Holomorphic bundles and commuting difference operators. Two-point constructions

I. M. Krichever and S. P. Novikov

This paper is closely related to our previous paper [1]. As is well known, in current mathematical physics the theory of commuting one-dimensional operators appears as an auxiliary algebraic aspect of the theory of integration of non-linear soliton systems and the spectral theory of periodic finite-zone operators [2]–[5].

As a purely algebraic problem, the problem of classifying ordinary scalar differential operators had already been posed in the 1920s by Burchnall and Chaundy [6], who advanced a long way towards a solution of the problem in the case of operators of mutually prime orders (in which the rank is always equal to 1), completed in [3]. However, they remarked that the general problem for rank $r > 1$ seemed extraordinarily difficult.

The first steps were taken in [7] and [8]. A method of effective classification of commuting differential operators of rank $r > 1$ in general position was created by the authors in [9] and [10]. Commuting pairs of rank $r > 1$ depend on $(r-1)$ arbitrary functions of one variable, a smooth algebraic curve Γ with *one* distinguished point P and a set of Tyurin parameters (characterizing a framed stable holomorphic bundle). We call these *one-point* constructions.

For difference operators the whole theory, which has already become classical, of pairs of commuting operators of rank $r = 1$ was based only on *two-point constructions* [11], [12]. Rings of such operators turned out to be isomorphic to the rings $A(\Gamma, P^{\pm})$ of meromorphic functions on an algebraic curve Γ with poles at a pair of distinguished points $P^{\pm}$.

In our previous paper [1] we showed that for rank $2l \geq 2$ a broad class of commuting difference operators can be obtained from the one-point construction. As in the continuous case, these operators depend on arbitrary functions of one variable $n \in \mathbb{Z}$.

In the present paper we have obtained a description of a broad class of commuting difference operators constructed starting from two-point constructions. In contrast to one-point constructions, there are no arbitrary functions here; the coefficients of the operators can be calculated by means of the Riemann theta function. As in the rank 1 case, these operators lead to solutions of the equations of the 2D Toda lattice and the whole hierarchy connected with them.

We consider a smooth algebraic curve Γ of genus g with two distinguished points $P^{\pm}$. Let (γ) be a collection of rg points γ_s, $s = 1, \dots, rg$, on Γ, and (α) be a collection of $(r-1)$-dimensional vectors: $\alpha_s = (\alpha_{sj})$, $j = 1, \dots, r-1$. According to [9], [10], these parameters (γ, α) are called Tyurin parameters. In the general case they determine a stable framed bundle $\mathcal{E}$ over Γ of rank r and degree $c_1(\det \mathcal{E}) = rg$.

Lemma 1. *For any choice of Tyurin parameters (γ, α) in general position there exists a unique meromorphic vector function $\psi_n(Q) = (\psi_n^i(Q))$, $i = 1, \dots, r$, $Q \in \Gamma$ such that*: (i) *outside $P^{\pm}$ the functions $\psi_n^i(Q)$ have no more than simple poles at the points γ_s, and their residues at these points satisfy the relation $\alpha_{si} \operatorname{res}_{\gamma_s} \psi_n^i = \operatorname{res}_{\gamma_s} \psi_n^r$*; (ii) *in a neighbourhood of the distinguished points $\psi_n(Q)^i$, $n = kr + j$, $0 \leq j < r$, has the form*

$$\psi_n^i = z_{\pm}^{\mp k} \big(\xi_{n,\pm}^i + O(z_{\pm}) \big), \tag{1}$$

where $\xi_{kr+j,+}^j = 1$, $\xi_{kr+j,+}^i = 0$ for $i > j$, and $\xi_{kr+j,-}^i = 0$ for $i < j$ (here $z_{\pm}$ are local coordinates in neighbourhoods of $P^{\pm}$).

The vector function defined above is the analogue of the Baker-Akhiezer difference function for the higher-rank case.

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Theorem 1. *Let $\psi_n^i(Q)$ be the Baker-Akhiezer vector function corresponding to the choice of algebro-geometric data $\{\Gamma, P^\pm, (\gamma,\alpha)\}$. Then for any function $f(Q) \in A(\Gamma, P^\pm)$ there exists a unique difference operator L_f of the form*

$$L_f = \sum_{i=-rn_-}^{rn_+} u_i(n)T^i, \qquad u_{\pm rn_\pm} = f_\pm \neq 0, \tag{2}$$

such that $L_f\psi_n^i(Q) = f(Q)\psi_n^i(Q)$, $i = 1,\dots,r$. Here $Ty_n = y_{n+1}$ is the shift operator and $n_\pm$ are the orders of the function $f(Q)$ at the points $P^\pm$.

Both the function ψ and the coefficients of the operators can be explicitly calculated in terms of the Riemann theta function corresponding to the curve Γ. The coefficients of the operators L_f are periodic functions if and only if $A(P^+) - A(P^-)$ is a point of finite order on the Jacobian $J(\Gamma)$.

Remark 1. We call attention to the fact that there are no functional parameters in this construction even in the case of rank $r > 1$. It follows from (2) that among the operators L_f there are both those for which all shifts T^i are positive, $i > 0$, and those for which all shifts are negative, $i < 0$. One-point constructions never lead to such operators. Here we also arrive naturally at the construction of representations of a version of the Kac-Moody algebra $\widehat{sl}(r, C)$ associated with the algebraic curve Γ with distinguished points $P^\pm$ following the plan of [13].

Remark 2. For any rank $r > 1$ the authors have also constructed two-point Baker-Akhizer vector functions giving the solutions of the complete hierarchy of equations of the $2D$ Toda lattice. They depend on $2(r-1)$ arbitrary functions. This construction contains the theory of integrable potentials of the two-dimensional Schrödinger operator connected with bundles of higher rank begun by the authors in [14].

We consider the ring $\mathcal{D}$ of difference operators of finite order. By Theorem 1 a choice of Tyurin parameters (γ,α) in general position defines a homomorphism $G_{(\gamma,\alpha)}\colon A(\Gamma, P^\pm) \mapsto \mathcal{D}$, whose image is a maximal commutative subring. The following theorem shows that the proposed construction describes all similar rings in general position.

Theorem 2. *For any ring monomorphism $G\colon A(\Gamma, P^\pm) \mapsto \mathcal{D}$ such that the operators $L_f = G(f)$ have the form* (2), *the normalized joint eigenfunctions determine a framed holomorphic bundle of rank r and degree rg. In general position, when this bundle is described by Tyurin parameters (γ,α), the homomorphism G coincides with the homomorphism $G = G_{(\gamma,\alpha)}$ defined by virtue of* Theorem 1 *with the aid of the corresponding Baker-Akhiezer vector function.*

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Columbia University, New York
University of Maryland, College Park
Landau Institute of Theoretical Physics

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