

ON THE EQUATION $[L, A] = \varepsilon \cdot 1$

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We shall investigate the equation $[L, A] = \varepsilon \cdot 1$ for the second order Schrödinger operator $L = -\partial_x^2 + u$ and some odd-order operator A using the ideas originated from soliton theory [?] (after the recent papers [?]-[?]).

The first nontrivial case is

$$(1) \quad L = -\partial_x^2 + u, \quad A = -4\partial_x^3 + 6u\partial_x + 3u',$$

$$(2) \quad [L, A] = \varepsilon \cdot 1 \leftrightarrow u''' = 6uu' + \varepsilon.$$

Consider the operator \tilde{L} and Eq. (3):

$$(3) \quad \tilde{L} = L - (\lambda + \varepsilon t) \cdot 1, \quad \partial \tilde{L} / \partial t = [\tilde{L}, A].$$

Equation (2) is obviously the stationary equation for (3). As normally in the soliton theory [?]-[?] we shall present Eq. (3) in the form

$$(4) \quad \left[\frac{\partial}{\partial t} - \Lambda, \frac{\partial}{\partial x} - Q \right] = 0, \quad \lambda \rightarrow \lambda + \varepsilon t,$$

where the matrices (Λ, Q) depend polynomially on $\lambda + \varepsilon t$ and $(u(x, t), u'(x, t), \dots)$. In the case of Eqs. (1) and (2) we have

$$(5) \quad \Lambda = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \begin{aligned} a &= -u', & b &= 2u + 4\lambda, \\ c &= -4\lambda^2 + 2\lambda u + 2u^2 - u'', \end{aligned}$$

$$(6) \quad Q = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix}, \quad \lambda \rightarrow \lambda + \varepsilon t.$$

The stationary problem (2) we shall obtain from (4) by replacing

$$\frac{\partial}{\partial t} \rightarrow \varepsilon \frac{\partial}{\partial \lambda}, \quad t = 0.$$

So Eq. (2) is equivalent to the Lax-type equations:

$$(7) \quad [\varepsilon \partial_x - \Lambda, \partial_x - Q] = 0,$$

$$(8) \quad \partial \Lambda / \partial x = [Q, \Lambda] + \varepsilon \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

We have to find the common solution of two linear equations (9) and (10):

$$(9) \quad \varepsilon \frac{\partial \psi}{\partial \lambda} = \Lambda \psi,$$

$$(10) \quad \frac{\partial \psi}{\partial x} = Q \psi.$$

For $\varepsilon = 0$ we have the situation standard in the “finite-zoned” periodic theory [?]-[?]. Our main idea in studying (2) is to introduce the Riemann surfaces analogous to Riemann surfaces in the “finite-zoned” periodic theory.

STEP 1. Introduce the matrix U such that $U\Lambda U^{-1} = \text{diagonal}$, $U^{-1}\tilde{\psi} = \psi$, and we find U^{-1} :

$$(11) \quad U^{-1} = \begin{pmatrix} 1 & 1 \\ \delta & f \end{pmatrix}, \quad U\Lambda U^{-1} = \begin{pmatrix} \sqrt{R} & 0 \\ 0 & -\sqrt{R} \end{pmatrix},$$

$$\delta = -(a - \sqrt{-\det \Lambda})/b, \quad f = -(a + \sqrt{-\det \Lambda})/b,$$

$$(12) \quad R(\lambda) = -\det \Lambda = a^2 + bc = -16\lambda^3 - 4C\lambda - \mathcal{D}.$$

From (2) and (12) we obtain the new useful form of this equation:

$$(13) \quad \mathcal{D}' = 2\varepsilon u, \quad C = \varepsilon x + C_0, \quad C_0 = \text{const},$$

$$(14) \quad \mathcal{D} = 2u^3 + (\varepsilon x + C_0)u - (u')^2, \quad C = u'' - 3u^2,$$

$$(15) \quad R'(\lambda) = -\varepsilon b, \quad (\mathcal{D}''/2\varepsilon)^2 = R(-\mathcal{D}'/4\varepsilon) = R(-u/2).$$

For Eqs. (9) and (10) we have the ‘‘gauge’’ transformations

$$(16) \quad \varepsilon\tilde{\psi}_\lambda = [-\varepsilon U(U^{-1})_\lambda + U\Lambda U^{-1}]\tilde{\psi} = (\Lambda_0 + \Lambda_1^*\varepsilon + \Lambda_1^d\varepsilon)\tilde{\psi},$$

$$(17) \quad \tilde{\psi}_x = [-U(U^{-1})_x + UQU^{-1}]\tilde{\psi} = (Q_0 + \varepsilon Q_1^*)\tilde{\psi},$$

$$\Lambda_0 = \begin{pmatrix} \sqrt{R} & 0 \\ 0 & -\sqrt{R} \end{pmatrix}, \quad \Lambda_1^* = \frac{1}{f - \delta} \begin{pmatrix} 0 & f_\lambda \\ -\delta_\lambda & 0 \end{pmatrix},$$

$$\Lambda_1^d = \begin{pmatrix} 2u'/b\sqrt{R} & 0 \\ 0 & -2u'/b\sqrt{R} \end{pmatrix} + t_\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} \sqrt{R}/b & 0 \\ 0 & -\sqrt{R}/b \end{pmatrix} + t_x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_1^* = \begin{pmatrix} 0 & -b/4R \\ b/4R & 0 \end{pmatrix},$$

$$t = -\frac{1}{2} \ln(f - \delta), \quad f - \delta = \det U^{-1}.$$

The quantities (δ, f) satisfy the ‘‘almost Riccati’’ equations:

$$(18) \quad \delta' + \delta^2 - (u - \lambda) = O(\varepsilon) = -\varepsilon b/4R,$$

$$f' + f^2 - (u - \lambda) = O(\varepsilon) = \varepsilon b/4R,$$

(because the matrix εQ_1^* is equal to (19)):

$$(19) \quad \varepsilon Q_1^* = \begin{pmatrix} 0 & f' + f^2 - (u - \lambda) \\ -\delta' - \delta^2 + (u - \lambda) & 0 \end{pmatrix}.$$

Important remark. The matrix $\tilde{\psi}$ should be 1-valued matrix function on the Riemann surface $\Gamma: \mu^2 = R(\lambda)$.

STEP 2. We solve Eq. (16) semiclassically in the standard form (see [?])¹

$$(20) \quad \tilde{\psi} = R^{1/4}(1 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots) \exp \left\{ \frac{1}{\varepsilon} [B_0 + \varepsilon B_1 + \dots] \right\} V,$$

$$V_\lambda = 0, \quad V = V(x),$$

such that:

- a) All matrixes $A_j(\lambda)$ are antidiagonal and algebraic functions on Γ .
- b) All matrixes $B_j(\lambda)$ are diagonal and all differential forms $d_\lambda B_j$ are algebraic on Γ .

¹Volowitch pointed out the author the factor $V(x)$ is important for the second equation in x (see Vladimirov and Volowitch, 1984, Trieste IC/84/128).

After the elementary calculations we obtain

$$(21) \quad \begin{aligned} B_{0\lambda} &= \Lambda_0, & B_{1\lambda} &= \Lambda_1^d, & B_{q\lambda} &= \Lambda_1^* A_{q-1}, & q &\geq 2, \\ A_q &= (\text{ad } \Lambda_0)^{-1}(H_q), & \text{ad } \Lambda_0(Y) &= [\Lambda_0, Y], \\ H_1 &= -\Lambda_1^*, & H_2 &= A_{1\lambda} + [A_1, \Lambda_1^d], & q &\geq 3, \end{aligned}$$

$$(22) \quad H_q = A_{q-1,\lambda} + [A_{q-1}, \Lambda_1^d] + \sum_{j=1}^{q-2} A_j \Lambda^* A_{q-1-j}.$$

From (21) and (22) we deduce the important conclusion: Matrix $\tilde{\psi}$ has the asymptotic (23) for $\lambda \rightarrow \infty$

$$(23) \quad R^{1/4} \tilde{\psi}_{11} = (\text{const}) k \exp \left\{ \frac{8}{5\varepsilon} k^5 + k \left(x + \frac{C_0}{\varepsilon} \right) \right\} \left(1 + \frac{\mathcal{D}}{4\varepsilon} k^{-1} + O(k^{-2}) \right), \\ k = (-\lambda)^{1/2}.$$

STEP 3. Our important conjecture is that the diagonal matrix-function $R^{-1/4} \tilde{\psi}_{11}^{(1)}(\lambda, x) = \exp\{B_0/\varepsilon + B_1\}$ may be found as the first 1-valued approximation in ε on the surface Γ for the function $\tilde{\psi}$ satisfying (16). We have the differential 1-form $\Omega = d_\lambda \ln \tilde{\psi}_{11}^{(1)} \Omega = B_{0\lambda}/\varepsilon + B_{1\lambda} = (\sqrt{R}/\varepsilon + u'/(u + 2\lambda)\sqrt{R} - (1/2)d_\lambda \ln t)$. Its analytical properties are:

(a) Ω has exactly 2 poles on Γ in the points (γ, ∞) :

- (1) $\gamma = (-u/2, +)$, $\text{res}_\gamma \Omega = +1$, $\text{Ord}_\gamma \Omega = 1$.
- (2) ∞ , $\text{res}_\infty \Omega = -1$, $\text{Ord}_\infty \Omega = 6$.

(b) $\exp(\int^\lambda \Omega) = R^{-1/4} \tilde{\psi}_{11}^{(1)}$ has the asymptotic (23) for $\lambda \rightarrow \infty$.

(c) All periods of Ω along the cycles on the surface Γ belong to $(2\pi i) \times \mathbb{Z}$.

We have therefore the special case of the so-called ‘‘Baker–Akhiezer functions’’ in the ‘‘finite-zoned’’ periodic theory (see [?, ?, ?]), but Riemann surface Γ depends on x ‘‘slowly’’ (g_2, g_3 depend on x). We have:

$$(24) \quad \begin{aligned} \mathcal{D}/4\varepsilon = F(x, g_2, g_3) &= -\zeta(x + A \mid g_2, g_3), & g_2, g_3 &\text{—fixed,} \\ A &= -x_0 + \frac{8}{5\varepsilon} U_6/U_2. \end{aligned}$$

We may deduce now from (12) and (13) that $g_2 = -(\varepsilon x + C_0)$, $4g_3 = -\mathcal{D}$,

$$U_{2j} = \oint_b \omega_{2j}, \quad \oint_a \omega_{2j} = 0, \quad \omega_{2j} \sim d(k^{2j-1}), \quad \lambda \rightarrow \infty.$$

Main result. Equation (??) is the transcendent equation for definition of \mathcal{D} .

After finding the solution of (21) we define:

$$(25) \quad 2\varepsilon u(x) = d\mathcal{D}/dx.$$

STEP 4. Only some very special solution of (2) is important in the ‘‘string’’ theory:²

$$(26) \quad \begin{aligned} u(x) &\sim u_0 = \sqrt{-\varepsilon x/3}, \\ \mathcal{D}(x) &\sim \mathcal{D}_0 = -4u_0^3. \end{aligned}$$

²Asymptotics for the general solution of (2) for $|x| \rightarrow \infty$, $x \in c$ has been investigated in 1913 by Boutroux (see remark in the Appendix).

Using the form (15) of (2), we shall see that for $x \rightarrow -\infty$ the Riemann surface Γ will have the form (see Fig. 1)

$$\begin{aligned}\mu^2 &= R(\lambda, x), & R_0 &= -16\lambda^3 - 4\varepsilon x\lambda - \mathcal{D}_0 = -16(\lambda + u_0/2)^2(\lambda - u_0), \\ R &= R_0 - (\mathcal{D} - \mathcal{D}_0), & \tau &= x^2 u_0 > 0, \\ \tilde{\mathcal{D}} &= \mathcal{D}/\mathcal{D}_0 = 1 + \sum_{n \geq 1} p_n \tau^{-n} = 1 + p \pmod{O(\tau^{-\infty})}, \\ \tilde{u} &= u/u_0 = 1 + \sum_{n \geq 1} q_n \tau^{-n} = 1 + q \pmod{O(\tau^{-\infty})}, \\ p_1 &= (16\tau)^{-1}, & q_1 &= -(24\tau)^{-1}, \\ \mathcal{D} - \mathcal{D}_0 &\sim +\varepsilon/12x, \\ u - u_0 &\sim -1/24x^2, & x &\rightarrow -\infty.\end{aligned}$$

FIGURE 1

For the important solution (26) the surface Γ is such that only one branching point is real and two are complex (near the point $-u_0/2$). The pole $(-u/2)$ of ψ is such that $-u/2 > -u_0/2$ on the real line. The exact position of the pole $(-u/2)$ may be found only from the exact solution. We shall compare the important solution (26) with the solutions (24) and (25) in the subsequent paper. *Our conjecture is that (??) and (??) describe exactly the important solution (??).*³

The general case $L = -\partial_x + u$, $A = \sum_{j=0}^n c_j A_{n-j}$ may be investigated exactly in the same way. The case of higher order operator is also clear (but less effective) if the orders of L and A are relatively prime (see [?]). For the operators of nonrelatively prime order see [?] and [?]. This case is more complicated.

APPENDIX (B. A. Dubrovin, S. P. Novikov)

We shall discuss here the “nonlinear semiclassics” for studying the equation $u''' = 6uu' + \varepsilon$ called “Whitham method”.

Theorem.⁴ *The solution may be found as a formal decomposition in ε*

$$(A1) \quad u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots,$$

such that

$$(A2) \quad \begin{aligned}u_0 &= 3\mathcal{P}\left(4g_2/5\varepsilon + \frac{i(\omega - \omega')}{\pi\varepsilon} + \frac{S_0(\omega + \omega')}{\pi\varepsilon} \mid g_2, g_3\right), \\ -g_2 &= \varepsilon x + g_2^0, & g_3 &= 2g_2 f - 5c/2(\omega + \omega'), & f &= \eta/\omega,\end{aligned}$$

g_2^0, c, S_0 being arbitrary constants, $\omega + \omega' \in R$, ω, ω' being periods.

Proof. As normally in Whitham method, we start from the family of exact solutions, $u = \varphi(k(x - x_0); g_2, g_3)$ of the equation with $\varepsilon = 0$, such that the function

$$(A3) \quad \varphi(\tau; g_2, g_3) = 2\mathcal{P}\left(\frac{\omega + \omega'}{\pi}\tau\right)$$

³P. Grinewitch pointed out to the author that this conjecture needs some corrections. Important result was obtained by Krichever.

⁴For $|x| \rightarrow \infty$ formula (A2) imply the old results of ... (1913).

is 2π -periodic in τ . The first term of the asymptotic decomposition (A1) has the form

$$(A4) \quad u_0 = \varphi \left(\frac{S(X)}{\varepsilon}; g_2(X), g_3(X) \right), \quad X = \varepsilon x.$$

$$\begin{aligned} a \cdot b &= 1 \\ a, b &\in H_1(\Gamma) \\ a &= [-\infty, \lambda_3], \quad = [\lambda_1, \lambda_3] \\ \lambda_1 \lambda_2 &\in C, \quad \lambda_3 \in R \end{aligned}$$

FIGURE 2

The averaged equations have the form

$$(A5) \quad dg_2/dX = -1, \quad d\bar{g}_3/dX = -f(g_2, \bar{g}_3).$$

For the phase we have

$$(A6) \quad \frac{dS}{dX} = k = -\frac{\pi}{\omega + \omega'}.$$

Lemma. *The averaged equations (??) may be presented in the “Flashka–McLaughlin form” (see Fig. 2)*

$$(A7) \quad -\frac{\partial w}{\partial X} = \frac{\partial p}{\partial \lambda}, \quad \oint_a dp(\lambda) = 0, \quad R(\lambda) = -4(\lambda^3 - g_2\lambda - g_3),$$

$w = \sqrt{R(\lambda)}$, $p(\lambda)$ —“quasimomentum”, i.e., $p(z) = (\zeta(z) - fz)$ and $\lambda = \mathcal{P}(z)$, $w = 2i\mathcal{P}'(z)$ on the surface Γ , $\mathcal{D} = -4g_3$.

The proof may be deduced from the Lax representation (3) of Eq. (2). \square

Corollary 1. *The quantity*

$$(A8) \quad 2ic = \oint_a w d\lambda = \int_0^{2(\omega+\omega')} (\mathcal{P}'(z))^2 dz, \quad c \in R$$

is the integral of (??).

From the value of this quantity we deduce the relations (A2).

Corollary 2. *The solution of (??) has the form*

$$(A9) \quad S = \frac{1}{2} \oint_b w d\lambda + S_0, \quad i \oint_b dp(\lambda) = \frac{\pi}{\omega + \omega'} = k.$$

We may see here that the result of Whitham-type method is in the good agreement with the results of more exact methods, developed in the main part of the paper for the special solution: phase S is the same as (23) in the leading term $\tilde{\psi}^{(1)}$ and the value of S (A9) is the averaged form of Eq. (24) in the main part of the paper.

We thank Krichever for the valuable advices and discussion.

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