Laplace transforms and simplicial connections

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Analogues of Laplace transforms for operators on the discrete lattice $\mathbb{Z}^2$ were constructed in [1], and it was shown that to do this we had to consider a *regular triangular lattice* generated by basic shifts T_1, T_2, such that $|T_1| = |T_2| = |T_1T_2^{-1}|$. Each vertex interacts with its six equidistant nearest neighbours. See [1] and [2] for the definition of a discrete Laplace transform and the notations.

As remarked in [1], a regular triangular lattice has six equivalent bases and this gives us three pairs of mutually inverse Laplace transforms. An operator transformation $L \mapsto fLf$, where f is a non-vanishing function on the lattice $\mathbb{Z}^2$, is called a *weak equivalence*. The Laplace transform is well defined on the weak equivalence classes [3].

Proposition 1. *For bases that differ by rotation through an angle of $2\pi/3$ the corresponding Laplace transforms on the weak equivalence classes are conjugate under a shift by a vector of the lattice $\mathbb{Z}^2$.*

Thus, the algebra of Laplace transforms on a regular lattice reduces (to within weak equivalence and shifts) to two mutually inverse transformations. These two transformations can be described in another way.

We colour the triangles forming the lattice like a chessboard: triangles of the form $\langle n, n+T_1, n+T_2\rangle$ are coloured black, and those of the form $\langle n, n-T_1, n-T_2\rangle$ white. Let V_1, V_2, V_3 be the spaces of functions of the vertices, the black triangles, and the white triangles respectively. Two vertices will be called *nearest neighbours* if they are joined by an edge, and two triangles are nearest neighbours if they are of the same colour and have a common vertex. By a *second-order operator* on V_i we mean an operator $L\colon V_i \to V_i$, the non-zero off-diagonal matrix elements of which correspond to pairs of nearest neighbours.

Proposition 2. *A self-adjoint operator $L\colon V_i \to V_i$ admits a representation in the form $L = Q_{ji}Q_{ji}^{+} + w$, where $i \neq j$, and $Q_{ji}\colon V_j \to V_i$ is an operator whose non-zero matrix elements correspond, depending on i and j, either to a pair of triangles of different colours having a common edge, or to a pair consisting of a triangle and one of its vertices. The Laplace transform $L \mapsto \widetilde{L} = Q_{ji}^{+}w^{-1}Q_{ji} + 1$ coincides (on weak equivalence classes) with the Laplace transform introduced earlier on a regular triangular lattice with a suitable identification of the spaces V_i and V_j.*

Such a Laplace transform associates an operator on V_i with an operator on V_j, giving a homomorphism of zero levels: $\psi \mapsto Q_{ji}^{+}\psi$.

Let M be an arbitrary two-dimensional surface with a triangulation in which the triangles are coloured black and white in such a way that any two triangles with a common edge have different colours. We define the spaces V_1, V_2, V_3 and the nearest neighbour relation as before.

Proposition 3. *If the triangulated surface M has only fourfold and sixfold vertices, then the self-adjoint operators of the second order on V_i, $i = 1, 2, 3$, admit a factorization and Laplace transforms just as in the case of a regular lattice in the plane. However, if the triangulation has fourfold vertices, then the factorizations of operators on V_2 and V_3 are not unique, and the degree of non-uniqueness is determined by the number of fourfold vertices.*

Remark. If the triangulation of M has vertices of multiplicity > 6, then the Schrödinger operator on the black (white) triangles only factorizes if additional conditions are satisfied; here the conditions for factorization 'through vertices' and 'through white (respectively, black) triangles' are incompatible. For more details see the survey [4].

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Let K be an arbitrary simplicial complex. Simplexes of dimension l will be denoted by σ^l. Let us fix some $k < l$. We shall call two simplexes σ_1^l and σ_2^l *nearest* if their intersection contains a simplex σ^k. Let L be a Schrödinger operator acting on functions of the l-simplexes in such a way that only nearest neighbours interact. By a *vector factorization of the operator L by means of k-simplexes* we mean a representation of L in the form: $L = \sum_{\alpha=1}^m Q^\alpha (Q^\alpha)^+ + w$, where w is the operator of multiplication by the function, and $Q^+ = ((Q^1)^+, \dots, (Q^m)^+)$ is an operator on functions of l-simplexes with values in the m-vector-functions of k-simplexes:

$$(Q^+\psi)^\alpha_{\sigma^k} = \sum_{\sigma^l \supset \sigma^k} b^\alpha_{\sigma^k:\sigma^l} \psi_{\sigma^l}. \tag{1}$$

We call a factorization of L *special* if $w = \text{const}$. To a vector factorization there corresponds a transformation of Laplace type: $L \mapsto \widetilde{L} = Q^+ w^{-1} Q + 1$, which associates with L an operator $\widetilde{L}$ on vector functions of k-simplexes.

Similarly we may define a factorization of an operator acting on l-simplexes by means of k-simplexes in the case $k > l$: we shall regard two l-simplexes as nearest neighbours if they are contained in some k-simplex.

As an example let us consider the *lattice of regular tetrahedra* in $\mathbb{R}^3$: as a basis T_1, T_2, T_3 of the lattice $\mathbb{Z}^3$ we choose the vectors joining a certain vertex of a regular tetrahedron in $\mathbb{R}^3$ to the remaining vertices. We consider the complex K consisting of the tetrahedra obtained from the given one by all possible shifts by vectors of $\mathbb{Z}^3$. Suppose that the operator L acts on $\mathbb{Z}^3$, where vertices only interact if they are joined by an edge.

Proposition 4. *A scalar factorization of the operator L by means of 3-simplexes of the complex K is possible if and only if for each tetrahedron in K the products of the connectedness coefficients situated on opposite edges coincide. A special m-vector factorization of L by means of 3-simplexes is possible for any $m > 1$, and moreover is not unique.*

In several cases the search for fundamental states of Schrödinger operators on l-simplexes reduces to the solution of the system of equations $Q^+\psi = 0$, where the operator Q^+ has the form (1). We call such a system of equations a *simplicial connection* if it satisfies the *non-degeneracy requirement* and the *localization requirement*, whose definitions can be found in [4]. The conditions for local compatibility of such a system lead to discrete analogues of curvature.

An example of a simplicial connection is given by the *triangle equations* considered in [3]. The *tetrahedra equations* in $\mathbb{R}^3$ are a generalization of them:

$$Q_1\psi = 0, \quad Q_2^+\psi = 0, \quad \text{where} \quad Q_{1,2} = 1 + x_{1,2}T_1 + y_{1,2}T_2 + z_{1,2}T_3. \tag{2}$$

Proposition 5. *A condition for the complete local compatibility of the system* (2) *is the relation* $(Q_1 - 1)(Q_2^+ - 1) - 1 = f((Q_2^+ - 1)(Q_1 - 1) - 1)$, *where f is a function on $\mathbb{Z}^3$.*

For other examples of simplicial connections and discrete analogues of the Laplace transform see [4].

Bibliography

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