

## ON EMBEDDING SIMPLY-CONNECTED MANIFOLDS IN EUCLIDEAN SPACE

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We shall consider compact smooth manifolds and smooth mappings. As usual, a mapping  $f: M^n \rightarrow W^m$  will be called *regular* if its Jacobian has rank  $n$  at every point, and *completely regular* if the inverse image  $f^{-1}(w)$  of every  $w \in W^m$  contains no more than two points. We shall study the possibility of smoothly embedding  $M^n \subset E^{2n-1}$ . Our method generalizes Whitney's in [2], where he considers embeddings  $M^n \subset E^{2n}$ , and is based on an idea of Pontryagin's concerning homotopy groups of spheres [1].

**Theorem 1.** *Every simply-connected odd-dimensional  $M^n$  ( $n > 6$ ) can be smoothly embedded in  $E^{2n-1}$ .*

The proof depends on a series of lemmas on regular mappings in the large. By means of a familiar technique elaborated by Whitney, one easily proves

**Lemma 1.\*** *For every regular  $f: M^n \rightarrow E^{2n-k}$  with  $k < [\frac{n}{2}]$  there exists a regular  $g: M^n \rightarrow E^{2n-k}$ , close to  $f$  in the  $C^1$  topology, such that*

- 1) *The equation  $g(x) = g(y)$  defines a compact submanifold  $\tilde{M}_g^k \subset M^n \times M^n - \Delta(M^n)$ , where  $\Delta$  is the diagonal map;*
- 2) *The projection  $p: M^n \times M^n \rightarrow M^n$ , restricted to  $\tilde{M}_g^k$ , is a smooth mapping;*
- 3)  *$g$  is completely regular; the restriction of  $g$  to the singular submanifold  $M_g^k = p(\tilde{M}_g^k) \subset M^n$  is a two-sheeted covering.*

It follows from the lemma that the singular manifold  $M_g^k$  decomposes into some number  $s$  of special pairs of mutually homeomorphic connected components

$$\bigcup_{i=1}^s (M_{g,1}^{k,i} \cup M_{g,2}^{k,i})$$

such that  $g(M_{g,1}^{k,i}) = g(M_{g,2}^{k,i})$ , and some number  $t$  of connected manifolds  $\bigcup_{j=1}^t M_g^{k,j}$  on which  $g$  is a nontrivial 2-covering. Thus

$$M_g^k = \left( \bigcup_{j=1}^t M_g^{k,j} \right) \cup \left( \bigcup_{i=1}^s (M_{g,1}^{k,i} \cup M_{g,2}^{k,i}) \right).$$

**Definition.**  $M^n$  will be called *k-parallelizable* if an  $\epsilon$ -neighborhood  $U_\epsilon^{(k)}$  of the  $k$ -skeleton of a differentiable triangulation of  $M^n$  is parallelizable, for some sufficiently small  $\epsilon$ .

It is obvious that for  $n > 2k + 2$  our definition does not depend on the triangulation, and that a  $k$ -connected manifold is  $k$ -parallelizable. For  $k = 1$ , our definition gives simply orientability. Note also that a  $k$ -parallelizable manifold is  $(k - 1)$ -parallelizable. Let us now take  $n \geq 2k + 3$ .

**Lemma 2.** *If  $M^n$  is  $k$ -parallelizable, then the singular submanifold  $M_g^k$  has trivial normal bundle in  $M^n$  and is a  $\pi$ -manifold (i.e., in an embedding  $M_g^k \subset E^m$  the normal bundle is trivial if  $m \geq 2k + 3$ ).*

The proof is based on the fact that  $M^n \times M^n - \Delta(M^n)$  is also  $k$ -parallelizable, while the normal

\*For  $k = 1$  this lemma is contained in Whitney's work, for example, in [3].

bundle of a submanifold of dimension  $\leq k$ , contained in a  $k$ -parallelizable manifold of greater dimension, is constructed exactly as in a euclidean space of the same dimension.

Now suppose  $n$  even,  $k = 1$ ,  $M^n$  orientable.

**Lemma 3.** *The singular manifold  $M_g^1 \subset M^n$  consists only of singular pairs of circles.*

Suppose on the contrary that  $M_g^1$  contains a circle  $S_g^1 \subset M_g^1$  on which  $g$  is a connected 2-covering. Obviously  $g(S_g^1) = S^1 \subset E^{2n-1}$ . Choose a system  $(W_1, \dots, W_{n-1})$  of independent vector fields tangent to  $M^n$  and transversal to  $S_g^1 \subset M^n$ .

Then, roughly speaking, there results a decomposition of the normal bundle of  $g(S_g^1) \subset E^{2n-1}$  into a sum of 2-plane bundles  $\mu_i^{(2)}$ ,  $i = 1, \dots, n-1$ , generated by the vectors  $W_i$ . Each of the bundles  $\mu_i^{(2)}$  is transversal to the circle  $g(S_g^1)$  and has transition matrix  $A_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , that is, is nonorientable. The Whitney sum of an odd number of such bundles is also nonorientable, therefore nontrivial. But since  $n-1$  is odd, while the normal bundle of a circle in euclidean space must be trivial, we obtain a contradiction, and the lemma is proved.

More generally, when  $k = 1$ , we have

**Lemma 4.** *The number  $t(g)$  of connected singular coverings is always even. The map  $g$  is regularly homotopic to a map  $g_1$  having no singular nontrivial 2-coverings.*

The proof of this lemma is of rather different character, and is based on studying the projection  $M^n \xrightarrow{\tilde{g}} E^{2n} \xrightarrow{\pi} E^{2n-1}$  where  $g = \pi \circ \tilde{g}$  and  $\tilde{g}$  is completely regular. (We can always reach this situation by means of a small deformation of the mapping in  $E^{2n}$ , and a projection of this small deformation into  $E^{2n-1}$ , which obviously preserves the properties mentioned in Lemma 1.)

The behavior of this projection is further described in the following (trivial) lemmas:

**Lemma 4a.** *If  $\tilde{g}: M^n \rightarrow E^{2n}$  and  $\pi\tilde{g}: M^n \rightarrow E^{2n-1}$  are regular, then  $\tilde{g}$  is regularly homotopic to an embedding and has an even number of pairs of singular points.*

**Lemma 4b.** *A connected singular covering for  $g$  can arise under the projection only from an odd number of pairs of singular points of  $\tilde{g}$ .*

*A singular pair can arise in the projection only from an even number of pairs of singular points of  $\tilde{g}$ .*

**Lemma 4c.** *There exists a regular homotopy  $\tilde{g}_t$  of  $\tilde{g} = \tilde{g}_0$  such that*

- 1)  $\tilde{g}_t$  and  $\pi\tilde{g}_t$  are regular for  $t \leq 1$ , and completely regular for  $t = 1$ ;  $\pi\tilde{g}_1$  satisfies Lemma 1;
- 2) coverings of the map  $\pi\tilde{g}_1$  can arise under the projection from a pair of singular points, while singular pairs of circles can come from nothing.

From now on we consider only mappings  $g: M^n \rightarrow E^{2n-1}$  which have no connected singular coverings. Suppose also that  $\pi_1(M^n) = 0$ . Following Pontryagin [1], we define an invariant of a singular pair and an invariant of the mapping  $g$ .

**Definition of the invariant of a singular pair.** Let  $S_1^1, S_2^1 \subset M^n$ ,  $g(S_1^1) = g(S_2^1)$ . Consider a pair of discs  $\sigma_1^2, \sigma_2^2 \subset M^n$  such that  $\sigma_1^2 \cap \sigma_2^2 = \emptyset$  and  $\partial\sigma_1^2 = S_1^1$ ,  $\partial\sigma_2^2 = S_2^1$ . We define a system of vector fields  $W_j^{(i)}$ ,  $i = 1, 2$ ;  $j = 1, \dots, n-2$  on  $M^n$ , orthogonal to the  $\sigma_i^2$ . We put  $W_{n-1}^{(i)} = \partial\sigma_i^2 / \partial t$  where  $t$  denotes the radii of the films (i.e., these are transversal to the set  $S_i^1$  and to the  $W_j^{(i)}$ ,  $j \leq n-2$ ). We obtain vectors  $\dot{g}(W_j^{(i)}) = V_{j+(i-1)}$ , transversal to  $g(S_i^1)$  and independent. These correspond to an element  $\alpha \in \pi_1(GL(2n-2)) = Z_2$ .

**Lemma 5.** *If the generator of  $H^n(M^n, Z_2)$  has the form  $Sq^2(x)$ ,  $x \in H^{n-2}(M^n, Z_2)$ , then the discs*

$\sigma_i^2$  and the fields  $\mathbb{W}_j^{(i)}$  can be chosen that  $\alpha = 0$ .

In case  $H^n(M^n, Z_2)/\text{Im } Sq^2 = Z_2$  the invariant  $\alpha$  of the singular pair does not depend on the choice of the discs  $\sigma_i^2$ . In this case the sum  $\sum_k \alpha_k$  of the invariants of all the singular pairs  $S_k = (S_{g,1}^{1,k} \cup S_{g,2}^{1,k})$  can be considered as an invariant of  $g: M^n \rightarrow E^{2n-1}$  at least when the latter has no connected singular coverings.

**Lemma 6.** *If  $M^n$  is simply-connected and  $n = 4l + 3$ , then the invariant  $\sum_k \alpha_k$  vanishes for any regular  $g: M^n \rightarrow E^{2n-1}$  with the properties of Lemma 1.\**

This lemma is an important step, and its proof, which is direct and geometrical, is rather complicated. However, it also follows easily from recent work by Hirsch on regular mappings [4].

Let  $S_1, S_2$  be two singular pairs of the mapping  $g: M^n \rightarrow E^{2n-1}$  such that  $\alpha(S_1) = \alpha(S_2)$ .

**Lemma 7.** *There exists a regular homotopy  $g_i$  of  $g = g_0$  such that  $g_1$  satisfies Lemma 1 and has two fewer singular pairs than  $g = g_0$ .*

The proof generalizes a familiar proof of Whitney's [2] for pairs of singular points. We attach rings  $B_1 = S_1^1 \times I$  and  $B_2 = S_2^1 \times I$  to  $M^n$  in such a way that  $S_i^1 \times \epsilon$  form the pair  $S_1$  and  $S_i^1 \times (1 - \epsilon)$  form  $S_2$ . We can arrange  $B_1 \cap B_2 = \emptyset$ . On these rings we take vector fields  $\mathbb{W}_j^{(i)}$ ,  $i = 1, 2$ ;  $j = 1, \dots, n - 2$  extending those on the discs which induce the invariants  $\alpha(S_i)$ . We can easily ensure that the frames  $(\tau_i, \dot{g}(\mathbb{W}_j^{(1)}), \dot{g}(\mathbb{W}_j^{(2)}))$  (where the  $\tau_i$  are the fields tangent to the  $g(S_i)$ ) define opposite orientations, for  $i = 1, 2$ . Now we attach a "Whitney cell"  $\psi: \sigma^2 \times S^1 \rightarrow E^{2n-1}$  such that

$$\psi(\sigma^2 \times S^1) \cap g(M^n) = g(B_1) \cup g(B_2);$$

we must also choose  $\psi$  to satisfy certain compatibility conditions on the boundaries. Now, since the invariants of  $S_1$  and  $S_2$  coincide, we can pick in a small neighborhood  $U(\psi(\sigma^2 \times S^1))$  a suitable system of coordinates, one of which is the coordinate on the circle, two others are the standard 2-frames on  $\sigma^2$ , and the remainder satisfy our boundary conditions. With these coordinates in  $U(\psi(\sigma^2 \times S^1))$ , we can perform Whitney's deformation, for a constant circle-coordinate.

Repeating this construction and applying Lemma 6, we obtain finally a map  $g_S: M^n \rightarrow E^{2n-1}$  which has singular pairs with zero invariant only.

With  $g_S$  we may now proceed in either of two ways: following [2], we may attach complementary pairs with zero invariants, and apply Lemma 7, or we may simply carry out a direct separation of a pair with zero invariants.\*\* In either case we arrive at an embedding. The theorem thus follows from the preceding lemmas.

Note that the lemmas imply the following conditional

**Theorem 2.** *Suppose  $n = 2l$ ,  $n \geq 6$ ,  $\pi_1(M^n) = 0$ . There is an embedding  $M^n \subset E^{2n-1}$  if and only if there is an immersion  $M^n \rightarrow E^{2n-2}$ . (See [4].)\*\*\**

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\*If  $n = 4l + 1$  one can assert the existence of a  $g: M^n \rightarrow E^{2n-1}$  with zero invariant, since there exists an immersion  $M^n \rightarrow E^{2n-2}$  [4].

\*\*A method for separating a singular pair with zero invariant has been indicated to me by D. B. Fuks, who kindly took an interest in the present article.

\*\*\*Closer study shows that for  $n = 4l + 2$ , the invariant  $\alpha$  is a homotopy invariant of  $M^n$ : it does not depend on the immersion  $g: M^n \rightarrow E^{2n-1}$  when  $n \not\equiv 1 \pmod{4}$ .

