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Topology of the Generic Hamiltonian Foliations on the Riemann Surfaces

Abstract. Topology of the Generic Hamiltonian Dynamical Systems on the Riemann Surfaces given by the real part of the generic holomorphic 1-forms, is studied. Our approach is based on the notion of Transversal Canonical Basis of Cycles (TCB). This approach allows us to present a convenient combinatorial model of the whole topology of the flow, especially effective for g=2. A maximal abelian covering over the Riemann Surface is needed here. The complete combinatorial model of the flow is constructed. It consists of the Plane Diagram and g straight line flows in the 2-tori "with obstacles". The Fundamental Semigroup of positive closed paths trasversal to foliation is studied. This work contains an improved exposition of the results presented in the authors recent preprint [1] and new results calculating all TCB in the 2-torus with obstacle, in terms of Continued Fractions.

Introduction. The family of parallel straight lines in the Euclidean Plane R^2 gives after factorization by the lattice $Z^2 \subset R^2$ the standard straight line flow in the 2-torus T^2. It is a simplest ergodic system for the irrational direction. This system is Hamiltonian with multivalued Hamiltonian function H and standard canonically adjoint euclidean coordinates x, y (i.e. the 1-form dH is closed but not exact, and $x_t = H_y, y_t = -H_x$ where H_x, H_y are constant). Every smooth Hamiltonian system on the 2-torus without critical points with irrational "rotation number" is diffeomorphic to the straight line flow. Every C^2-smooth dynamical system on the 2-torus without critical points and with irrational rotation number is C^0-homeomorphic to the straight line flow according to the famous classical theorem. Various properties of the generic Hamiltonian Systems on the 2-torus were studied in [20], ergodic properties were found in [21].

Question: What is going on in the Riemann Surfaces with $g > 1$?

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We consider Hamiltonian Systems with multivalued Hamiltonian H (i.e. given by the nonexact form $dH = 0$). In many cases we ignore time dependence of the trajectories and discuss only the properties of foliation $dH = 0$ given by the closed 1-form. *The present author started to investigate such foliations in early 1980s as a part of the newborn Topology of The Closed 1-Forms. An important example was found in the Quantum Solid State Physics describing the motion of semiclassical electrons along the so-called Fermi Surface $M^2 \subset T^3$ for the single crystal normal metals and low temperature in the strong magnetic field (see [2]). The electron trajectories exactly coincide with connectivity components of the sections of Fermi Surface by the planes orthogonal to magnetic field. One might say that they are the levels of quasiperiodic function on the plane with 3 periods. An extensive study of that class was performed by the author's Moscow Topological Seminar since early 1980s and was continued in Maryland. Remarkable "topological complete integrability" and "topological resonance" properties in the nonstandard sense were discovered here, for the set of directions of magnetic field of the full measure on the 2-sphere S^2. These properties play a key role in the physical applications [3].*

Following class of generic Hamiltonian systems on the Riemann Surfaces was studied by many mathematicians (no applications outside of pure mathematics were found for them until now, unfortunately): Take any nonsingular compact Riemann Surface V, with genus g. Every holomorphic 1-form $\omega \in C^g$ defines a Hamiltonian system (foliation) $\Re$ on the manifold V:

$$\Re = \{\omega^R = 0\}$$

because $d(\omega^R) = 0$. Here $\omega = \omega^R + i\omega^I$. Most people considered a more general class of "foliations $\Re$ with invariant transversal measure" on Riemann Surfaces. Every holomorphic quadratic differential Ω defines such foliation $\Re$ by the formula $(\sqrt{\Omega})^R = 0$. This is a locally hamiltonian foliation (i.e. it admits a transversal measure) but non-orientable (it does not admit global time direction if $\Omega \neq \omega^2$). We do not consider nonorientable case. It does not define Hamiltonian Dynamical System (even locally it has different generic singularities).

The systems with transversal invariant measure were studied since early 1960s by the following method (see in the book [7]): Take any closed curve γ transversal to our foliation $\Re$. Assume that almost every nonsingular trajectory is dense. For every point $Q \in \gamma$ except finite number the trajectory started in the point Q returns to the curve γ first time at the new point P (in positive direction of time). We define a "Poincare map" $Q \to P$. This map preserves transversal measure equal to the restriction of the form dH on the curve γ. **It is the Energy Conservation Law** for Hamiltonian System. So our transversal closed curve γ is divided into $k = k_\gamma$ intervals $\gamma = I_1 + I_2 + \ldots + I_k$. The Poincare map looks as a permutation of k intervals on the circle; this map is ill-defined in the finite number of points only. The time t_Q varies continuously in the interior of each interval $Q \in I_j$.

No doubt, the use of closed transversal curves is extremely productive. At the same time, we are not satisfied by this approach; Following questions can be naturally asked:

1.This method essentially ignores time/length and homology/homotopy classes of trajectories starting and ending in γ. Our goal is to present some sort of global topological description of the flow (or foliation $\Re$) on the algebraic curve V similar to the case of genus 1 as much as possible.

2.There are many different closed transversal curves in the foliation $\Re$. How this picture depends on the choice of tranversal curve γ? Nobody classified them yet as far as I know. Indeed, in the theory of codimension 1 foliations developed by the present author in 1960s (see[4]) several algebraic structures were defined for the closed transversal curves:

Consider all closed positively (negatively) oriented transversal curves starting and ending in the point $Q \in V$. We can multiply them. Transversal homotopy classes of such curves generate **A Transversal Fundamental Semigroup** $\pi_1^+(\Re, Q)$ and its natural homomorphism into the fundamental group (even, into the fundamental group of the unit tangent S^1-bundle $L(V)$

$$\psi^\pm : \pi_1^\pm(\Re, Q) \to \pi_1(L(V), Q) \to \pi_1(V, Q)$$

The set of all closed transversal curves naturally maps into the set of conjugacy classes in fundamental group. We denote it also by ψ. The Transversal Semigroups might depend of the leaf where the initial point Q is chosen.

How to calculate these invariants for the Hamiltonian Systems on the algebraic curves?

The simplest fundamental properties of these foliations are following:

Property 1. They have only saddle type critical points (no centers). In the generic case such foliation has exactly 2g-2 nondegenerate saddles.

Property 2. Every nonempty closed transversal curve γ is non-homologous to zero. Every period of the form dH is positive $\oint_\gamma dH > 0$ for the positive transversal curve. So the composition

$$\pi_1^+(\Re, Q) \to \pi_1(V, Q) \to H_1(V, Z) \to R$$

does not map any element into zero. Every periodic trajectory γ is such that $\oint_\gamma dH = 0$. Therefore for the generic case every periodic trajectory is homologous to zero. We call systems without periodic trajectories **Irreducible**.

Definitions

a.We say that foliation on the Riemann surface V belongs to the class T if there exist a **Transversal Canonical Basis** of curves

$$a_1, b_1, \ldots, a_g, b_g$$

such that all these curves are non-selfintersecting (simple), transversal to foliation, and the curves a_j and b_j transversally cross each other exactly in one point. Other curves do not cross each other.

b.We say that foliation belongs to the class T^0 if there exists a canonical basis such that all a-cycles $a_1, \ldots, a_g$ are simple, do not cross each other and are transversal to foliation. We say that foliation belongs to the **Mixed Class** of the Type $T^k, k = 0, 1, \ldots, g$, if there exists an incomplete canonical basis $a_j, b_q, j \leq g, q \leq k$, such that all these cycles are transversal to foliation, simple, and only pairs crossing each other are a_j and b_j for $j = 1, 2, \ldots, k$. All intersections are transversal and consist of one point each. The complete case is $T = T^g$

Remark 1 *A number of people including Katok, Hasselblatt, Hubbard, Mazur, Veech, Zorich, Konzevich, McMullen, Smillie, Eskin and others wrote a lot of works related to study the ergodic properties of foliations with "transversal measure" on the Riemann Surfaces, and their total moduli space (see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]). Many researchers investigated recently closed geodesics of the flat Riemannian Metric* $ds^2 = \omega\bar{\omega}$ *singular in the critical points of the holomorphic 1-form* ω. *These geodesics consists of all trajectories of the systems* $(\exp\{i\phi\}\omega)^R = 0$ *on* V. *Closed geodesics appear for the special number of angles* ϕ_n *only. Therefore we don't have them describing the generic Hamiltonian systems. Beautiful analogs of the Poincare 4-gons associated with flat metrics with singularities were invented and used. Our intention is to describe topology of the generic foliations given by the equation* $\omega^R = 0$ *for the holomorphic form* ω. *As A.Zorich* *pointed out to us, some features were recently observed which may illustrate our ideas (see [7, 19]). Our key idea of transversal canonical basis did not appeared before.*

Section 1.
Examples: The Real Hyperelliptic Curves.

Let us show the examples of foliations in the classes T, T^k. Consider any real hyperelliptic curve of the form

$$w^2 = R_{2g+2}(z) = \prod_j (z - z_j), z_j \neq z_l$$

where all roots z_j are real and ordered naturally $z_1 < z_2 < \ldots < z_{2g+2}$.

Lemma 1 *For every complex non-real and non-imaginary number $u+iv, u \neq 0, v \neq 0$, generic polynomial $P_{g-1}(z)$ and number ϵ small enough, the Hamiltonian system defined by the closed harmonic 1-form below belongs to the class T^0. All cycles $a_1, \ldots, a_g, c_1, \ldots, c_{g+1}$ are transversal to foliation and cross each other transversally,*

$$\omega = \frac{(u + iv + \epsilon P_{g-1}(z))dz}{\sqrt{R_{2g+2}(z)}}, \omega^R = 0$$

For $g = 1, 2$ this foliation belongs to the class T. For $g > 2$ it belongs to the class T^2.

Proof. The polynomial $R = R_{2g+2}(z)$ is real on the cycles $a_j = p^{-1}[z_{2j} z_{2j+1}], j = 1, ..., g$ and purely imaginary on the cycles c_q located on the real line x immediately before and after them $c_q = p^{-1}[z_{2q-1} z_{2q}], q = 1, 2, \ldots, g + 1$.

For $g = 1$ we take a canonical basis $a_1, b_1 = c_1$. For $g = 2$ we take a canonical basis $a_1, a_2, b_1 = c_1, b_2 = c_3$. We shall see that all these cycles are transversal to foliation.

For $\epsilon = 0$ we have

$$\omega^R = (udx - vdy)/R^{1/2} = 0, y = 0, x \in a_j$$

$$\omega^R = (vdx + udy)/iR^{1/2} = 0, y = 0, x \in c_j$$

In both cases we have for the second component $dy \neq 0$ for the direction of Hamiltonian system. So the transversality holds for all points on the cycles except (maybe) of the branching points z_j. The cycles c_l, a_k are orthogonal to each other in the crossing points (i.e. in the branching points). We need to check now that in all branching points z_j the angle between trajectory and both cycles a, c is never equal to $\pm\pi/2$ because they meet each other with this angle exactly. After substitution $w'^2 = z - z_j$ we have

$$\omega = (u + iv)2w'dw'/w'F_j^{1/2} = 2(u + iv)dw'/F_j^{1/2}$$

where $F_j = \prod_{l \neq j}(z - z_l)$. The real function $F_j(x)$ for real $x \in R$ does dot change sign passing through the point $x = z_j \in R$: $F_j(z_j) \neq 0$. For $w' = f + ig$ we have $\omega^R = 2(udf - vdg)/F_j^{1/2}$ if $F_j(z_j) > 0$ or $\omega^R = 2i(udg + vdf)/F^{1/2}$ if $F_j(z_j) < 0$. We have $x = f^2 - g^2, y = 2fg$. The condition $y = 0$ implies the equation $fg = 0$, i.e. the union of the equations $f = 0$ and $g = 0$. It is exactly an orthogonal crossing of our cycles. In both cases our system $\omega^R = 0$ implies transversality of trajectories to both cycles $df = 0$ and $dg = 0$ locally.

We can choose now the a-cycles as any subset of g cycles out of a, c not crossing each other, for example $a_1, \ldots, a_g$. All of them are transversal to foliation. Therefore we are coming to the class T^0. Now we choose following two b-cycles: $b_1 = c_1$ and $b_g = c_{g+1}$. According to our arguments, this choice leads to the statement that our foliation belongs to the class T for $g = 1, 2$ and to the class T^2 for all $g \geq 2$.

Lemma is proved now because small ϵ-perturbation cannot destroy transversality along the finite family of compact cycles.

We choose now the generic perturbation such that **all critical points became nondegenerate, and all saddle connections and periodic trajectories nonhomologous to zero disappear.**

Let now $R = R_{2g+2} = \prod_{j=1}^{2g+2}(z - z_j)$ is a polynomial of even degree as above with real simple roots $z_j \in R$, and $\omega = P_{g-1}(z)dz/R^{1/2}$ is a generic holomorphic 1-form. Let $P = u + iv$ where u, v are real polynomials in the variables x, y. Only the zeroes $v = 0$ in the segments $[z_{2q-1}z_{2q}], q = 1, ..., g+1$, and the zeroes $u = 0$ in the segments $[z_{2j}z_{2j+1}], j = 1, ..., g$, are important now. We assume that $u(z_k) \neq 0$ and $v(k) \neq 0$ for all $k = 1, \ldots, 2g + 2$.

Lemma 2 *Remove all open segments containing the important zeroes, i.e. the open segments* $[z_{2q-1}z_{2q}]$ *containing the zeroes* $v = 0$*, and all open segments* $[z_{2j}z_{2j+1}]$ *containing the zeroes* $u = 0$*. If remaining segments are* *enough for the construction of the half-basis $a_1, \dots, a_g$ (i.e. there exists at least g disjoint closed segments between them), then our foliation belongs to the class T^0. In particular, it is always true for $g = 2$ where u and v are the linear functions (and have no more than one real zero each). Let there is no real zeroes $v = 0$ in the segments $[z_1 z_2], [z_5 z_6]$, and no real zeroes $u = 0$ in the segments $[z_2 z_3], [z_4 z_5]$ for $g = 2$; Then this foliation belongs to the class T. Let $g > 2$, all real zeroes $v = 0$ belong to the open intervals*

$$(-\infty, z_1), (z_3 z_4), (z_5 z_6), \dots, (z_{2g-1} z_{2g}), (z_{2g+2}, +\infty)$$

and all real zeroes $u = 0$ are located in the open segments

$$(-\infty, z_2), (z_3 z_4), ..., (z_{2g-1} z_{2g}, (z_{2g+2}, +\infty)$$

In this case foliation $\omega^R = 0$ belongs to the class T^2.

Proof is exactly the same as above.

For every foliation of the class T^0 we cut Riemann Surface V along the transversal curves a_j. The remaining manifold $\tilde{V}$ has a boundary $\partial \tilde{V} = \bigcup_j a_j^{\pm}$ where $a_j^{\pm} = S^1$; the foliation enters it from inside for (a_j^-) and from outside for (a_j^+). The domain $\tilde{V} = D_*^2 \subset R^2$ is a disc with $2g - 1$ holes removed from inside: the external boundary is exactly a_1^+. The boundaries of inner holes are $a_1^-, a_j^{\pm}, j > 1$. All boundaries are transversal to our system (see Fig 1) with periods

$$\oint_{a_j} \omega^R = |a_j| > 0, |a_1| \geq |a_2| \geq \dots \geq |a_g| > 0$$

The system does not have critical points except $2g - 2$ nondegenerate saddles. Every trajectory starts at the **in-boundary** $\bigcup a_j^+$ and ends at the **out-boundary** $\bigcup a_j^-$ (see Fig 4).

Consider a maximal abelian Z^{2g}-covering $V' \to V$ with basic shifts $a_j, b_j : V' \to V'$ for every class T foliation. Its fundamental domain can be obtained cutting V along the Transversal Canonical Basis. The connected pieces A_j of the boundary exactly represent free abelian groups Z_j^2 generated by the shifts a_j, b_j. Every boundary component $A_j = A_j^+ \bigcup A_j^-$ looks like the standard boundary square on the plane (see Fig 2). The foliation near the boundary A_j looks like standard straight line flow at the space R_j^2 with lattice Z_j^2. Our fundamental domain $\bar{V} \subset V', \partial \bar{V} = \bigcup A_j$, is restricted to the inner part of

2-parallelogram in every such plane R^2_j. Topologically this domain $\bar{V}$ is a 2-sphere with g holes (squares) , with boundaries A_j.

Algebraic Geometry constructs V' analytically (the Abel Map):

$$A = (A^1, \ldots, A^g) : V \to V' \subset C^g, A^j(Q) = \int_P^Q \omega_j, j = 1, \ldots, g$$

Here ω_k is a normalized basis of holomorphic forms:

$$\oint_{a_j} \omega_k = \delta_{jk}, \oint_{b_j} \omega_k = b_{jk} = b_{kj}$$

The form $\omega = \sum u_k \omega_k$ is generic here. It defines a one-valued function F:

$$F : V' \to C; F(Q') = A^\omega(Q') = \sum u_k A^k(Q')$$

where Q' corresponds to Q under the projection $V' \to V$. The basic shifts are $a_j, b_j : V' \to V'$:

$$F(a_j(Q')) = F(Q') + u_j, F(b_j(Q')) = F(Q') + \sum_k u_k b_{kj}$$

The levels $F^R = const$ are exactly the leaves of our foliation $\omega^R = 0$ on the covering V'. The map $\pi_1(V) \to H_1(V, Z) = Z^{2g} \to R$ is defined by the correspondence:

$$a_j \to u_k^R, b_j \to (\sum_k u_k b_{kj})^R$$

The map $\psi : \pi_1^+(\Re) \to \pi_1(V) \to R^+$ of the positive transversal semigroup (above) certainly belongs to the $(n, m) \in Z_+^{2g}(\omega^R)$ if

$$\omega^R(\sum_{m_k, n_l} m_k a_k + n_l b_l) > 0$$

Section 2. Some General Statements.

Let us introduce the class of **Positive Almost Transversal Curves**. They are piecewise smooth and consist of the smooth pieces of the First and Second Type: The First Type pieces are positive and transversal to foliation. Every Second Type piece belongs to one trajectory.

Every almost transversal curve can be approximated by the smooth transversal curve with the same endpoints (if there are any).

In many cases below we construct closed almost transversal curves and say without further comments that we constructed smooth transversal curve.

The **Plane Diagram** of foliation of the type T^k with incomplete transversal canonical basis is **Topological Type** of foliation on the plane domain $\bar{V}$ obtained from V by the cuts along this basis (see Fig 10).

Theorem 1 *Every generic foliation given by the holomorphic 1-form $\omega^R = 0$ on the algebraic curve of genus 2 admits Transversal Canonical Basis*

Lemma 3 *Every generic foliation $\omega^R = 0$ belongs to the class T^0 for genus equal to 2*

Proof. Take any trajectory such that its limiting set in both directions contains at least one nonsingular point. Nearby of the limiting nonsingular point our trajectory appears infinite number of times. Take two nearest returns and join them by the small transversal segment. This closed curve consists of the piece of trajectory and small transversal segment. It is an almost transversal curve. Therefore it can be approximated by the closed non-selfintersecting smooth transversal curve. Take this curve as a cycle a_1. Now cut V along this curve and get the surface $\tilde{V}$ with 2 boundaries $\partial\tilde{V} = a_1^+ \bigcup a_1^-$. Take any trajectory started at the cycle a_1^+ and ended at a_1^-. Such trajectory certainly exists. Join the ends of this trajectory on the cycle a_1^- by the positive transversal segment along the cycle a_1 in V. We get a transversal non-selfintersecting cycle b_1, crossing a_1 transversally in one point. Cut V along the pair a_1, b_1. We get a square $\partial D^2 = a_1 b_1^{-1} a_1^{-1} b_1 \subset R^2$ with 1-handle attached to the disk D^2 inside. Choose such notations for cycles that our foliation enters this square along the piece $A_1^+ = a_1 b_1^{-1} \subset \partial D^2$, and leaves it along the piece $A_1^- = a_1^{-1} b_1$ (see Fig 3). Nearby of the angle these pieces $R^{\pm}$ are attached to each other. The trajectories here spend small time inside of the square: they enter and leave it cutting the angle (see Fig 3). Move inside from the ends of the segment A_1^+. Finally we find 2 points $x_{1,1}^+, x_{2,1}^+ \in A_1^+$ where this picture ends (because topology is nontrivial inside of the square). These points are the ends of the separatrices of two different saddles $x_{1,1}^+ \neq x_{2,1}^+$. Take any point y between $x_{1,1}^+$ and $x_{2,}^+$ (nearby of the left one $x_{1,1}^+$). The trajectory passing through y crosses A_1^- in the point y' (see Fig 3). Join y' with point $\tilde{y}$ equivalent to y by the transversal segment σ along the curve A_1^- in positive direction. This is a closed curve c_1 almost transversal to our foliation. If initial point is located on the cycle a_1, the curve c_1 does not cross the cycle b_1. We take cycles b_1, c_1 as a basis of the transversal a-cycles. If initial point is located on the cycle b_1, the curve c_1 does not cross a_1. In this case we take a_1, c_1 as a basis of the transversal a-cycles. The cycle c_1 is non-homologous to b_1 in the first case. Therefore it is a right basis of the a-cycles which is transversal. The second case is completely analogous. Our lemma is proved.

Lemma 4 *For every generic foliation $\omega^R = 0$ of the class T^0 on the algebraic curve of genus 2, the transversal basis of a-cycles can be extended to the full Transversal Canonical Basis a, b, so every T^0-class foliation belongs to the class T.*

Proof. Cut the Riemann Surface V along the transversal cycles a_1, a_2. Assuming that $\oint_{a_1} \omega^R = |a_1| \geq |a_2| = \oint_{a_2} \omega^R$, we realize this domain as a plane domain D^2_* as above (see Fig 4 and the previous section). Here an external boundary $\partial_{ext} D^2_* = a_1^+$ is taken as our maximal cycle. There are only two different topological types of the plane diagrams (see Fig 4, a and 4,b): The **first case** is characterized by the property that for each saddle all its separatrices end up in the four different components of boundary $a_1^{\pm}, a_2^{\pm}$. We have following matrix of the trajectory connections of the type $(k,l) : a_k^+ \to a_l^-$ for the in- and out-cycles and their transversal measures:

$a_1^+ \to a_1^-$ (with measure a), $a_k^+ \to a_l^-, k \neq l$ (with measure b for $(k,l) = (1,2), (l,k) = (2,1)$), and $a_2^+ \to a_2^-$ with measure c. All measures here are positive. We have for the measures of cycles: $|a_1| = a + b, |a_2| = c + b$. This topological type does not have any degeneracy for $a = c$.

The diagonal trajectory connections of the type (l,l) generate the transversal b-cycles closing them by the transversal pieces along the end-cycles in the positive direction.

In the second case we have following matrix of trajectory connections:

$a_1^+ \to a_1^-$ with measure $a > 0$, $a_1^+ \to a_2^+$ with measure $b > 0$, $a_2^+ \to a_1^-$ with the same measure $b > 0$. We have $b = |a_2|, a + b = |a_1|$. So we do have the trajectory-connection $a_1^+ \to a_1^-$, but we do not have the second one, of the type $(2,2)$. However, we may connect a_2^+ with a_2^- by the almost transversal curve as it is shown in the Fig 4,b), black line γ. So we construct the cycles b_1, b_2 as the transversal curves crossing the cycles a_1, a_2 only. Our lemma is proved.

Therefore the theorem is also proved.

Example: The case $g = 3$. A family of concrete foliations of the class T^2 were demonstrated above for the real nonsingular algebraic curves

$$w^2 = \prod(z - z_1) \dots (z - z_8) = R(z), z_j \in R$$

with the cycles

$$a_1 = [z_2 z_3], a_2 = [z_7 z_8], a_3 = [z_4 z_5], b_1 = [z_1 z_2], b_2 = [z_6 z_7]$$

and $\omega = P_2(z)dz/R(z)^{1/2}$ The polynomial $P_2(z) = u + iv$ should be chosen such that $u \neq 0$ in the segments $[z_2 z_3], [z_4 z_5], [z_6 z_7]$, and $v \neq 0$ in the segments $[z_1 z_2], [z_7 z_8]$.

Question: How to extend this basis to the Transversal Canonical Basis?

After cutting the Riemann Surface along the cycles a_j, b_q we realize it as a plane domain with following components of the boundary:

The external boundary

$$A_1 = (a_1 b_1^{-1}) \bigcup (a_1^{-1} b_1) = A_1^+ \bigcup A_1^-$$

The internal boundary

$$A_2 = (a_2 b_2^{-1}) \bigcup (a_2^{-1} b_2) = A_2^+ \bigcup A_2^-$$

The interim boundary $a_3^+ \bigcup a_3^-$ inside.

Our notations are such that trajectories enter this domain through the pieces with sign $+$ and leave it through the pieces with the sign $-$. We make a numeration such that:

$$2|A_1^{\pm}| = A_1 > A_2 = 2|A_2^{\pm}|, |a_3| = |a_3^{\pm}| = a$$

Here A_k means also the measure of this boundary component. Nearby of the ends of the segments A_l^+ the trajectories enter our domain and almost immediately leave it through the piece A_l^-. Therefore, there exist the first points in $A_l^{\pm}$ where this picture ends. These are the endpoints $x_{1,j}^{\pm}, x_{2,j}^{\pm} \in A_j^{\pm}$ of the pair of separatrices of saddles (see Fig 5).

A complete list of topologically different types of the Plane Diagrams in the class T^2 for the genus $g = 3$ can be presented. It shows that there is only one type such that we cannot extend the incomplete transversal basis to the complete transversal basis (see Fig 6)): no closed transversal curve exists in this case crossing the cycle a_3 and not crossing other curves a_1, b_1, a_2, b_2 (i.e. joining a^+ and a^- on the plane diagram). In all other cases such transversal curve b_3 can be constructed.

Consider now this special case where the transversal incomplete basis cannot be extended (Fig 6).

Lemma 5 *There exists a reconstruction of this basis such that new basis can be extended to the complete transversal canonical basis.*

For the proof of this lemma, we construct a closed transversal curve γ such that it crosses the cycle $a = a_3$ in one point and crosses the segment A^+ leaving our plane diagram (see Fig 7). It enters A^- in the equivalent point, say, through the cycle b_1. We take a new incomplete transversal basis $a_2, b_2, a_3, \gamma, a_1$. If γ crosses $A_1^{\pm}$ through the cycle a_1, we replace a_1 by the cycle b_1 as a last cycle in the new incomplete basis of the type T^2. The proof follows from the plane diagram of the new incomplete basis (let us drop here these technical details).

Comparing these lemmas with the construction of special foliations in the previous section on the real Riemann Surface, we are coming to the following

Conclusion. For every real hyperelliptic Riemann Surface of the form $w^2 = \prod(z - z_1) \dots (z - z_8) = R(z), z_k \neq z_l \in R$, every generic form $\omega = P_2(z)dz/R(z)$ defines foliation $\omega^R = 0$ of the class T if real and imaginary parts of the polynomial $P = u + iv$ do not have zeroes on the cycles indicated above for genus $g = 3$.

Extending these arguments, the present author proved the existence of Transversal Canonical Basis for $g = 3$ (see Appendix in [1], version 1). Very soon G.Levitt informed us that he can prove the existence of TCB for all g (see Appendix in [1], versions 2 and 3). We present here this proof based on the result of [22].

Consider any C^1-smooth vector field on the compact smooth Riemann Surface M_g of the genus $g \geq 1$, with saddle (nondegenerate) critical points only. We introduce **the class G of vector fields** requiring that there is no saddle connections and no periodic trajectories for the vector fields in this class. Let a finite family of smooth disjoint

Theorem 2 *For every generic dynamical system of the class G there exist Transversal Canonical Basis*

Proof. The result of [22] is following: For every dynamical system of the class G there exist $3g-3$ non-selfintersecting and pairwise non-intersecting transversal cycles $A_i, B_i, C_i, i=1,...,g-1$, where $C_{g-1}=C_1$, with following properties: These cycle bound two sets of surfaces: $A_i \bigcup B_i \bigcup C_i = \partial P_i$ where P_i is a genus zero surface ("pants"); The trajectories enter pants P_i through the cycles A_i, B_i and leave it through the cycle C_i; There is exactly one saddle inside of P_i for each number i. The cycles A_i, B_i, C_{i-1} bound also another set of pants Q_i such that $\partial Q_i = A_i, B_i, C_{i-1}$. The trajectories enter Q_i through C_{i-1} and leave it through A_i and B_i. There is also exactly one saddle inside Q_i.

The construction of Transversal Canonical Basis based on that result follows: Define a_g as $a_g = C_1$. Choose a segment of trajectory γ starting and ending in C_1 assuming that it passed all this "necklace" through P_k, Q_k. For each i this segment meets either A_i or B_i. Let it meets B_i (it does not matter). We define a_i as $a_i = A_i, i=1,...,g-1$. Now we define b_g as a piece of trajectory γ properly closed around the cycle $C_1 = a_g$. This curve is almost transversal in our terminology (above), so its natural small approximation is transversal. We are going to construct the cycles b_i for $i=1,...,g-1$ in the union $P_i \bigcup Q_i$. Consider the saddle $q_i \in Q_i$. There are two separatrices leaving q_i and coming to A_i and B_i correspondingly. Continue them until they reach C_i through P_i. Denote these pieces of separatrices by $\gamma_{1,i}, \gamma_{2,i}$. Find the segment S_i on the cycle C_i not crossing the curve γ chosen above for the construction of the cycle b_g. The curve $\gamma_{1,i} S_i \gamma_{2,i}^{-1}$ is closed and transversal everywhere except the saddle point q_i. We approximate now the separatrices $\gamma_{1,i}$ and $\gamma_{2,i}^{-1}$ by the two pieces of nonseparatrix trajectories γ'_i, γ''_i starting nearby of the saddle q_i from one side of the curve $\gamma_{1,i}\gamma_{2,i}^{-1}$. Close this pair by the small transversal piece s_i. There are two possibilities here (two sides). We choose the side such that the orientation of the transversal piece s_i near the saddle q_i is agreed with the orientation of the segment S_i, so the whole curve $b_i = s_i \gamma'_i S_i \gamma''_i$ is closed and almost transversal. We define b_i as a proper small transversal approximation of that curve. So our theorem is proved because every cycle b_i crosses exactly one cycle a_i for $i = 1, 2, ..., g$.

Section 3. Topological Study of the Class T.

Let us describe here some simple general topological properties of the class T foliations After cutting the surface V along the transversal canonical basis a_j, b_j, we are coming to the fundamental domain $\bar{V}$ of the group Z^{2g} acting on the maximal abelian covering V'. There is exactly g boundary "squares" $\partial\bar{V} = \bigcup_j A_j$ where $A_j = a_j b_j^{-1} a_j^{-1} b_j = A_j^+ \bigcup A_j^-$. Every piece $A_j^{\pm}$ consists of exactly two basic cycles a_j, b_j attached to each other. These pieces are chosen such that trajectories enter A_j^+ from outside through $a_j b_j^{-1}$ and enter the fundamental domain $\bar{V}$. In the areas nearby of the ends the trajectories enter A_j^- from inside and leave fundamental domain. There is exactly $2g-2$ saddle points inside of $\bar{V}$. They are no saddles located on the selected transversal cycles a_j, b_j. Our foliation nearby of each boundary square A^+ looks exactly as a straight line flow. It means in particular that there exist two pairs of points $x_{1,j}^{\pm}, x_{2,j}^{\pm} \subset A_j^{\pm}$ which are the endpoints of separatrices in $\bar{V}$, nearest to the ends at the each side $A_j^{\pm}$ (see Fig 3 and 5). We call them **The Boundary Separatrices** belonging to **The Boundary Saddles** $S_{j,1}, S_{j,2}$ for the cycle A_j. We can see $2g$ of such saddles $S_{j,1}, S_{j,2}$ looking from the endpoints of $A_j^{\pm}$, but some of them are in fact the same. At least two of them should coincide. We call corresponding saddles **The "Double-boundary" saddles**.

Definitions.

1.The saddle point $S \in \bar{V}$ has a type $< jklm >$ if it has two incoming separatrices starting in A_j^+, A_l^+ and two outgoing separatrices ending in A_k^-, A_m^-. The indices are written here in the cyclic order corresponding to the orientation of $\bar{V}$. Any cyclic permutation of indices defines an equivalent type. we normally write indices starting from the incoming separatrix, $< jklm >$ or $< lmjk >$.

2. We call foliation **Minimal** if all saddle points in $\bar{V}$ are of the boundary saddles. In particular, their types are $< jjkl >$. We call foliation **Simple** if there are exactly two saddle points of the double-boundary types $jjkk$ and $jjll$ correspondingly. The index j we call **Selected**. We say that foliation has a rank equal to r, if there exist exactly r saddles of the types $< jklm >$ such that all four indices are non-boundary. In particular, we have $0 \leq r \leq g-2$. There is exactly t saddles of the double-boundary types where $t - r = 2$. There is also $2g - 2t$ other saddles of the types like $< jjkl >$ where only index j corresponds to the boundary separatrices. The extreme cases are

$r = 0, t = 2$ which we call **special** (above), and $r = g-2, t = g$ which we call **maximal**. In the maximal case there exists a maximal number of saddles of the nonboundary types, and all boundary type saddles form the pairs. One might say that for the maximal type every index is selected. **For maximal type the genus should be an even number** because the boundaries A^j form the cycles. Every cycle should contain even number of them, by the elementary orientation argument. The relation $2g - t + r = 2g - 2$ for the total number of saddles gives $t - r = 2$. For the case $g = 2$ we obviously have $r = 0$. For the case $g = 3$ the only possible case is $t = 2, r = 0$ (the simple foliations); the case $t = 3, r = 1$ cannot be realized for $g = 3$ because it is maximal in this case but the maximal case corresponds to the even genus only.

How to Build Hamiltonian Systems from the elementary pieces. Let us introduce following **Building Data** (see Fig 9):

I.**The Plane Diagram** consisting of the generic Hamiltonian System on the 2-sphere S^2 generated by the hamiltonian h with nondegenerate critical points only (centers and saddles) sitting on different levels (see fig 9a,9b,9c). Let minimum of h is located in the point 0, and maximum in ∞. There are $t-2$ other centers and $r = t-2$ saddles. Let exactly g oriented segments be given transversal to the system everywhere except centers $t_1, ..., t_g \subset S^2$ with measures $m_1, m_2, ..., m_g$ provided by hamiltonian h, such that:

a.They do not cross each other; the values of Hamiltonian in all their ends, centers and saddles are distinct except that exactly two of them meet each other in every center; They do not touch any saddle on the two-sphere.

b.Every cyclic and separatrix trajectory of the hamiltonian system on S^2 meets at least one of these segments.

We make cuts along these segments and define the sides $t_j^{\pm}$ where trajectories leave and enter them correspondingly.

II.**The Torical Data** consisting of the g tori $T_j^2, j = 1, ..., g$, with irrational straight line flows and selected oriented transversal segments $s_1, ..., s_g$ (one for each torus). Their transversal measures are equal to the same numbers $m_1, ..., m_g$. The Transversal Canonical Basis a_k, b_k in every torus $T_{m_k}^*$ minus the segment s_k, is selected. Here a_k are positive, b_k are negative, and $|a_k| + |b_k| > m_k$. We define the sides $s_k^{\pm}$ of the boundary ∂T_k^* as for the segments t_j.

Identifying the segments s_j^+ on the tori with t_j^- on the sphere S^2 and vice versa, we obtain a Riemann surface $V = M_g^2$ with Hamiltonian System (Foliation). Every end of the segment $t_j \subset S^2$ located on the trajectory $\gamma \subset S^2$, defines exactly one saddle of the boundary type $< jjkl >$. Here t_k, t_l are the segments where this piece of the trajectory γ ends on the 2-sphere S^2. Two pieces of γ divided by the segment t_j, provide a pair of non-boundary separatrices for the saddle on the Riemann Surface defined by this picture.

By construction, every center generates a double-boundary saddle of the type $< jjkk >$. So we have $t - r = 2$.

Let us perform following operations:

1.Cut our surface along the selected TCB. The boundary of the domain $\bar{V}$ is equal to the union $\bigcup_j (A_j^+ \bigcup A_j^-) = \partial\bar{V}$. Every component is presented as boundary of the Fundamental Parallelogram P_j associated with the 2-torus T_j^2. Our flow covers the boundary of P_j as a straight line flow: the trajectories enter through the pair of cuts $a_j b_j^{-1}$ and leave through the pair $a_j^{-1} b^j$ (see Fig 2).

2.Find for every 2-torus T_j^2 the pair of boundary saddles in P_j and join them by the pair of transversal segments $s_j^{\pm} \subset P_j$ along the parts $A_j^{\pm}$ (see Fig 5). We perform this operation in the fundamental parallelogram P_j. These segments should be chosen in such a way that outside of them in P_j near the one-skeleton, we have a straight line flows.

3.Cut our surface $V = M_g^2$ along the segments $s_j^+ \bigcup s_j^-$. It is divided now into the torical pieces T_j^2 and one piece S^2 whose boundary consists of the curves $\bigcup_j (t_j^+ \bigcup t_j^-)$ for S^2, and $s_j^+ \bigcup s_j^-$ for the tori T_j^2. After cutting the surface M_g^2 along the pieces $s_j^{\pm}$ we keep the notation $s_j^{\pm}$ for the curves in the tori, but for the part S^2 we denote them by $t_j^{\pm}$.

4.Glue t_j^+ with t_j^- for the sphere S_j^2, and s_j^+ with s_j^- for the tori T_j^2, preserving the transversal measure. The system on 2-sphere appears with g selected transversal segments t_j and 2-tori T_j^2 with straight line flows where the transversal pieces s_j with m_j are marked.

5.Near the double-boundary saddle this picture is topologically equivalent to the center (but this equivalence in non-smooth).

We can see that our construction preserves the measure-type invariants.

Therefore we are coming to the following

Theorem 3 *Every generic foliation given by the real part of holomorphic one-form, can be obtained by the measure-preserving gluing of the pieces* $(S^2, h, t_1, ...t_g)$ *and* $(T_1^2, s_1), ..., (T_g^2, s_g)$ *along the transversal segments* t_j *and*

s_j, as it was described above. For the genus $g = 2$ we can remove sphere S^2 from the description: Every generic foliation given by the real part of holomorphic one-form, can be obtained from the pair of tori (T_1^2, s_1) and (T_2^2, s_2) with different irrational straight line flows and transversal segments s_1, s_2 with transversal measure $m_1 = m_2 = m$

Example. The Topological Types of the Minimal Foliations.

For the Minimal Foliations above we have $t = 2, r = 0$. The hamiltonian system on the 2-sphere is trivial (see Fig 9,a): It can be realized by the rotations around the points 0 and ∞. There are no saddles on the sphere. For the simplest case $g = 2$ we have two segments t_1, t_2. Both of them join 0 and ∞. So they form a cycle of the length 2. The difference between the values of Hamiltonian $h(\infty) - h(0)$ is equal to $m_1 = m_2$. All possible pictures of the transversal segments can be easily classified here for every genus g (see Fig 9,a and b).

There exist following types of topological configurations only:

a.The Plane Diagram has exactly one "cycle" t_1, t_2 of the length two (like for $g = 2$) and $g - 2$ disjoint segments $t_j, j \geq 3$; This type is available for all $g \geq 2$ (see fig 9a for $g = 2$).

b.The Plane Diagram has two pairs t_1, t_2 and t_3, t_4 where the members of each pair meet each other either in the center 0 or in ∞. Other $g - 4$ segments $t_j, j \geq 5$ are disjoint. In the second case we have $g \geq 4$.

c.The Plane Diagram has exactly one connected set consisting of 3 segments $t_1 t_2 t_3$ passing through $0, \infty$. Other $g - 3$ segments $t_j, j \geq 4$ are disjoint. For this type we have $g \geq 3$.

Theorem 4 *For $g = 2, 3$ every class T foliation is simple. The Maximal type exists only for even genus $g \geq 4$*

Proof. For $g = 2$ it was already established above: all generic (irreducible) hamiltonian foliations are simple. Both indices $k = 1, 2$ are selected. Consider now the case $g = 3$. Our foliation can be either simple ($t = 2, r = 0$) or maximal ($t = 3, r = 1$) in this case. We have $t = 3, 2g - 2t = 0$ for the maximal case. If it is so, every boundary saddle should be paired with some other. So there is a cyclic sequence of boundary saddles containing all three boundary components. However, every cyclic sequence should contain even number of boundary components (and the same number of boundary saddles), otherwise the orientation of foliation is destroyed. This is possible only for even number of indices which is equal to genus. Our conclusion is that $g \geq 4$. This theorem is proved.

The 2-sphere S^2 is covered by the non-extendable **" corridors"** between two segments $t_j, t_k \subset S^2$ (see fig 9b) consisting of the non-separatrix trajectories moving from the inner points of t_j to the inner points of t_k not meeting any points of t_l and the saddles inside. The right and left sides of corridors are either separatrices of the saddles in S^2 or the trajectories passing through the ends of some segments t_l.

Classification of the generic Morse functions h on the sphere S^2 is following: Take any connected trivalent finite tree R (see fig 9c). It has vertices divided into the r **Inner Vertices** and t **Ends**. Assign to each vertex $Q \in R$ a value $h(Q)$; these values are not equal to each other $h(Q) \neq h(P), P \neq Q$; therefore the edges become oriented looking "up" where h is increasing. Every inner vertex Q is a "saddle", i.e.

$$\min_i h(Q_i) < h(Q) < \max_i h(Q_i), i = 1, 2, 3$$

where Q_i are the neighbors. The function h on the graph should be monotonic for every edge $[Q_1 Q_2] \subset \tilde{R}$.

Let us describe the set of segments $t_j \subset S^2$. They do not cross each other and never meet saddles. They are transversal to the flow everywhere except centers where exactly 2 of them meet each other and end up. Every trajectory meets at least one segment t_j. Its transversal measure is equal to $m_j = h(P_2) - h(P_1) > 0$. Here P_1, P_2 are the ends of this segment.

Example: The Maximal Foliations for $g = 4$.

For the maximal foliations of the class T all boundary saddles are paired with each other. It means that all of them organize a system of cycles where the next boundary saddle is paired with the previous one. The length of every cycle is an even number $2l_q, q = 1, \ldots, f$, so $2l_1 + \ldots + 2l_f = g$. We say that system has a cycle type $(l_1, \ldots, l_f)$. The maximal system also $g - 2$ saddles such that all 4 separatrices are nonboundary. For the case $g = 4$ we have two possibilities for the cycles, namely $(1, 1)$ and (2). The type (2) is especially interesting. This cycle separates a 2-sphere on the South and North Hemisphere (see Fig 12) where A_1, A_2, A_3, A_4 are located along the equator. The additional two saddles are sitting in the poles with separatrix curves going to the each "country" $A_k, k = 1, 2, 3, 4$ along the 4 selected meridians from the north and south poles.

Lemma 6 *The transitions* $A_k^+ \to A_l^-$ *with measures* m_{kl} *visible from the poles (i.e. located in the corresponding hemisphere) are the following:*

From the North Pole we can see the transitions

$$A_k^+ \to A_l^-, k = 1, 3, l = 2, 4$$

From the South Pole we can see the transitions

$$A_k^+ \to A_l^-, k = 2, 4, l = 1, 3$$

They satisfy to the Conservation Law $\sum_k m_{kl} = |A_l|, \sum_l m_{kl} = |A_k|$, *where* k *and* l *are neighbors in the cyclic order* ...12341.... *We have* $A_1 - A_2 + A_3 - A_4 = 0$. *The constant flux in one direction of the cycle* 1234 *is defined provided by the asymmetry* m *of the measures* $m_{kl} - m_{lk}$:

$$m_{12} - m_{21} = m_{23} - m_{32} = m_{34} - m_{43} = m_{41} - m_{14} = m$$

We say that the system is rotating clockwise if $m > 0$. *It is rotating contr-clockwise if* $m < 0$.

Section 4. Three-Street Picture on the torus with obstacle.

We are dealing now with one torus $T^2 = T_k^2$ with generic straight line flow. The torus is presented as a lattice in $C = C_k$ given by the parallelogram $P = P_k$, the flow is a set of vertical lines. A transversal segment $s = s_k$ with measure $m = m_k$ is marked in T^2.

Question. How long the trajectory can move in the plane C (i.e. in the torus T^2) until it hits some periodically repeated copy of the segment s ?

Every trajectory starts and ends in some segments of the selected periodic family generated by the segment s. Family of such pieces with fixed ends form the connected strip. We require that these strips cannot be extended to the left and right: every trajectory in the strip ends in the same segments. The right and the left boundaries of the strip are separatrices of some saddles. Every such strip has **Width** w. It is equal to the transversal measure.

Definition.We call the unextendable strips by **The Streets** and denote them $p^\tau, \tau = 0, 1, 2$. We denote a longest unextendable strip by p^0. Its right and left boundaries meet the ends of some segments s', s'' of our family strictly inside (see Fig 11). The lower and upper segments of this strip should be located strictly inside of the corresponding segments s, s'''. Their widths we denote by $|p^\tau|$ correspondingly. The street number 2 is located from the right side from the longest one, the street number 1–from the left side. We assign also homology classes to the street $h^\tau \in H_1(T^2, Z)$ naturally.

Lemma 7 *For every straight-line foliation of T^*_m, i.e. of the torus T^2 with transversal canonical basis a, b and obstacle $s \subset T^2$ with measure $m < |a|+|b|$, there exist exactly three streets $p^\tau, \tau = 0, 1, 2$ such that $\sum_\tau |p^\tau| = m$ and $h^1 + h^2 = h^0$. Two smaller streets are attached to the longest one from the right and left sides. This picture is invariant under the involution changing time and orientation of the transversal segments. The union of the three streets started in the segment s is a fundamental domain of the group Z^2 in the plane C (see Fig 11).*

Proof. Construct first the longest street p^0. We start from any non-separatrix trajectory ending in the segments of our family inside both of them. Extend this strip to the left and right directions as far as possible. We either meet the ends of upper or lower segments or the end of some segment strictly between. In the first case we see that extending beyond the left end we construct longer strips. Start the same process with longer strip. Finally, we reach the "locally maximal" strip ss'''. By construction, we meet from the left and right sides of it the ends of some segments s', s'' correspondingly, otherwise it would be extendable. So the locally maximal street is constructed. Consider the neighboring streets from the right and left sides. They are shorter by construction. Denote the left one by p^1 and the right one by p^2. Let us prove that their width can be extended to the left and right boundaries of the locally longest one. As it follows from the Z^2- periodicity, all segments are Z^2-equivalent. In particular, they have equal lengths and are parallel to each other on the plane. Our flow is realized as a straight line flow here, and Z^2 acts as a group of shifts. Denote the lengths of the streets along the flow p^1, p^2 by the letters l_l, l_r correspondingly. For the generic flow they are not equal $l_l \neq l_r$. Let $l_l > l_r$. Take the right end of the right segment s''' and consider the piece of trajectory τ of the length l_r ending in it (i.e. coming to it from below). By periodicity no one segment of our family crosses this piece of the length $l_l > l_r$ (because it is Z^2- equivalent to the right side τ' of the left street p^1). So the right street p^2 is extendable to the very right end of the segment s. We have also $l_l + l_r = l$ where l is the length of the longest street p^0 along the flow. Consider now a piece of trajectory κ of the length l_l joining the left end of the segment s'' with the upper street s'''. No one segment crosses it by construction. It is Z^2-equivalent to the piece of trajectory κ' of the length l_l starting in the left end of the segment s and going up. By the same reason, no one segment Z^2-equivalent to s, crosses it. Therefore the left street p^1 is extendable to the very left end of s. Our statement is proved. The equality $l_l + l_r = l$ implies that our picture is invariant under the simultaneous change of time and direction of segments (i.e. π-rotation of the plane).

We can see now that our locally longest street is surrounded by the exactly four unextendable streets (two streets from each side). They are restricted from the right and left sides by the ends of the lower or upper segments s, s'''(see Fig 11). The pairs of streets located across the diagonal of each other are also equal. So, there are no other unextendable streets except these three.

Lemma is proved.

Section 5. The Homology and Homotopy Classes.

How to describe the image of Fundamenal Transversal Semigroups in fundamental group $\pi_1^+(\Re) \to \pi_1(V)$ and in homology group $H_1(V, Z)$?

There are three types of the transversal curves generating all of them: (I) The "Torical Type" transversal curves $\gamma \subset T_m^* \subset V$ in one 2-torus not touching the obstacle s of the transversal measure m; (II) The "Trajectory Type" transversal curves $\gamma \in V$ (or the Poincare Curves). They coincide with some trajectory of the Hamiltonian System started and ended in the transversal interval s_k^+ in the the torus T_k^2. We make it closed joining the endpoints by the shortest transversal interval along s_k^+; (III) The general non-selfintersecting transversal closed curves.

Step 1: Classify all closed transversal curves $\gamma \subset T_m^*$.

Let $|a|, |b|$ denote the measures of the basic transversal closed curves a, b. **We made our notations above such that the cycles a, b^{-1} are positive and transversal.** According to our construction, the measure m of the segments s satisfies to the inequality $0 < m < |a| + |b|$. Define minimal nonnegative pairs of integers $(u > 0, v \geq 0), (w \geq 0, y > 0)$ such that

$$m > u|a| - v|b| > 0, m > y|b| - w|a| > 0$$

(see Fig 13). It means exactly that these new lattice vectors represent the shifts s', s'' of the segment s visible directly from s along the shortest trajectories looking to the positive time direction.

Lemma 8 *The new lattice vectors h^1, h^2 have transversal measures equal to $|a^*| = u|a| - v|b| < m$ and $|b^*| = y|b| - w|a| < m$ where $|a^*| + |b^*| > m$. Their homology classes $h^1 = u[a] + v[b]$ and $-h^2 = w[a] + y[b]$, generate homological image of semigroup of positive closed transversal curves not crossing the segments s (i.e. of the Torical Type (I)). These classes have canonical lifts a^*, b^* to the free group F_2 generated by a, b; The lifts $a' = a^*, b' = (b^*)^{-1}$ generate free semigroup of all homotopy classes of positive closed transversal curves not crossing the segments $s_k^{\pm}$, starting and ending in the street number 0. These semigroups depend on the triple $|a|, |b|, m$ only. Similar classes h_-^1, h_-^2 can be constructed for the negative time direction: their homology classes are opposite to the positive ones. They represent the same streets going back. Their lifts a'_-, b'_- to the free fundamental group $\pi_1(T_m^*, s^-)$ of torus with cut along s, are defined as a mirror symmetry of the lifts a', b'. Here $t \to -t, s^+ \to -s^-$.*

Proof. We have the streets p^α in the plane C where the longest one is p^0. It is located between two others. The right one is p^2, and the left one is p^1. For the homology classes we have $h^1 + h^2 = h^0$, and for the transversal measures $\sum_\alpha |p^\alpha| = m$. Their bottom parts cover together the segment s. This picture is invariant under the change of direction of time and simultaneous permutation of the lower and upper segments s. All positive transversal paths in the plane C_k can be written combinatorially, numerating the streets which they cross, in the form

$$\ldots \to \alpha_q \to 0 \to \alpha_{q+1} \to 0 \to \alpha_{q+2} \to 0 \to \ldots$$

where $\alpha_q = 1, 2$. We have a pair of positive "basic cycles" $[0 \to 1 \to 0] = (b^*)^{-1}$ and $[0 \to 2 \to 0] = a^*$. All other transversal cycles not touching the segment s and its shifts, starting and ending in the longest strip 0, have a form of the arbitrary word in the free semigroup generated by $a^*, (b^*)^{-1}$ in the plane C. The measures of these new m-dependent basic cycles are

$$|a^*| = |p^0| + |p^2|, |b^*| = |p^0| + |p^1|$$

Topologically the cycles a^*, b^* represent some canonical m-dependent basis of elementary shifts in the group Z^2. To define the elements a^*, b^*, lifting them to fundamental group, we choose the simplest transversal paths joining the initial point in s^+ with its image in C avoiding obstacles. New basic shifts map the segment $s \subset P_{0,0}$ exactly into the the segments s', s'' attached to the middle part of the long street from the left side (for a^*) and from the right side (for b^*)–see Fig 13. Homologically the elements $[a^*] = h^1, [b^*] = h_k^2$ are calculated in the formulation of lemma using the transversal measures $|a|, |b|$ of the initial basic cycles and the measure m. From geometric description of new paths a^*, b^* we can see that they satisfy to the same relation as the original a, b: Their commutator path exactly surrounds one segment s on the plane. Therefore our lemma is proved.

This is the end of the Step 1.

Our 2-torus with cut along the obstacle is a part of the Riemann surface $T^*_{m_k} \subset V$. As we know, in order to have well defined element $\{\gamma\} \in \pi_1(V)$, we need to have both ends belonging to s_k^+. To define homology class $[\gamma]$ is enough to have both ends belonging to $s_k^+ \bigcup s_k^-$ because $H_1(V, s_k^+ \bigcup s_k^-) = H_1(V)$. Therefore we arrive to the following

Conclusion: *Every trajectory of Hamiltonian System on the Riemann Surface V constructed out of Building Data above, defines an infinite "sum" of homology classes* $(... + [\gamma_N] + [\gamma_{N+1}] + [\gamma_{N+2}] + ...)$ *where* $[\gamma_N] \in H_1(V) = H_1(V, S^2) = \sum_k H_1(T_k^2)$. *Every class* $[\gamma_N]$ *is equal to the homology class of the street* $h_k^\tau \in H_1(T_k^2)$ *for* $\tau = 0, 1, 2$. *So we have the exact coding of trajectories by the sequences of symbols* τ, k *if these homology classes are known.*

The case $g = 2$. For the special case $g = 2$ we can assign the invariant $\phi_{\alpha\beta} \in \pi_1(V, s_1^+)$ to every piece of trajectory passing two streets

$$\gamma_q \subset p_1^\alpha p_2^\beta, \phi : \gamma_q \to \pi_1(V, s_1^+)$$

So every infinite trajectory γ can be viewed as an infinite product of pieces

$$\gamma = \dots \phi_{\alpha_q \beta_q} \phi_{\alpha_{q+1} \beta_{q+1}} \dots$$

Every finite connected piece γ' of this sequence defines the element $\phi(\gamma') \in \pi_1(V, s_1^+)$. Its ends can be joined by the shortest transversal piece along the segment s. Depending on orientation of this piece, either $\phi(\gamma')$ or $\phi(\gamma')^{-1} \in \pi_1(V, s_1^+)$ defines a closed positive transversal curve of the Poincare Type in the Fundamental Transversal Semigroup.

Reduction to the Standard Model: The segment s of the length m is divided on 5 connected open pieces τ_q: There exist exactly 9 possible types of trajectory pieces $\{\alpha\beta\}, \alpha, \beta = 1, 0, 2$ with measures $p_{\alpha\beta}$ but 4 of them are in fact empty. They have measure equal to zero. In order to see that, we remind how these pieces were constructed. A segment $s = s_1$ of the total measure m is divided into 3 pieces by the points $0, 1, 2, 3$ for $k = 1$ and by the points $0', 1', 2', 3'$ for $k = 2, s = s_2$: For the streets we have:

$$\alpha = 1, 0, 2 = [01], [12], [23], \beta = 1', 0', 2' = [0'1'], [1'2'], [2'3']$$

So, the index $\alpha = 1, 0, 2$ corresponds to the segments $[01], [12], [23]$ with measures p_1^α, and the index $\beta = 1', 0', 2'$ corresponds to the segments $[0'1'], [1'2'], [2'3']$ with measures p_2^β. The positions of the points $0 = 0', 3 = 3'$ are fixed. Other points never coincide for the generic foliations.

Every jump from C_1 to C_2 is accompanied by the permutation of 3 segments: the left street number 1 in C_1 ends up in the extreme right part of $s = s'$ before making jump to C_2. The right street number 2 in C_1 ends up in the extreme left part of $s = s''$ in C_2. So jumping from C_1 to C_2, we should permute the segments $1 = [01] = 1$ and $2 = [23] = 2$ preserving orientation:

$$\eta_{12} : 2 \to 0 = 2^*, 3 \to 3^* \in s, 1 \to 3 = 1^*, 0 \to 0^* \in s$$

. Here $|[23]| = |[2^*3^*]|, |[01]| = |[0^*1^*]|$. The segment s is divided by the points $0 = 2^*, 1', 2', 3^*, 0^*, 3 = 1^*$, so it is presented as a union of 5 sub-segments

$$s = \tau_1 + ... + \tau_5$$

In order to return back from the second torus (plane) C_2 back to C_1, we need to apply the similar map η_{21} based on the permutation of the streets $2' = [2'3']$ and $1' = [0'1']$. Our broken isometry i_σ based on the permutation σ of 5 pieces, is defined as a composition

$$i_\sigma = \eta_{21}\eta_{12}$$

Lemma 9 *There exist six Topological Types of Foliations (the measures $p_{\alpha\beta}$ of the sub-segments $\tau_q, q = 1, 2, 3, 4, 5$ are given in the natural order on the segment s):*

$$(I) : 0 = 2^* < 1' < 2' < 3^* < 0^* < 1^* = 3; \sigma = (32541);$$

$$p_{12} + p_{02} + p_{21} + p_{20} + p_{22} = m$$

$$(II): 0 = 2^* < 1' < 3^* < 2' < 0^* < 1^* = 3; \sigma = (24153);$$

$$p_{12} + p_{01} + p_{02} + p_{21} + p_{20} = m$$

$$(III): 0 = 2^* < 1' < 3^* < 0^* < 2' < 1^* = 3; \sigma = (41523);$$

$$p_{10} + p_{12} + p_{00} + p_{21} + p_{20} = m$$

$$(IV): 0 = 2^* < 3^* < 1' < 2' < 0^* < 1^* = 3; \sigma = (25314);$$

$$p_{12} + p_{01} + p_{00} + p_{02} + p_{21} = m$$

$$(V): 0 = 2^* < 3^* < 1' < 0^* < 2' < 1^* = q3; \sigma = (31524);$$

$$p_{10} + p_{21} + p_{01} + p_{00} + p_{21} = m$$

$$(VI): 0 = 2^* < 3^* < 0^* < 1' < 2' < 1^* = 3; \sigma = (52134);$$

$$p_{11} + p_{10} + p_{12} + p_{01} + p_{02} = m$$

All other measures $p_{\alpha\beta}$ are equal to zero. We have

$$\sum_{\alpha} p_{\alpha\beta} = p_2^{\beta}; \sum_{\beta} p_{\alpha\beta} = p_1^{\alpha}$$

As we can see, our 3-street model automatically creates the standard type "broken (i.e. discontinuous) isometry" $\eta_{21}\eta_{12} = i_\sigma : s \to s$ generated by the permutation σ of 5 pieces τ_q of the segment s. Such systems were studied by the researchers in the ergodic theory since 1970s. Let Let us point out that our combinatorial model easily provides full information about the topology lying behind this permutation. We know geometry of all pieces.

Let us define an algebraic object (no doubt, considered in the ergodic theory many years ago): **The Associative Semigroup** $S_{\sigma,\tau}$ with "measure". It is generated by the 5 generators $R_1, ..., R_5 \in S_{\sigma,\tau}$. There is also a zero element $0 \in S_{\sigma,\tau}$. We define multiplication in the semigroup $S_{\sigma,\tau}$ as in the free one but factorize by the "zero measure" words: they are equal to zero. In order to define which words are equal to zero, we assign to every generator an interval $R_q \to \tau_q, q = 1, ..., 5$. We assign to the word R the set τ_R where $\tau_{R_q} = \tau_q$ for generators. Let the word R has a length N. By induction, we assign to the word R_pR the set τ_{R_pR} such that

$$\tau_{R_pR} = i_\sigma^{-N}(\tau_p) \bigcap \tau_R$$

if it is nonempty. We put $R_pR = 0$ otherwise. This semigroup is associative. For every word in the free semigroup $R = R_{q_1}...R_{q_N}$ we assigne a set

$$i_\sigma^{-N+1}(\tau_{q_1})\bigcap i_\sigma^{-N+2}(\tau_{q_2})\bigcap ...\bigcap \tau_{q_N} = \tau_R$$

whose measure is well defined. We put R equal to zero if its measure is equal to zero.

The ordered infinite sequence $R_\infty = \prod_{p\in Z} R_{q_p}$, defines trajectory if and only if every finite sub-word is nonzero. The measure is defined for the "cylindrical" sets U_R consisting of all "trajectories" with the same sub-word R sitting in the same place for all of them. It is equal to τ_R. The "Shift Function" is well-defined by the semigroup. **Our goal is to calculate representation of this semigroup in the fundamental group of the Riemann Surface generated by the Hamiltonian System.**

Our global reduction of the flow is based on the pair of non-closed transversal segments $s^\pm$ leading from one saddle to another. This construction seems to be the best possible genus 2 analog of the reduction of straight line flow on the 2-torus to the rotation of circle. Many features of this construction certainly appeared in some very specific examples studied before.

We always choose initial point in the segment $s_1^+ \subset C_1$. It is equivalent to the choice of initial point in the segment $s_2^- = s_1^+ \subset C_2$. **The New m-Dependent Transversal Canonical Basis** is (see Fig 13):

$$a_1^*, b_1^*, a_2^* = a_{2,-}^*, b_2^* = b_{2,-}^*$$

This basis is attached to the segment $s_1^+ = s_2^-$, so we treat them as the elements of fundamental group $\pi_1(V, s_1^+)$. It was proved before that homology classes of the (positive) streets are following:

$$h_k^1, h_k^2, h_k^0 = h_k^1 + h_k^2 \in H_1(V, Z)$$

Here $k = 1, 2$. For the negative time we simply change sign $[a_-^*] = -h_1^k, [b_-^*] = -h_2^k$.

Let us describe the homotopy classes $\phi_{\alpha\beta} \in \pi_1(V, s_1^+)$ of the two-street paths $\alpha\beta = p_1^\alpha p_2^\beta$ with ends in the same open segment $s = s_1^+ = s_2^- \in T_1^2 \bigcap T_2^2$.

Consider first the plane $C_1 \to T_1^2$. In order to make closed paths out of the streets p_1^α, we join ends going around the segment s from s_1^- to s_1^+ from the right or from the left side of it (see Fig 14):

For the street number 1 we construct the path a_1^* extending the street around s' from the right side along the path κ_1, circling contr-clockwise around s'; For the street number 2 we construct the path b_1^* closing street around s'' from the left side along the path κ_2, circling clockwise around s''; So we have $p_1^1 \sim a_1^* \kappa_1^{-1}$, $p_1^2 \sim b_1^* \kappa_2^{-1}$. Here and below **symbol $\sim$ means "homotopic with fixed ends"**. We define a closed path circling clockwise around the segment s in C_1:

$$\kappa = \kappa_2 \kappa_1^{-1} \sim (s^+ \bigcup s^-) \sim a_1^* b_1^* (a_1^*)^{-1} (b_1^*)^{-1}$$

We assign element $p_1^0 \sim a_1^* b_1^* \kappa_2^{-1}$ to the street number zero. Same description we have also for the streets $p_2^\alpha(-) = (p_2^\alpha)^{-1}$ in the plane C_2 going in the opposite direction, replacing (+) by (-) and the paths $\kappa, \kappa_1, \kappa_2$ by the similar paths $\delta, \delta_1, \delta_2$:

$$(p_2^1)^{-1} \sim a_2^* \delta_1^{-1}, (p_2^2)^{-1} b_2^* \sim \delta_2^{-1}, (p_2^0)^{-1} \sim a_2^* b_2^* \delta_2^{-1}$$

. We have

$$\kappa_1 = \delta_2, \kappa_2 = \delta_1, \delta = \delta_1 \delta_2^{-1} = \kappa^{-1}$$

We use the new basis $a_1^*, b_1^*, a_2^*, b_2^*$ defined above, dropping the measure m and signs $\pm$, as it was indicated above. **All formulas are written in the new m-dependent basis**.

Performing very simple multiplication of paths, we obtain following formulas in the group $\pi_1(V, s_1^+)$:

Lemma 10 *The homotopy types of all nonnegative almost transversal two-street passes in the positive time direction starting and ending in the segment $s_1^+ = s_2^-$, including the trajectory passes, are equal to the following list of values of the elements $\phi_{\alpha\beta} \in \pi_1(V, s_1^+)$ where $\alpha = 1, 0, 2$, $\beta = 1', 0', 2'$ and $\alpha\beta$ means $\phi_{\alpha\beta}$:*

$$11' \sim a_1^* \kappa^{-1} (a_2^*)^{-1}; 10' \sim a_1^* b_1^* (a_2^*)^{-1}; 12' \sim b_1^* (a_2^*)^{-1}$$

$$01' \sim a_1^* (b_2^*)^{-1} (a_2^*)^{-1}; 00' \sim a_1^* b_1^* \kappa (b_2^*)^{-1} (a_2^*)^{-1}; 02' \sim b_1^* \kappa (b_2^*)^{-1} (a_2^*)^{-1}$$

$$21' \sim a_1^* (b_2^*)^{-1}; 20' \sim a_1^* b_1^* \kappa (b_2^*)^{-1}; 22' \sim b_1^* \kappa (b_2^*)^{-1}$$

For the negative time direction the two-street passes have homotopy classes equal to $\phi^{-1}_{\alpha\beta}$ *in the same group* $\pi_1(V, s_2^-) = \pi_1(V, s_1^+)$

For the Topological Types I–VI of Foliations following homotopy classes of almost transversal two-street passes have nonzero measure:

$$\phi_{\alpha\beta} = p_1^\alpha x p_2^\beta, \phi^*_{\alpha\beta} = (p_1^\alpha)^{-1} x (p_2^\beta)^{-1}$$

(here $x \geq 0$ *means positive transversal shift along the segment* s*):*

Type (I): All 9 classes $\phi_{\alpha\beta}$*, and all classes* $\phi^*_{\alpha\beta}$ *except* $11', 10', 01', 00'$

Type (II): All classes $\phi_{\alpha\beta}$ *except* $22'$*, and all classes* $\phi^*_{\alpha\beta}$ *except* $11', 10', 01'$

Type (III): All classes $\phi_{\alpha\beta}$ *except* $22', 02'$*, and all classes* $\phi^*_{\alpha\beta}$ *except* $11', 01'$

Type (IV): All classes $\phi_{\alpha\beta}$ *except* $20', 22'$*, and all classes* $\phi^*{}_{alpha\beta}$ *except* $11', 10'$

Type (V): All classes $\phi_{\alpha\beta}$ *except* $22', 20', 02'$*, and all classes* $\phi^*_{\alpha\beta}$ *except* $11'$

Type (VI): All classes $\phi_{\alpha\beta}$ *except* $20', 22', 00', 02'$*, and all 9 classes* $\phi^*_{\alpha\beta}$

So we are coming to the following

Combinatorial Model of Foliation/Dynamical System: *It involves 1.The semigroup* $S_{\sigma,\tau}$ *and its representation in the fundamental group. All positive finite words*

$$R = R_{j_1} R_{j_2} \dots R_{j_N} \in \pi_1(V)$$

are written in the new m*-dependent Transversal Canonical Basis* $a_1^*, b_1^*, a_2^*, b_2^*$ *where every symbol* R_j *is equal to one of the elements* $\phi_{\alpha\beta} \in \pi_1(V), j = 1, 2, 3, 4, 5$ *found above. 2.The permutation* σ *and measures of the sub-segments* $\tau_q, q = 1, 2, 3, 4, 5, \sum_q \tau_q = m$*. They represent the types* I, II, III, IV, V, VI *described above. The permutation* σ *defines a broken isometry* $i_\sigma : s \to s$ *well-defined for the inner points of the sub-segments* τ_q*. The measure is extended to the whole semigroup according to the rules formulated above. 3.Every finite word* $R \in S_{\sigma,\tau}$ *defines a connected strip of the nonseparatrix trajectories of the Hamiltonian System corresponding to our data, with the transversal measure* τ_R*, starting and ending in the transversal segment* s_1^+*. It defines a closed transversal curve* γ_R *with transversal measure equal to the shift of the end* $r(R)$*. It is positive if* $r(R) < 0$*, and negative if* $r(R) > 0$*. The transversal measure of these transversal curves are equal to* $-r(R)$*. The infinite trajectories are presented by the infinite sequences of the symbols* R_j *such that every* *finite piece of the sequence is nonzero as an element of the semigroup $S_{\sigma,\tau}$. The measure on the set of trajectories is defined by the transversal measure of the finite words in $S_{\sigma,\tau}$. The basic measurable sets are "cylindrical" (i.e. they consists of all trajectories with the same finite word R, and measure is equal to τ_R).*

Therefore the Step 2 is realized.

Problem. How to classify simple closed transversal curves? We consider this problem for the special case of torus with obstacle. Let a straight line flow on the 2-torus with irrational rotation number and fixed positive transversal canonical basis a, b^{-1} is given as above. Every indivisible homology class in $H_1(T^2, Z)$ can be realized by the non-selfintersecting closed curve transversal to foliation. We can see that **the semigroup of positive closed transversal curves has infinite number of generators containing the elements with arbitrary small transversal measure**. The same result remains true after removal of one point $T \to T^* = T^*_0$, but the semigroup became non-abelian. Remove now transversal segment s with positive measure $m > 0$ from the torus $T^*_0 \to T^0_{(m)}$. As we shall see below, the situation changes drastically: the semigroup became finitely generated with two free generators.

Start with Fundamental Domain in C consisting of the union of the three streets $D_m = p^1 \bigcup p^0 \bigcup p^2$. We use an extension $D_m \subset D' \subset C$ of it adding two more streets– one more copy of the street p^1 over p^2 and one more copy of the street p^2 over p^1. Every positive simple transversal curve can be expressed in the m-dependent basis $a^*, b^* \in \pi_1(T^2_m)$. Its homology class is $[\gamma] = ka^* + l(-b^*) \in H_1(T^*_m, Z), k > 0, l > 0, (k, l) = 1$. We take the cycles $a = a^*, b^* = b^{-1}$ as basic positive closed transversal curves.

Every non-selfintersecting positive closed transversal curve $\gamma \subset T^*_m = T^2 minus(s)$ crosses fundamental domain and especially the street p^0 exactly $k + l$ times from the right to the left. We denote its segments by $t_j, j = 1, ..., k + l$ ordering them from the segment $s^+_1 = s$ up. It crosses the street p^2 exactly k times, entering the domain p^0 in the points $y_1, ..., y_k$; they are naturally ordered by height. It crosses p^1 exactly l times, entering the domain 0 from p^1 in the points $y_{k+1}, ..., y_{k+l}$, We denote the points on the equivalent segments from the left side of fundamental domain by the same figures with symbol ': on the (lower) left side of the street p^0 there are the points $y'_{k+1}, ..., y'_{k+l}$; the points $y'_1, y'_2, ..., y'_k$ we have on the segment separating the upper copy of the street p^2 from the street p^0. Assume that $k > l$.

The sequence of segments ordered by height,is following:

$$I : 2 \to 0 \to 1 : t_1 = [1, (k+1)'], ..., t_l = [l, (k+l)']$$

$$II : 2 \to 0 \to 2 : t_{l+1} = [l+1, 1'], ..., t_k = [k, (k-l)']$$

$$III : 1 \to 0 \to 2 : t_{k+1} = [k+1, (k-l+1)'], ..., t_{k+l} = [k+l, k']$$

We assign to every segment following homotopy class depending on the group I,II,III and their product to the whole curve γ:

$$\phi : t_j \to b', t_j \in (I); \phi : t_j \to a', t_j \in (II), (III)$$

$$\phi(\gamma) = \phi(t_{q_1})...\phi(t_{q_{k+l}}) \in \pi_1(T_m^*)$$

where $\gamma \sim t_{q_1}...t_{q_{k+l}}$ in the natural order along the curve. As a result of the previous lemmas, we obtain following

Theorem 5 *The invariant $\phi(\gamma)$ describes homotopy classes of all closed non-selfintersecting positive transversal curves in T_m^* as a positive words written in the free group with two m-dependent generators $a' = a^*, b' = (b^*)^{-1}$ whose transversal measures are smaller than m . In particular, every indivisible homology class $k[a'] + l[b'], k > 0, l > 0$, has m-dependent simple positive transversal representative as a positive word unique up to cyclic permutation (i.e. it defines exactly $k+l$ equivalent positive words). This word is calculated above in the three street model. Every such curve can be taken as a part of new Transversal Canonical Basis in T_m^*.*

Description of all Transversal Canonical Bases on the torus with obstacle. We can see that every integer-valued 2×2-matrix $T, \det T = 1$, with positive entries $k, l, p, q \geq 0$

$$[A] = T(a') = k[a'] + l[b'], [B] = T(b') = p[a'] + q[b']$$

determines finite number of different transversal canonical bases $A, B \in \pi_1(T_m^*)$. They are represented by the curves A, B crossing each other transversally in one point and representing the homology classes $T(a'), T(b')$. Consider the segments of both these curves in the street p^0 as above. Let the curves A, B are represented by the ordered sequences of pieces $t_1, ...t_{k+l}$ and $t''_1, ...t''_{p+q}$ correspondingly going from the right to the left side. We require existence of one intersection point for the selected pair $t_i \bigcap t''_j \neq \emptyset$. All other intersections should be empty. Every such configuration determines transversal canonical basis A, B. Its equivalence class is completely determined by the relative order of segments t, t'' taking into account that t_i crosses t''_j only once and t_i is "higher" than t''_j from the left side). Starting from the selected point $t_i \bigcap t''_j$, we apply the procedure described in the theorem. It gives us two positive words A, B in the free group F_2. By definition, the map $\hat{T} : a' \to A, b' \to B$ defines a lift from homology to fundamental group, for every pair of words A, B constructed in that way. Consider first a **Reducible** case where the left ends of the segments t_i and t''_j crossing each other are located on the same connected part of boundary (i.e. where the street p^0 meets the same street p^1 or p^2). If it is the street p^1, than the both words A and B start with the same letter b'. We can deform our crossing point along the cycle b'. After this step we are coming to the conjugated pair A', B' where b' is sent to the end:

$$(A = b'\tilde{A}, B = b'\tilde{B}) \to (A' = \tilde{A}b', B' = \tilde{B}b')$$

. If the pair t_i, t''_j ends up in the street p^2, we replace b' by a'. After the series of such steps, we are coming to the case where t_i ends in p^1, t''_j ends in p^2. This process cannot be infinite because the words A, B are not powers of the same word. So this process ends. We call **reduced** the final state where the process ended up.

Example: The pair of words $A = b'a'b'a'b', B = b'a'$ requires five conjugations to arrive to the reduced case.

In the final reduced state the relation: $ABA^{-1}B^{-1} = a'b'a'^{-1}b'^{-1}$ can be easily seen on the plane (with periodic set of segments removed), looking on the 3-street decomposition of the plane.

According to the classical result of A.I.Maltsev, the semigroup of unimodular 2×2-matrices with nonnegative integer-valued entries, is free, with two generators T_1, T_2 such that

$$T_1(a') = a' + b', T_1(b') = b', T_2(a') = a', T_2(b') = a' + b'$$

It is important to us that they can be lifted to the "positive" automorphisms of the free group F_2 mapping free "positive" semigroup into itself. Their lifts to the automorphisms of the free group are following:

$$\hat{T}_1(a') = a'b', \hat{T}(b') = b'; \hat{T}_2(a') = a', \hat{T}_2(b') = b'a'$$

They both preserve the word $a'b'a'^{-1}b'^{-1}$ so all semigroup of nonnegative unimodular matrices preserves this word. It is isomorphic to the semigroup of all positive automorphisms of free group F_2 preserving this word.
The semigroup G of all positive automorphisms of the free group F_2 in the alphabet a', b' preserving the conjugacy class of the word $\kappa = a'b'a'^{-1}b'^{-1}$, contains following parts: 1.A free semigroup G_κ of transformations preserving this word κ exactly; It is isomorphic to the semigroup of matrices T with $\det T = 1$ and nonnegative integer entries;

2. Every element $T \in G_\kappa$ defines finite number of positive transformations $T' \in G$ such that the corresponding pair of positive words $A' = T'(a'), B' = T'(b')$ is simultaneously conjugate to the pair of **reduced** words $A = T(a), B = T(b)$ for $T \in G_\kappa$. All these conjugations are the simultaneous cyclic permutation of the reduced words A, B: every time we remove one letter from the left ends and sending it to the right ends of both words. It is possible until both words ends up from the left with the same letter. **Total number $||T||$ of these conjugations is equal to the sum of matrix elements of T minus 2**. The natural projection of semigroups $h : G \to G_\kappa$ is defined such that the inverse image $h^{-1}(T)$ of matrix T contains exactly $||T|| = k + l + p + q - 2$ positive automorphisms. In the example above we have $||T|| = 5$. We proved these simple statements jointly with I.Dynnikov. Certainly, they are not new but but I cannot give any good reference to the reader. All standard algebraic literature refers to the groups only ignoring semigroups. Everybody except Maltsev, ignored semigroups which we need. I leave these statements here as exercises because they are very simple.
Let me describe now **the "m-cutted Euclidean Algorithm"** allowing to find minimal transversal basis a', b' starting from the arbitrary TC basis A, B in the 2-torus with the straight-line flow and transversal segment s of the measure m. Let $A, B \subset T^*_m, |A| + |B| > m$ and $|A| > |B|$. Take **the minimal integer** $l_1 \geq 0$ such that

$$|A| - l_1|B| = |A_1| = r_1|B| < \max(m, |B|)$$

For the new pair $(|A_1|, |B_0| = |B| > |A_1|)$ take minimal integer $l_2 \geq 0$ such that $|B_1| = |B_0| - l_2|A_1| < \max(m, |A_1|)$ and so on. We have a sequence of segments $A = A_0, B = B_0, A_1, B_1, A_2, B_2, ...$ and sequence of corresponding natural numbers $l_1, l_2, \ldots$. The process stops at the first moment q when the segment A_q or B_q appears which is less than m: For the case $q = 0$ we have $max(A, B) < m$ and $(a', b') = (A, B)$. Let $q > 0$. If $A_q < m$ but $B_{q-1} > m$, we calculate $B_q = B_{q-1} - l_{2q}A_q$ where l_{2q} is minimal and such that $B_q < m$. We define $l_r = 0$ for all $r > 2q$. After that we define a pair of cycles a', b' as A_q, B_q. If $B_q < m$ but $A_q > m$, we have minimal positive l_{2q+1} such that $A_{q+1} = A_q - l_{2q+1}B_q < m$. We define a pair of cycles a', b' as B_q, A_{q+1} and put $l_r = 0$ for all $r > 2q + 1$.

Theorem 6 *The transformation $T = T_1^{l_1}T_2^{l_2}T_1^{l_3}T_2^{l_4} \ldots T_{l_r}^{r}$ maps the minimal TC basis a', b' into the reduced TC basis $A = A_0, B = B_0$ satisfying to the condition $|A| + |B| > m$. Here $T_{2p-1} = T_1, T_{2p} = T_2$. Coefficients of the word T and coefficients of the Continued Fraction $|A|/|B| = l'_1 + 1/(l'_2 + 1/(l'_3 + 1/(...)))$ coincide for $j = 1, ..., r$. Here $r = 2q - 1$ for the case such that $A_q < m$ but $B_{q-1} > m$, and $r = 2q$ for the case $A_q > m$ but $B_q < m$. For the remaining numbers we have $0 < l_{2q} \leq l'_{2q}, l_j = 0, j > 2q$, for the first case, and $0 < l_{2q+1} \leq l'_{2q+1}, l_j = 0, j > 2q + 1$, for the second case. All reduced TC bases in the torus minus obstacle T^*_m of the transversal measure m, can be obtained by that transformation. All other TC bases can be obtained from the reduced ones by the sequence of conjugations in the free semigroup with 2 generators a', b' such that resulting words are also positive. There are exactly $||T||$ such conjugations where $||T||$ is equal to the sum of all matrix elements minus 2.*

Proof of this theorem immediately follows from the previous results. On the homological level all information about relationship between the original TC basis $a = A, b = B$ and minimal basis a', b' was clarified before, at the beginning of section 5 (see lemma 8). The matrix T constructing minimal basis a', b' was described in different terms. Our theorem here describes it as a sequence of elementary matrix transformations in the free semigroup. Easy to see that this presentation is exactly equivalent to the language of Continued Fractions. Let us explain this situation on the noncommutative level also. We always choose initial point in the street p^0. According to this choice, our semigroup of positive transversal curves is a free semigroup with two generators a', b'. All closed transversal curves were described in the theorem 5. Every TCB in the torus with obstacle is presented by the pair of simple close curves intersecting each other transversally in one point only. Such pairs were classified above in geometrical terms. Expressing this result in algebraic terms, they can be obtained from the reduced ones by conjugations such that result is positive. These arguments immediately implies our theorem.

References

[1] S.P.Novikov. Topology of Foliations given by the real part of holomorphic one-form, preprint, 2005, arXive math.GT/0501338

[2] S.P.Novikov. The Hamiltonian Formalism and Many-Valued Analogs of the Morse Theory, Russian Math Surveys (1982) v 37 n 5 pp 1-56 (in English)

[3] I.Dynnikov, S.Novikov. Topology of Quasiperiodic Functions on the Plane, Uspekhi Math Nauk=Russian Math Surveys (2005) v 60 n 1

[4] S.P.Novikov. Topology of Foliations, Trudy MMO=Transactions of the Moscow Math Society, 1965, v 14 (in Russian)

[5] A. Katok. Invariant measures of flows on oriented surfaces. Soviet Math. Dokl., v 14, pp 1104–1108 (1973)

[6] E. Sataev. The number of invariant measures for flows on orientable surfaces. Izv. Akad. Nauk SSSR Ser. Mat. v 39, no. 4, pp 860–878 (1975)

[7] B. Hasselblatt, A. Katok (ed). Handbook of Dynamical Systems, Vol. 1A, pp 1015–1089. Elsevier Science B.V. (2002)

[8] J. Hubbard, H. Masur. Quadratic differentials and measured foliations. Acta Math., v 142, pp 221–274 (1979)

[9] S. Kerckhoff, H. Masur, J. Smillie. Ergodicity of billiard flows and quadratic differentials. Annals of Math., v 124, pp 293–311 (1986)

[10] H. Masur. Interval exchange transformations and measured foliations. Ann. of Math., v 115, pp 169-200 (1982)

[11] W. A. Veech. Gauss measures for transformations on the space of interval exchange maps. Annals of Math., v 115, pp 201–242 (1982)

[12] H. Masur, S. Tabachnikov. Rational Billiards and Flat Structures. In: B. Hasselblatt and A. Katok (ed), Handbook of Dynamical Systems, Vol. 1A, 1015–1089. Elsevier Science B.V. (2002)

[13] A.Zorich. Proceedings of the Conference "Geometric Study of Foliations-Tokyo, 1993/ edited by Muzutani et al, Singapore, World Scientific (1994)

[14] A. Zorich. How do the leaves of a closed 1-form wind around a surface. In the collection: "Pseudoperiodic Topology", *AMS Translations*, Ser. 2, vol. 197, AMS, Providence, RI, pp 135–178 (1999)

[15] A. Zorich. Flat Surfaces. "Frontiers in Number Theory, Physics and Geometry", Proceedings of Les Houches winter school-2003, Springer Verlag (2005) (to appear)

[16] M. Kontsevich, A. Zorich: Connected components of the moduli spaces of Abelian differentials. Invent. Math., 153:3, pp 631–678 (2003)

[17] M. Kontsevich. Lyapunov exponents and Hodge theory. "The mathematical beauty of physics" (Saclay, 1996), (in Honor of C. Itzykson) pp 318–332, Adv. Ser. Math. Phys., 24. World Sci. Publishing, River Edge, NJ (1997)

[18] G. Forni. Deviation of ergodic averages for area-preserving flows on surfaces of higher genus. Annals of Math., v 155, no. 1, pp 1–103 (2002)

[19] C. McMullen. Dynamics of $SL_2(R)$ over moduli space in genus two. Preprint (2003)

[20] V. I. Arnold. Topological and ergodic properties of closed 1-forms with rationally independent periods. Functional Anal. Appl., v 25, no. 2, pp 81–90 (1991)

[21] K. M. Khanin, Ya. G. Sinai. Mixing of some classes of special flows over rotations of the circle. Functional Anal. Appl. v 26, no. 3, pp 155–169 (1992)

[22] G.Levitt, Topology, vol 21 (1982), pp 9-33

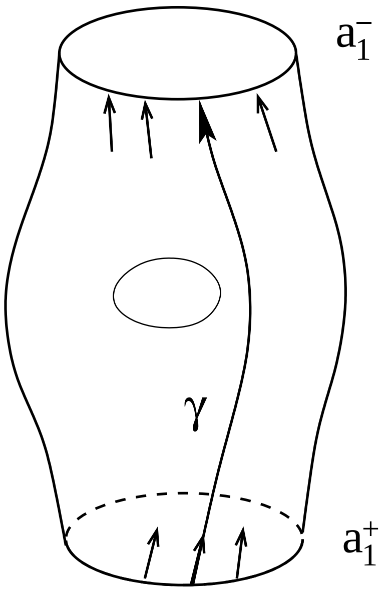

Fig 1: Constructing TCB: The first pair of cycles

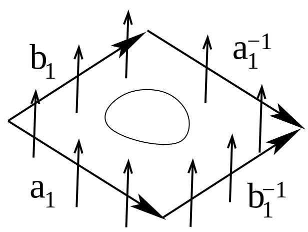

Fig 2: Cutting Riemann Surface along TCB.

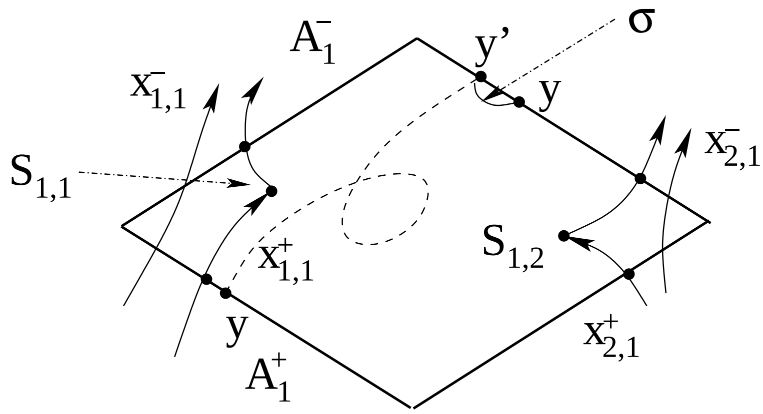

Fig 3: Boundary Saddles

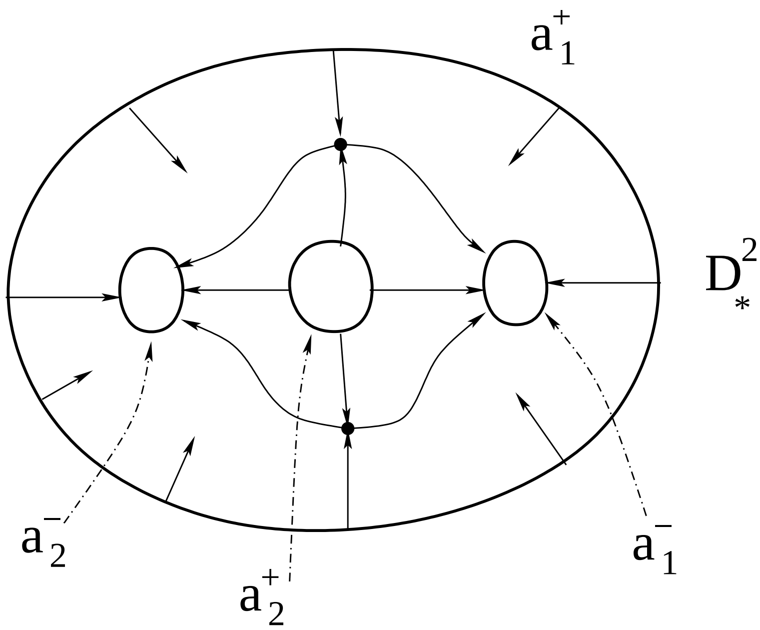

Fig 4a: Trajectory connections: $a_1^+ \to a_1^-$, $a_2^+ \to a_2^-$.

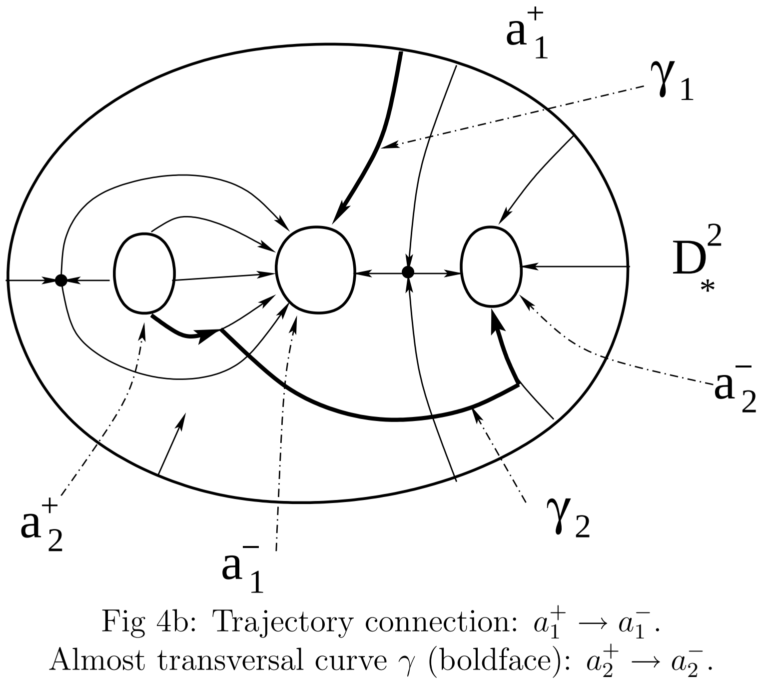

Fig 4b: Trajectory connection: $a_1^+ \to a_1^-$.
Almost transversal curve γ (boldface): $a_2^+ \to a_2^-$.

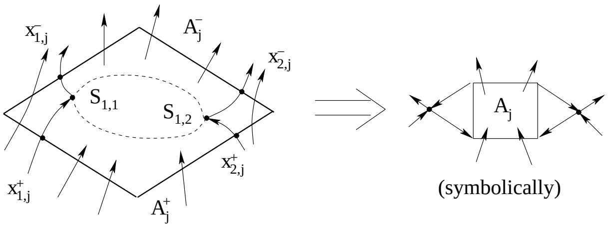

Fig 5: Transversal Cuts in the Riemann Surface

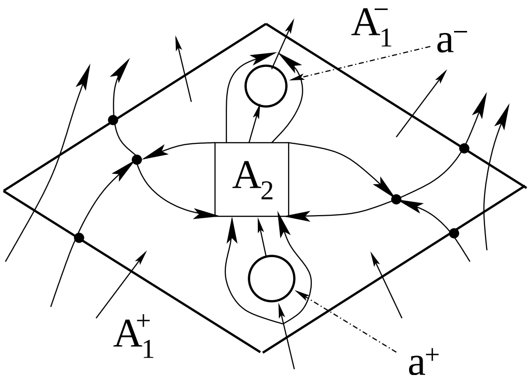

Fig 6: Non-extendable diagram of the type T^2.

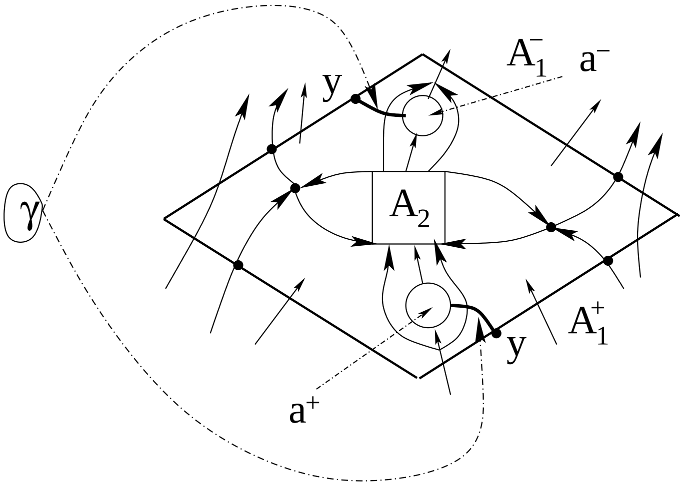

Fig 7: Reconstruction: new transversal curve $\gamma : a_1^+ \to a_1^-$ is passing through the boundary.

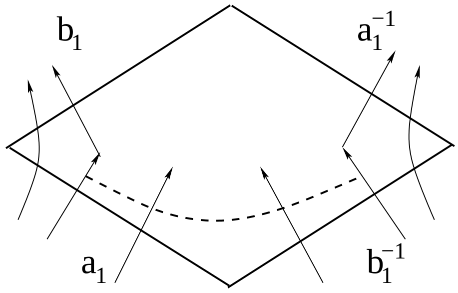

Fig 8: Transversal segment joining two saddles

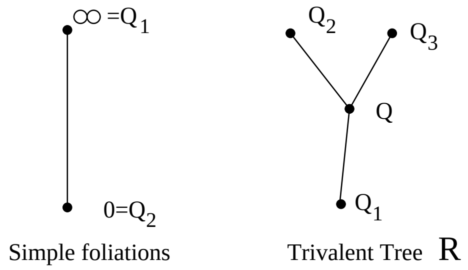

Fig 9c: Coding Morse Functions on S^2 by Graphs

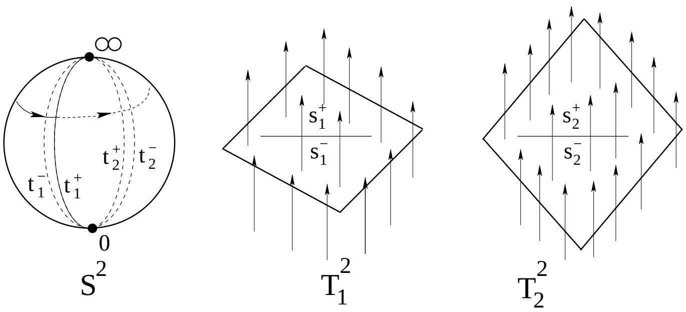

Fig 9a: The Building Data: $g = 2$

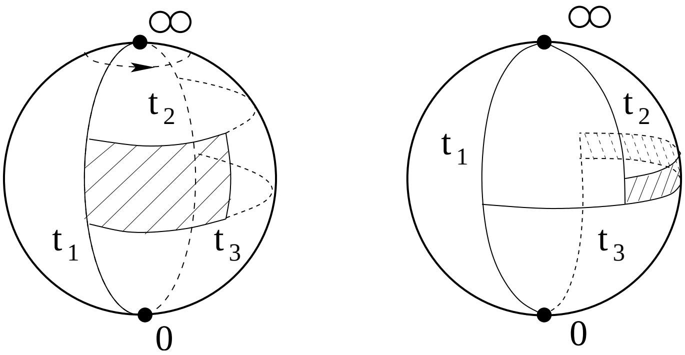

Fig 9b: The Corridors in S^2 $(g = 3)$

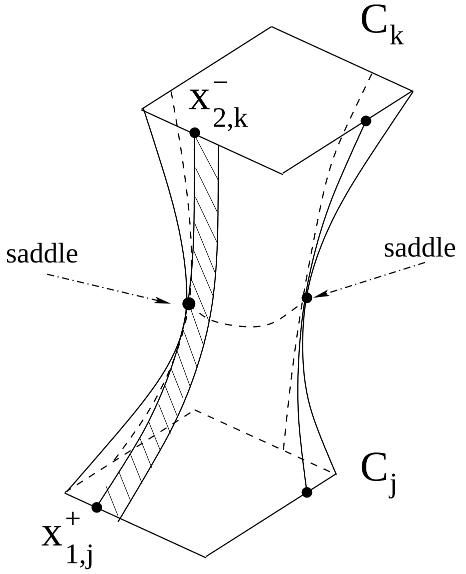

Fig 10: The Abelian Fundamental Domain $(g = 2)$

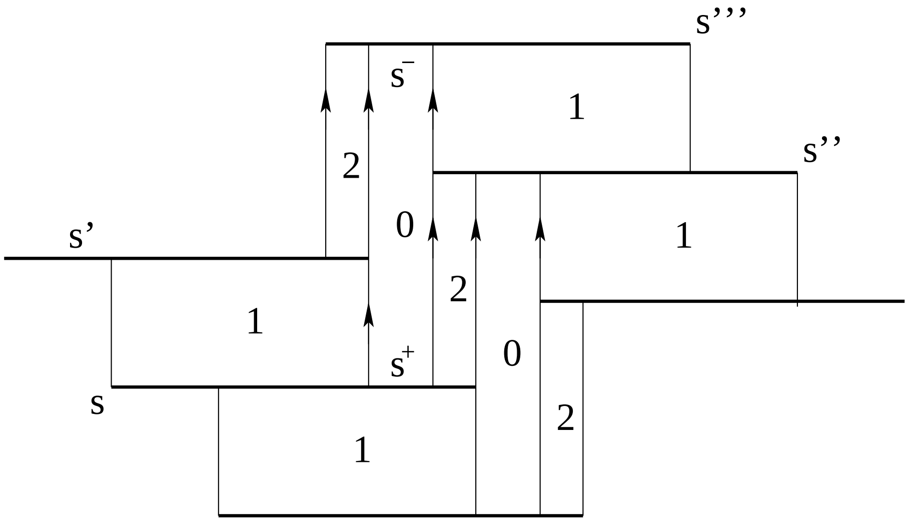

Fig 11: The 3-street model.

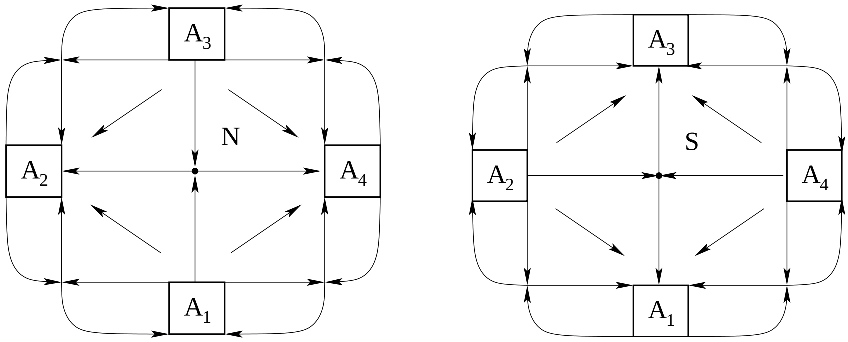

Fig 12: The Maximal Foliation: North Hemisphere(left) and South Hemisphere(right)

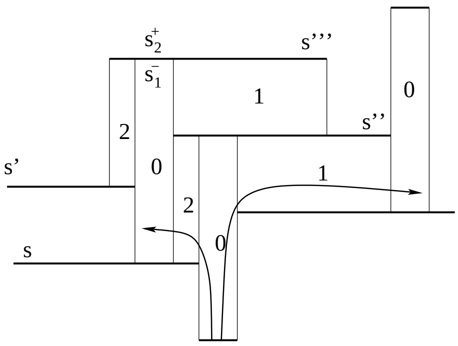

Fig 13: $a^* : 0 \to 2 \to 0$, $b^* : 0 \to 1 \to 0$.

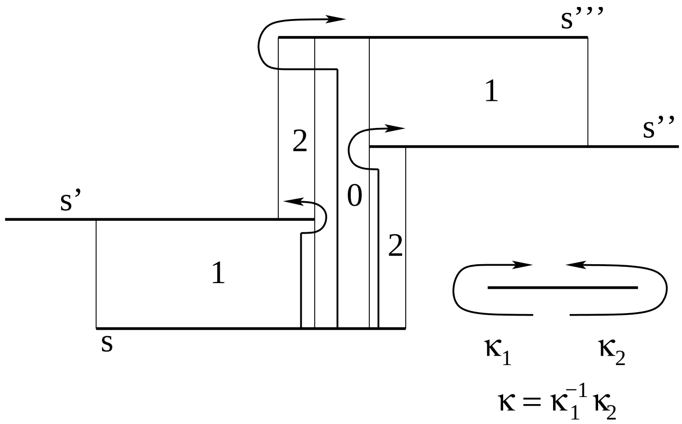

Fig 14. The elements of π_1 assigned to streets

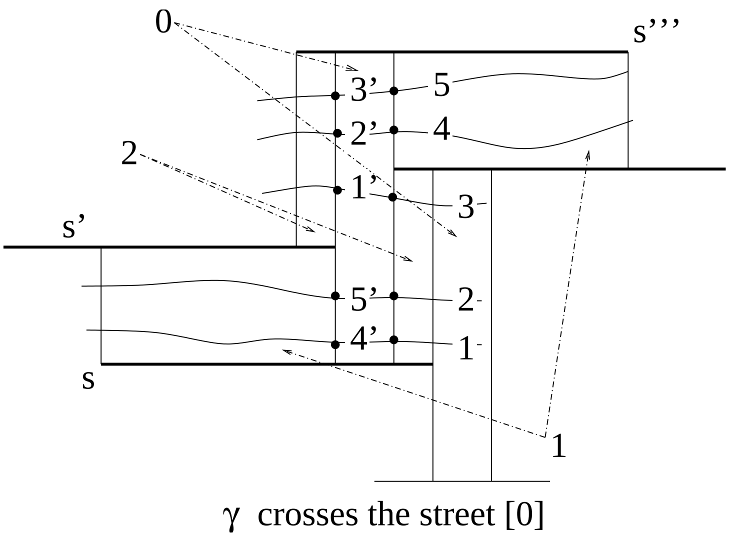

Fig 15:The simple transversal curve: $k = 3$, $l = 2$.