Exactly soluble periodic two-dimensional Schrödinger operators

A. P. Veselov and S. P. Novikov

In this paper, originating from the results [1] on cyclic chains of Backlund–Darboux transforms in the one-dimensional case, we use a *Laplace transform* to construct exactly soluble periodic operators $2L = (\bar\partial + B)(\partial + A) + 2V$. Here A, B, V are functions on the (x, y)-plane, $z = x + iy$, $\partial = \partial_x - i\partial_y$. We call V the *potential* and $H = (B_z - A_{\bar z})/2$ the *magnetic field*. The Laplace transform associates the solution of the equation $L\psi = 0$ with the solution of a new equation $\widetilde{L}\widetilde{\psi} = 0$, where $\widetilde{\psi} = (\partial + A)\psi$, $2\widetilde{L} = (\bar\partial + \widetilde{B})(\partial + \widetilde{A}) + 2\widetilde{V} = (\partial + \widetilde{A})(\bar\partial + \widetilde{B}) + 2V$, $\widetilde{A} = A - (\ln V)_z$, $\widetilde{B} = B$. For the potential and the magnetic field the transform has the form $\widetilde{V} = V + \widetilde{H}$, $\widetilde{H} = H + \frac{1}{2}(\ln V)_{z\bar z}$. It appeared in classical geometry at the beginning of the 19th century with aims far removed from ours and where the operator L was always hyperbolic: $z \to \xi$, $\bar z \to \eta$ (see [2]). An infinite chain of Laplace transforms $V_n = V_{n-1} + H_n$, $H_n = H_{n-1} + \frac{1}{2}\partial\bar\partial f_{n-1}$, $V_n = \exp f_n$, after transition to functions $g_n - g_{n-1} = f_n$, coincides with a *two-dimensionalized Toda chain*, which is widely known in the theory of solitons. As Ferapontov has informed the authors, this circumstance was known to A. M. Vasil'ev but was not utilized in any way. Cyclic chains of Laplace transforms were considered by Darboux and Tzitzéica [3] who made a number of valuable formal calculations. These authors did not, however, pose global problems.

In formulating global problems we assume that the operator L is elliptic and that the potential V and magnetic field H are doubly-periodic real functions on a plane with vectors of periods T_1, T_2 and an elementary cell K. We call the integral $\iint_K H\,dx\,dy = [H]$ the *flow of the magnetic field*. Suppose that $[H]$ exists. We call the case $[H] = 0$ *topologically trivial* and the case $[H] \neq 0$ *topologically non-trivial*.

Proposition 1. *Every finite-gap (or algebraic-geometric) solution of a two-dimensionalized Toda chain for any n determines a two-dimensional periodic (quasi-periodic) topologically trivial Schrödinger operator that is finite-gap with respect to the zero level of energy (that is, the Fermi curve has finite genus), and conversely, where the displacement $n \to n+1$ corresponds to a shift by a constant vector (the difference of two points at infinity) on the Jacobi manifold for the Fermi curve.*

The corresponding Schrödinger operators were first constructed in [4]. For the finite-gap solutions of a two-dimensionalized Toda chain see [5].

Proposition 2. *Suppose that a chain of Laplace transforms is cyclic: $V_{i+n} \equiv V_i$, and all the potentials $V_i = \exp f_i$ are doubly-periodic smooth real positive functions. Then for all k the Schrödinger operators L_k are algebraic-geometric (finite-gap) with respect to the zero level of energy $L_k\psi = 0$. All these operators are topologically trivial.*

Examples. 1) $n = 1$. Here $V_1 = V_0$, $H_1 = H_0 = 0$, $\partial\bar\partial f_0 = 0$. In the smooth doubly-periodic case we have $f_0 = \text{const}$.

2) $n = 2$. Here $V_2 = V_0$, $H_2 = H_0 + \frac{1}{2}\partial\bar\partial(f_0 + f_1)$. In the smooth doubly-periodic case $f_0 = -f_1 + a = g + a/2$, $\partial\bar\partial g = -8e^{a/2}\sinh g$, $a = \text{const}$. Proposition 2 for $n = 2$ follows from the results stated in [6].

3) Let n be arbitrary but $H_0 = H_n = 0$, that is, $2L_0 = \partial\bar\partial + 2V = 2L_n$, $V = V_0 = V_n$. For $n = 3$ we obtain first the non-trivial operators that in the smooth doubly-periodic positive case satisfy Tzitzéica's equation $\partial\bar\partial g = 2e^{a/3}(e^{-2g} - e^g)$, $f_0 = g + a/3$, $a = \text{const}$. For $n = 4$ under the same conditions we obtain $f_3 = f_0$, $f_0 + f_1 + f_2 + f_3 = a$, $3f_0 + 2f_1 + f_2 = b$, $\partial\bar\partial g = -4e^{(b-a)/2}\sinh g$, $g = f_0 + (b - a)/2$, a and b are constants.

In the topologically non-trivial case $[H_0] \neq 0$ we introduce two types of chains.

I. *Semicyclic chains*, where $V_n = V_0 + C_n$, $H_n = H_0$. Here $|K|C_n = n[H_0]$. For $n = 1$ we have only the Landau operator $V_0 = \text{const}$, $H_0 = \text{const}$. For $n = 2$ we arrive at the equation

$\Delta g = -2C_2 - 8e^{a/2}\sinh g$, $g + a/2 = f_0$ in the smooth doubly-periodic case. Curiously this system was pointed out in [7] as a reduction of the two-dimensionalized Toda chain. Here two levels $L_0\psi_0 = 0$ and $L_0\psi_n = -2C_n\psi_n$ are connected by the relation $\psi_n = Q_{n-1}\dots Q_0\psi_0$, where $Q_j = e^{f_j/2}(\partial + A_j)$ in the *real* calibration $B_j = -\overline{A}_j$, $\operatorname{Im}(\overline{\partial}A_j) = 0$. **If zero is a point of the spectrum, then $-2C_n$ is also a point of the spectrum.**

II. *Quasi-cyclic chains* (principal case). Here $V_0 = H_0$, $V_n = H_n + C_n$, where $[H_0] = [H_n] > 0$. The Pauli vector operator P for a particle with spin 1/2 in the magnetic field and zero electric field is the direct sum of two scalars: $P_0 = \nabla_0\nabla_0^+ \oplus \nabla_0^+\nabla_0$, $P_n = \nabla_n\nabla_n^+ \oplus \nabla_n^+\nabla_n$, $\nabla_j = \overline{\partial} + B_j$ in the 'real' calibration, $L_0 = \nabla_0^+\nabla_0$, $L_n = \nabla_n^+\nabla_n + 2C_n$. For $[H_j] > 0$ the spectrum $\nabla_j^+\nabla_j$ begins with zero; the eigenfunctions of $L_0\psi_0 = 0$ and $(L_n - 2C_n)\psi_n = 0$ in the rapidly decreasing [8] and periodic [9] cases are found in the form $\nabla_j\psi_j = (\overline{\partial} + B_j)\psi_j = 0$, $j = 0, n$. In the periodic case these are magnetic Bloch waves for an integral value of the flow $[H_0] = [H_n] = 2\pi m$, $m > 0$ of the form $\psi_j = (\exp\varphi_j)\sigma(z - a_1)\cdots\sigma(z - a_m)e^{\alpha_j z}$, $j = 0, n$, where $\Delta\varphi_j = -H_j$, $a = (a_1, \dots, a_m)$. We obtain the level of $L_n\widetilde{\psi}_n = 0$ in the form $\widetilde{\psi} = Q_{n-1}\dots Q_0\psi_0$.

Theorem. *Suppose that all the potentials V_k, $k = 0, \dots, n$, of a quasi-cyclic chain are smooth, positive and doubly-periodic on the (x, y)-plane, where $V_0 = H_0$, $V_n = H_n + C_n$ and the flow $[H_0] = [H_n]$ is integral. Then the operator L_n has two exactly soluble isolated levels in $\mathcal{L}_2(\mathbb{R}^2)$, $L_n\psi = 0$ and $L_n\psi = 2C_n\psi$, that are isomorphic to the Landau levels for the case $H_0 = H_n =$* const, $V_0 = V_n =$ const.

Example. Let $n = 2$. The hypothesis of the theorem reduces to the equation $\Delta f_0 = 2C_2 - 4e^{f_0}$. For the operator L_2 we have $V_2 = \exp f_2 = 2C_2 - e^{f_0}$, $H_2 = C_2 - e^{f_0}$, $[H_0] = [H_2] = C_2/2$. From the substitutions $z' = 2z$, $g + \text{const} = f_0$ we obtain $\Delta g = 1 - e^g$. This equation has many solutions periodic in x and not depending on y. In the y-calibration we have a reduction to the one-dimensional problem: $\psi = e^{iky}\varphi(x)$ and obtain new *oscillator-like* families of Schrödinger operators $M = \partial_x^2 - V(x, k)$ that for all $k \in \mathbb{R}$ have two levels, exactly soluble and not depending on k. De Vega has pointed out to the authors that the equation $\Delta g = 1 - e^g$ arose as an instanton reduction of the Ginzburg–Landau equation for a critical value of the parameter separating types I and II (see [10]). One of the authors (Novikov) has suggested a curious method of solving this equation and also a difference analogue of the theory of Laplace transforms.

Bibliography

[1] A. P. Veselov and A. B. Shabat, *Funktsional. Anal. i Prilozhen.* **27**:2 (1993), 1–21; English transl. in *Functional Anal. Appl.* **27** (1993).

[2] S. P. Finikov, *The theory of congruences*, Gostekhizdat, Moscow–Leningrad 1950. (Russian)

[3] G. Tzitzéica, *Géométrie différentielle projective des réseaux*, Paris–Bucarest 1924.

[4] B. A. Dubrovin and I. M. Krichever, *Dokl. Akad. Nauk SSSR* **229** (1976), 15–18; English transl. in *Soviet Math. Dokl.* **17** (1976).

[5] I. M. Krichever, *Uspekhi Mat. Nauk* **36**:2 (1981), 72–80[1]; English transl. in *Russian Math. Surveys* **36**:2 (1981).

[6] A. I. Bobenko, *Uspekhi Mat. Nauk* **46**:4 (1991), 3–42; English transl. in *Russian Math. Surveys* **46**:4 (1991).

[7] D. Levi, O. Ragnisco, and A. B. Shabat, *Canad. J. Phys.* **72** (1994), 439–441.

[8] Y. Aharonov and A. Casher, *Phys. Rev. A* **19** (1979), 2461–2462.

[9] B. A. Dubrovin and S. P. Novikov, *Dokl. Akad. Nauk SSSR* **253** (1980), 1293–1297; English transl. in *Soviet Math. Dokl.* **22** (1980).

[10] H. J. de Vega and F. A. Schaposnik, *Phys. Rev. D* **14** (1976), 1100–1106.

Institute of Theoretical Physics
Russian Academy of Sciences

Received 12/SEP/95

[1] This is an appendix to an article by B. A. Dubrovin.