z_2)B_U(z_1),$$
$$-\bar{B}_U(z_1)\bar{B}_U(z_2) = \bar{B}_U(z_2)\bar{B}_U(z_1).$$

If $z^*_1, \dots, z^*_s$ is a basic set of linearly independent integral functionals over Z^s with the generators $z_1, \dots, z_s$, then the product $B_U(z^*_{i_1}) \circ \dots \circ B_U(z^*_{i_k})$ depends on the subgroup $Z^{s-k}_{(z^*_{i_1}, \dots, z^*_{i_k})}$ on which $(z^*_{i_q}, Z^{s-k}_{(z^*_{i_1}, \dots, z^*_{i_k})}) = 0$; this product is denoted by

$$B_U = (z^*_{i_1} \wedge \dots \wedge z^*_{i_k}) = B_U(z^*_{i_1}) \circ \dots \circ B_U(z^*_{i_k}),$$

where $z^*_{i_1} \wedge \dots \wedge z^*_{i_k} \in \Lambda^{k,*}$; $\Lambda^{k,*}$ is the exterior power over $\operatorname{Hom}_Z(Z^s, Z)$.

Theorem 6.5. *The following formula holds for all k:*

$$U^k_i(A[z, z^{-1}]) = U^k_i(A) + U^{k-1}_i(A).$$

The proof of this theorem for odd k was fundamental to Theorem 5.4. When k is even (it is sufficient to take U^2_i, $k = 2$, $i = 1, 2$) the result follows easily from Theorems 6.4 and 6.3. In fact, when we apply the operator $B_U(z_2)$ to both sides of the formula with $k = 2$ and introduce a new variable z_2, since the Bass operators anticommute, we reduce the proof of Theorem 6.5 to Theorem 5.4, because $U^3_i = U^1_j$

[5] Let us recall (see the Introduction) that all these theorems refer to the theory of $U^*_j \otimes Z[1/2]$.

and $U_i^2 = U_j^0$ for $i \neq j$, $i, j = 1, 2$. Thus Theorem 6.5 follows from Theorems 6.4 and 6.3.

Before proving Theorems 6.3 and 6.4 we give an important consequence, the algebraic construction of a “homeomorphism of higher signatures”

$$\sigma_k \colon U_i^j(A[Z^s]) \to \Lambda^{j-4k}(Z^s).$$

Here U_1^* denotes U-theory in a symmetric category and U_2^* in a skew-symmetric category; $U_2^k = U_1^{k+2}$. If $\alpha \in U_1^j(A[Z^s])$ and $\gamma \in \Lambda^{j-4k}(Z^s)^*$, then on the basis of Theorem 6.4 we have the operator $B(\gamma) \colon U_1^j \to U_1^{4k}(A[Z^{4k}])$ if we annihilate the generator of Z^s entering into γ. Next there is a natural “homomorphism of the usual signature” σ: $U_1^{4k} \to Z$ for any $\tilde{A}$. Namely, if $\beta \in U_1^{4k}(\tilde{A})$ is represented by a quadratic form on an $\tilde{A}$-module M, then in view of the augmentation $\nu \colon \tilde{A} \to K$, where K is the ring of scalars with an involution, we have the K-module $M \otimes_\nu K$. If $K = Z, R, Q$ or C (with the natural involution), the usual signature of the induced form on $M \otimes_\nu K$ is denoted by $\sigma(\beta)$ where $\beta \in U_1^{4k}(\tilde{A}) = U_2^{4k+2}(\tilde{A})$. By definition, for $\alpha \in U_1^j(A[Z^s])$ we put

$$(\sigma_k(\alpha), \gamma) = \sigma B(\gamma)[\alpha] \in Z,$$

where $\gamma \in \Lambda^{j-4k}(Z^s)^*$, $B(\gamma)$ is the product $B(z_{i_1}^*) \circ \cdots \circ B(z_{i_{4k}}^*)$ of the Bass operators and $\gamma = z_{i_1}^* \wedge \cdots \wedge z_{i_{4k}}^*$.

Thus we have given a closed algebraic construction of the homomorphisms σ_k, since the Bass operators were defined above.

Remark 6.6. The present author has previously (see [13] and [14]) pointed out the existence of these σ_k, for even j from topological considerations and absolutely noneffectively. Later Shaneson [16] proved that they are isomorphic for $\pi = Z^s$ and the groups $L_n(\pi)$ at least on the groups $L_n \otimes Q$. However, Shaneson [16] developed an idea of Browder [5] concerning the reduction of the problem to a simply-connected topology, without using any algebraic definitions of the groups $L_n(\pi)$. Hence the question as to which “homology theory”, from an algebraic point of view, this noneffective “existence and uniqueness theorem” for the higher signatures and Bass type operators refers to, remains open, and the noneffective geometrical study of Shaneson [16] can be remedied and made correct for the individual case $L_n(Z^s) \otimes Z[1/2]$, where all these theories coincide after they have been constructed algebraically.

Remark 6.7. For the group ring $A = Z[\pi]$ of a free abelian group π with the generators $z_1, \dots, z_s$, Theorems 6.3 and 6.4 show that $U_1^*(A)$, constructed by the operators $\bar{B}_U(z_j)$ and $B_U(z_j)$ from $U_1^*(A_0)$, where $A_0 = Z[1]$, from a purely formal point of view correspond to the construction by the “creation operators” $\bar{B}_U$ and the “annihilation operators” $B_U(z_j)$ with anticommuting relations (Theorem 6.4). Here, naturally, the relation $B_U(z_j)^2 = 0$, or $\bar{B}_U(z_j)^2 = 0$, is taken by definition, since the repeated application of these operators is meaningless. For the group ring of the unit group $A_0 = Z[1] = Z$ the theory (with accuracy up to the tensor multiplication by $Z[1/2]$)

$$U_1^*(A_0) = \begin{cases} Z, & j = 4k, \\ 0, & j \neq 4k, \end{cases}$$

is generated by the “Milnor matrix” $\Phi \in U_1^0(Z)$ with signature 8 over the “ring of scalars” $Z[x, x^{-1}]$, where $U_1^j \xrightarrow{x} U_1^{j+4}$ is the periodicity operator which commutes with the "annihilation and creation operators" $\bar{B}_U, B_U$, and $B_U(z_j)\Phi = 0$ by definition for all z_j; that is, Φ corresponds formally to the zero vector, from which all the basic elements in $U_1^*(A)$ can be obtained by an iteration of the different operators $\bar{B}_U(z_j^*)$.

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