OPERATION RINGS AND SPECTRAL SEQUENCES OF THE ADAMS TYPE IN EXTRAORDINARY COHOMOLOGY THEORIES. U-COBORDISMS AND K-THEORY

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I. Let $X = (X_n)$ be a spectrum of $(n-1)$-connected complexes whose maps $f_n \colon EX_n \to X_{n+1}$ are homotopy equivalences up to a high dimension. $X^*(K, L)$ will denote the cohomology of the pair with respect to the spectrum X. We let $H^i(X, Z)$ be the limit $\lim_{n\to\infty} H^{n+i}(X_n, Z)$ and $X^i(X) = \lim_{n\to\infty} X^i(X)$. $X^*(X) = \sum_i X^i(X)$ is the Steenrod ring A^X.

We can now consider $X^*(K) = \sum_i X^i(K)$ as an A^X-module. As in [2] one can sometimes construct the sequence (E_r, d_r), where $E_2 = \mathrm{Ext}_{A^X}(X^*(K), X^*(L))$. Under what conditions does this spectral sequence converge to $\mathrm{Map}^S[L, K]$? We are interested in the case $L = S^0$.

Theorem 1. *If for a point P the group $X^0(P) = Z$ and $H^*(X, Z)$ has no torsion and the spectral sequence (E_r, d_r) with $E_2 = H^*(K, X^*(P))$, which converges to $X^*(K)$ has the property that all the differentials vanish for complexes with torsion free cohomology, then for such complexes the Adams spectral sequence with $E_2 = \mathrm{Ext}_{A^X}(X^*(K), X^*(P))$ exists and converges to the stable homotopy groups $\pi_*^S(K)$.*

Corollary 1. *For the spectrum $X = MU$, $X_{2n} = MU_n$ the Adams spectral sequence exists for any torsion free complex K and converges to the stable homotopy groups $\pi_*^S(K)$.*

Remark. For the spectrum $X = k$, $X_0 = BU \times Z$, $X_{2n} = BU^{(2n)}$, where $BU^{(2n)}$ is the space BU made $(2n-1)$-connected, the hypotheses of Theorem 1 are not satisfied. The spectrum k is the spectrum of the "stable" K-theory; here $\Omega^{2n} X_{2n} = BU \times Z$ (ordinary K-theory, where $X_{2n} = BU \times Z$ and $X_{2n-1} = U$ does not satisfy the "stabilization" condition $\pi_{i-k}(X_i) = 0$).

II. The main problem is computing the Steenrod rings A^U and A^k of U-cobordism and k-theory. We first compute A^U.

Consider the cobordism ring $\Omega_U = U^*(P) = Z[x_1, \ldots, x_i, \ldots]$ with $\dim x_i = -2i$ (see [9–11]). The ring $X^*(K)$ for $X = MU$ will be denoted by $U^*(K)$. Since $\Omega_U = U^*(P)$ we have operations of multiplication by an element x of the cohomology of a point: $U^j(K, L) \to U^{j-2k}(K, L)$, $\dim x = -2k$, $x \in \Omega_U^{2k}$, where $\Omega_U^{2k} = U^{-2k}(P)$. For any $\alpha, \beta \in U^*(K, L)$ we clearly have $x(\alpha\beta) = (x\alpha)\beta = \alpha(x\beta)$. The ring A^U can be considered as a free left Ω_U-module.

We further note that $N = A^U \otimes_{\Omega_U} A^U$ is also a left A^U-module since $N = U^*(MU \wedge MU)$. The multiplication $MU \wedge MU \to MU$ defines a "diagonal" $\Delta \colon A^U \to A^U \otimes_{\Omega_U} A^U$.

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We have the Kunneth formula in U-theory: the natural homomorphism of A^U-modules

$$U^*(K_1, L_1) \otimes_{\Omega_U} U^*(K_2, L_2) \to U^*(K_1 \times K_2 / K_1 \times L_2 \cup L_1 \times K_2),$$

is an isomorphism for torsion-free complexes, where the action of the algebra A^U on $U^*(K_1, L_1) \oplus_{\Omega_U} U^*(K_2, L_2)$ is given by the diagonal Δ.

We use an analog of characteristic classes of Chern (see [7] §§1, 2) to construct the operations of A^U.

Lemma 1. *There exist unique characteristic classes of Chern* $\sigma_K \colon k^0(K) \to U^{2k}(K)$ *with the following properties*:

1. $\sigma_0 = 1$; *if* η *is a* U_1*-bundle, then* $\sigma_1(\eta) \in \mathrm{Map}(K, MU_1)$.
2. *The Whitney formula* $\sigma(\xi \oplus \eta) = \sigma(\xi)\sigma(\eta)$, $\sigma = \sum_i \sigma_i$.
3. *For any* U_1 *bundles* ξ, η *the class* $\sigma_1(\xi \oplus \eta)$ *can be represented as* $\sigma_1(\xi) + \sigma_1(\eta) + a(\sigma_1(\xi)\sigma_1(\eta))$ *where* $a \in A_2^U$.
4. $\sigma_1(\lambda_{-1}\xi) = a_{n-1}\sigma_n(\xi)$, $\xi \in \mathrm{Map}(K, BU_n)$, $a_{n-1} \in A^U$.

As always the classes σ_k give rise to classes σ_ω, where $\omega = (k_1, \ldots, k_s)$ is a splitting of k into positive integers (or $\omega = (0)$ and $k = 0$) and $\sigma_{(1,\ldots,1)} = \sigma_i$, $\sigma_\omega(\xi \oplus \eta) = \sum_{\omega=(\omega_1,\omega_2)} \sigma_{\omega_1}(\xi)\sigma_{\omega_2}(\eta)$.

In U-theory we have the natural Thom isomorphism $\phi_U \colon U^*(K) \to U^*(M\xi, P)$, where ξ is a complex vector bundle over K, P is a point and $M\xi$ is the Thom complex. For a manifold M^n we shall denote by $D\sigma_\omega(M^n)$ the dual (see [1]) bordism classes $D\sigma_\omega(M^n) \in U_{n-2k}(M^n)$.

Lemma 2. *For closed almost complex manifolds the bordism classes* $D\sigma_\omega(M^n, \eta)$, *where* η *is a normal complex vector bundle of* M^n, *determine monomorphisms* $\sigma^*_\omega \colon \Omega_U \to \Omega_U$ *where* $\sigma^*_\omega \Omega^n_U \subset \Omega^{n-2k}_U$, $\omega = (k_1, \ldots, k_s)$, $\sum k_i = k$, $\sigma^*_\omega[M^n] = \epsilon D\sigma_\omega(M^n, \eta)$, $[M^n] \in \Omega^n_U$ *and* $\epsilon \colon U_*(M^n) \to U_*(P)$ *is the natural homomorphism.*

Note that when $n = 2k$, $\sigma^*_\omega(M^n)$ are characteristic numbers. The homomorphisms σ^*_ω are easily computed for complex projective spaces $CP^n \colon \sigma^*_\omega(CP^n) = \lambda_\omega[CP^{n-k}]$, $\lambda_k = k$.

Lemma 1 implies the Leibnitz formula

$$\sigma^*_\omega(xy) = \sum_{(\omega_1,\omega_2)=\omega} \sigma^*_{\omega_1}(x)\sigma^*_{\omega_2}(y), \quad xy \in \Omega_U.$$

Lemmas 1 and 2 lead to an important theorem which gives a complete determination of the ring $A^U = \sum_i A^U_i$.

Theorem 2. 1) *There exist unique operations* $S_\omega \colon U^j(K, L) \to U^{j+2k}(K, L)$, $\omega = (k_1, \ldots, k_s)$, $\sum k_i = k$, *satisfying the following properties*: a) *the operations* S_ω *are natural with respect to continuous maps, commute with the suspension homomorphism and the homomorphism* $\delta \colon U^{j-1}(L) \to U^j(K, L)$; b) *for any* $\alpha, \beta \in U^*(K, L)$ *we have*

$$S_\omega(\alpha\beta) = \sum_{\omega=(\omega_1,\omega_2)} S_{\omega_1}(\alpha)S_{\omega_2}(\beta);$$

c) *if* $\alpha \in \mathrm{Map}(K, MU_1) \subset U^2(K)$ *then* $S_k(\alpha) = \alpha^{k+1}$, *and* $S_\omega(\alpha) = 0$ *if* $\omega \neq (k)$; d) *the composition* $S_{\omega_1} \circ S_{\omega_2}$ *in* A^U *is a linear combination of operations of the form* S_ω; e) *if* $K = CP^N_1 \times \cdots \times CP^N_n$ *where* n *and* N *are large,* $u_i \in U^2(CP^N_i)$ *elements dual to the submanifolds* $CP^{N-1}_i \subset CP^N_i$, $u = u_1 \circ \cdots \circ u_n$, *then the operations*

$S_\omega(u)$ are linearly independent for all ω with $\dim\omega < n$ *and the linear Z-space spanned by $S_\omega(u)$ is the ideal generated by u in the ring of symmetric polynomials generated by $u_1,\dots,u_n$;* f) *the Chern classes σ_ω are equal to $\phi_u^{-1}S_\omega\phi_u(1)$.*

2) *Every element $\gamma \in A^U_{2k}$ has a unique representation as a linear combination $\sum_{i\to\infty}\lambda_i x_i S_{\omega_i}$, where x_i is some additive homogeneous basis of the ring Ω_U,* $\dim(x_iS_{\omega_i}) = 2k$ *and λ_i are integers.*

The ring A^U is a graded topological ring with basis x_jS_ω and the series $\sum\lambda_i x_i S_{\omega_i}$ converges if $\dim\omega_i \to \infty$ *as $i\to\infty$. We have the equality $A^U = (\Omega_U \circ S)$ where S is the ring spanned by all S_ω.*

3) *There is a commutation relation of the subrings $\Omega_U \subset A$ and $S \subset A^U$:*

$$S_\omega \circ x = \sum_{\omega=(\omega_1,\omega_2)} \sigma^*_{\omega_1}(x)S_{\omega_2}.$$

The Adem formulas in A^U follow from b)–e) and 3) is proved by the standard method of Cartan [6].

Example. For $K = S^0$, $L = P$ we get: the module $M_U = U^*(P)$ is given by one generator $t \in M_U^0$ with the relation $S_\omega(t) = 0$ for $\dim\omega > 0$.

Theorem 3. *For a mapping $A^U \xrightarrow{d} A^U$, where $d(1) = S_\omega$, the corresponding mapping $d^*\colon \mathrm{Hom}_{A^U}(A^U, M_U) \to \mathrm{Hom}_{A^U}(A^U, M_U)$ coincides with $\sigma^*_\omega\colon \Omega_U \to \Omega_U$, where Ω_U is naturally isomorphic with the group $\pi^S_*(MU)$, $\pi^S_*(MU) = \mathrm{Hom}_{A^U}(A^U, M_U)$.*

III. Let Q_p be the ring of rational numbers whose denominators are not divisible by p. For the study of p-components of $\pi^S_*(K)$ it suffices to consider the ring $A^U \otimes_Z Q_p$ and the modules $U^*(K)\otimes_Z Q_p$. Let C be the class of finite groups whose order is prime to p. It follows from [9–11] that the spectrum MU is C-homotopically equivalent to the direct sum $\sum_\omega M_\omega$ where ω are the non-p-adic $(k_1,\dots,k_m)$ and the A-module $H^*(M_\omega, Z_p)$ is equal to $A/A\bar{B}$, B is generated by the elements $e'_r \in A^{2p-1}$ (recently Brown and Peterson have in [5] constructed the spectrum M_ω). Let $X = M_\omega$ and $A^U_p = X^*(X)\otimes_Z Q_p$ be the graded ring over Q_p. With this notation we have:

Theorem 4. *The ring $A^U \otimes_Z Q_p$ is isomorphic to $GL(A^U_p)$, where $GL(A^U_p)$ is the graded ring of matrices of the form $(a_{\omega_i,\omega_j}) \in GL(A^U_p)$, $a_{\omega_i,\omega_j} \in A^U_p$ and*

$$n = \dim\omega_i - \dim\omega_j + \dim a_{\omega_i,\omega_j},$$

is the dimension of the matrix.

B. *In the ring A^U_p we have the subring $\Omega_U(p) = Z[x_1,\dots,x_i,\dots] \subset \Omega_U\otimes Q_p$,* $\dim x_i = 2p^i - 2$, *a projection $\pi_p\colon \Omega_U\otimes Q_p \to \Omega_U(p)$, with $\pi_p(x,y) = \pi_p(x)\pi_p(y)$ is defined; the generators x_i are such that $\sigma^*_{(2p^i-2)}x_i = p$ and for all ω with* $\dim\omega = 2p^i - 2$, $\sigma^*_\omega x_i = 0 \mod p$ *(outside of these conditions the choice of x_i is arbitrary).*

C. *The algebra S_p consists of all elements α of $S\otimes Q_p$, two elements α_1 and α_2 are identified when $\pi_p\alpha_1^*\pi_p = \pi_p\alpha_2^*\pi_p$ in $\Omega_U(p)$; here $\alpha^*\colon \Omega_U \to \Omega_U$ is the homomorphism of Lemma* 2.

D. *$X^*(P) = M_U(p)$, where $M_U(p)$ has a generator t given by the relation $S_p(t) = 0$ ($M_U, M_U(p)$ are the corresponding modules for the sphere).*

Even though the Theorem 4 gives a complete algebraic description of the ring A^U_p, this is very awkward for computations and it would be much better if direct formulas for composition could be found.

IV. We now pass to k-theory, where $k_{2n} = BU^{(2n)}$ and $k_0 = BU \times Z$, $k_2 = BU$, $k_4 = BSU$, $\Omega^{2n} k_{2n} = BU \times Z$ (Bott) and inclusions $x\colon k_{2n} \to k_{2n-2}$ are defined for all n. For the k-theory defined in this way we find that the functors $k^i(K, L)$ are isomorphic to the usual ones for $i \leq 0$ and for $i > 0$, $k^{2i}(K, L)$ for torsion free complexes consists of elements from K^0 of filtration $\geq 2i$. We have the Bott operator $x\colon k^j(K, L) \to k^{j-2}(K, L)$ which represents an element of the Steenrod ring A^k. We shall use two methods for computing the ring A^k: the second one uses the Adams operations $\Psi^k\colon K^0 \to K^0$ and the Bott operator $x\colon k^j \to k^{j-2}$, the first one will be based on the inclusion $k^2 \to U^2$ [7]. Neither one of them is complete.

$1st$ *method*: we use the "fixed" Conner–Floyd operators $\lambda_{-1}\colon U^j \to k^j$, $|j| < \infty$ and $\sigma_1\colon k^2 \to U^2$ which gives a splitting of the cohomology theory $U^2 = k^2 + \cdots$, $\sigma_1 \circ \lambda_{-1}\colon U^2 \to U^2$ is the projection [7]; we have the inclusion $x^N A^k_j \to A^U_{j-2N}$ for N large. Note that $x^N \to a_N$ (see Lemma 1) and $\sigma_1 \lambda_{-1}(\xi) = a_{N-1}\sigma_N(\xi)$, where ξ is a U_N-bundle and x the Bott operator.

$2nd$ *method* for determining the operations in k-theory: the Adams operations Ψ^k do not exist in stable k-theory since $\Psi^k \circ x = kx \circ \Psi^k$, but we have nonstable operations $k^n \Psi^k\colon k^{2n}(K, L) \to k^{2n}(K, L)$. Let n be large. We pick a large integer m and form a linear combination $\sum_k \lambda_k^{(n)} k^n \Psi^k = a_n$ and such that the maps of homotopy groups $a_{n^*}^{(j)}\colon \pi_{2n+2j}(BU^{(2n)}) \to \pi_{2n+2j}(BU^{(2n)})$ for $j \leq m$ be independent of n in the sense that for all $N > n$ one can find numbers $\lambda_k^{(N)}$ so that the homomorphisms $a_{n^*}^{(j)} = \sum_k \lambda_k^{(n)} k^{n+j}$ coincide with $a_{N^*}^{(j)} = \sum_k \lambda_k^{(N)} k^{N+j}$ for $j \leq m$. A sequence $a = (a_n)$, where $a_{n^*}^{(j)}$ are independent of n for $j \leq m(n) \to \infty$ will be called an *operation* in k-theory. If $a_{n^*}^{(j)} = 0$ for $j < q$, then $a = x^q b$ where x is the Bott operator and $b\colon k^l \to k^{l+2q}$. The ring of these operations (together with x) we shall denote by A^k_Ψ. If we consider the module M_k^Ψ (of the sphere) with one generator t over A^t_Ψ with $bt = 0$ for positive dimensions, then we find that $\mathrm{Ext}^{1,2i}_{A^k_\Psi}(M_k, M_k)$ is a cyclic group of order d_i, where d_i is the greatest common divisor of all numbers $k^n(k^i - 1)$ as k runs over the integers and n is large.

V. We indicate some simple results of computations.

Theorem 5. 1) *The group* $\mathrm{Ext}^{1,2i}_{A^k_\Psi}(M_k, M_k)$ *is equal to* Z_{d_i} *where* d_i *is the greatest common divisor of all numbers* $k^n(k^i - 1)$ *over all* k, $n \to \infty$.

2) *The group* $\mathrm{Ext}^{1,4i+2}_{A^U}(M_U, M_U)$ *is equal to* Z_2.

3) *The groups* $\mathrm{Ext}^{1,4i}_{A^U}(M_U, M_U)$ *are equal to* Z_{d_i/a_i}, *where* $a_{2q} = 1$, $a_{2q+1} = 2$, $d_i/2$ *is the denominator of* $B_i/2i$ *where* B_i *are the Bernoulli numbers.*

4) $d_i/\mathrm{Ext}^{1,2i}_{A^U}(M_U, M_U) = 0$, $i \neq 4k - 1$; $d_3/E_3^{1,8k+6} \neq 0$, *where* d_i *are the differentials of the Adams spectral sequence* (E_r, d_r), $k \geq 0$.

5) *The homomorphism* $q = \mathrm{Ext}^1 \circ J\colon \pi_{2i-1}(SO) \to \pi_{N+2i-1}(S)^N \to \mathrm{Ext}^{1,2i}_{A^U}$ *coincides with the Hopf–Milnor–Kervaire invariant* [8].

References

[1] M. F. Atiyah, Proc. Cambridge Philos. Soc. **57** (1961), 200. MR 23 A4150.
[2] J. Adams, Comment. Math. Helv. **32** (1958), 180. MR 20 #2711.
[3] ———, Bull. Amer. Math. Soc. **68** (1962), 39. MR 24 #A3662.
[4] J. Adams and G. Walker, Proc. Cambridge Philos. Soc. **61** (1965), 81. MR 30 #1516.
[5] E. Brown and F. Peterson, Topology **5** (1966), 149. MR 33 #719.
[6] H. Cartan, Comment. Math. Helv. **29** (1955), 40. MR 16, 847.

[7] P. Conner and E. Floyd, Princeton Univ., Preprint 1–21, 1963.
[8] J. Milnor and M. Kervaire, Proc. Internat. Congr. Math. 1958, Cambridge Univ. Press, New York, 1960, p. 454. MR 22 #12531.
[9] J. Milnor, Amer. J. Math. **82** (1960), 505. MR 22 #9975.
[10] S. P. Novikov, Dokl. Akad. Nauk SSSR **132** (1960), 1031 = Soviet Math. Dokl. **1** (1960), 717. MR 22 #12545.
[11] S. P. Novikov, Mat. Sb. **57** (99) (1962), 407. MR 28 #615.

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