ANALYTICAL THEORY OF HOMOTOPY GROUPS

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This article summarizes and developes some ideas initiated by the author in the works [1, 2] and oriented to the future analytical applications of the homotopy theory.

1. J.H.C.Whitehead's Formula for the Hopf Invariant and its Properties. Generalization of H-Property

Consider any smooth mapping $F\colon S^3\times\mathbb{R}\to S^2$ and some 2-form ω on S^2, $f_t(x)=F(x,t)$. Whitehead's formula for the Hopf invariant is well-known (in fact, this quantity originated from the classical hydrodynamics of XIX century)

$$H(f_t,\omega)=\int_{S^3\times t} v\bar\omega=\int (v,\operatorname{rot} v)d^3x,\qquad \bar\omega=F^*\omega=dv. \tag{1}$$

The quantity (1) has the important properties:

A. *H-property* (Homotopy property)

$$\frac{\delta H(f,\omega)}{\delta f}=0 \;\leftrightarrow\; \frac{dH(f_t,\omega)}{dt}=0.$$

Proof. $d(v\bar\omega)=\bar\omega^2=F^*(\omega^2)=0$ in $\Lambda^*(S^3\times\mathbb{R})$. □

B. *R-property* (Rigidity property)

$$\delta H/\delta\omega=0,\quad \int_{S^2}\omega=0,\quad \int_{S^2}\delta\omega=0.$$

Proof. If $\omega'=\omega+\delta\omega=\omega+\delta\tau$, $\bar\omega'=\bar\omega+d\bar\tau$ we have $dv=\bar\omega$, $dv'=\omega'$, $v'=v+\bar\tau=v+F^*\tau$

$$H(f_t,\omega')=\int v'\bar\omega'=\int(v\,d\bar\tau+\bar\tau\bar\omega+\bar\tau\,d\bar\tau)+H(f_t,\omega).$$

After integrating by parts we obtain

$$H(f_t,\omega')=H(f_t,\omega)\pm\int_{S^2}2\bar\omega\bar\tau=H(f_t,\omega).$$ □

C. *V-property* (Variational property).

Consider the space of all smooth mappings $g\colon S^2\to S^2$, $\deg g=0$. For any continuation $f\colon D^3\to S^2$, $f|_{\partial D^3}=g$, the integral (2) determines nonlocal multivalued functional on the space

$$H^+(g,\omega)=\int_{D^3} v\bar\omega,\quad \int_{S^2}\omega=1,\qquad \bar\omega=f^*\omega=dv,\quad d^*v=(*d*)v=0. \tag{2}$$

(The example pointed out to the author by Polyakov and Wiegman—see [1, 2]). The first variation (δH^+) s well-defined non-local closed 1-form nontrivial in the group $H^1(F,\mathbb{Z})$. All local examples were described by the author in [6]—"The Maxwell–Weiss–Zumino terms". The idea of generalization of (1) was introduced briefly in [3], p. 312. Some partial results were obtained in [4]. These papers discussed only the H-property of (1). The work [3] used unnaturally complicated language. The elementary and general approach including all properties (above) was developped in [1, 2]. Consider any d-algebra—i.e. skew commutative ("super") $\mathbb{Z}^+$-graded differential algebra $A=\sum_{i\geqslant 0} A_i$, $A_iA_j=(-1)^{ij}A_jA_i$, $d\colon A_i\to A_{i+1}$ over some field $K=\mathbb{Q},\mathbb{R},\mathbb{C}$. Suppose $H^0(A)=K$, $H^1(A)=0$. The main example is either the algebra of differential forms $A=\Lambda^*(M,K)$ for any smooth simply connected manifold or some subalgebra $A\subset\Lambda^*$. We shall construct the minimal d-extension $C_qA\supset A$

$$C_qA=A[\dots,v_{j\alpha},\dots],\quad \dim v_{j\alpha}=j,$$

such that all new generators $v_{j\alpha}$ are free and the next properties holds:

a) $0<\dim v_{j\alpha}\leqslant q-1$,

b) $H^j(C_qA)=0$, $j=1,\dots,q$.

Definition 1. The homotopy group of d-algebra is:

$$\pi_{q+1}(A)\otimes K=H^{q+1}(C_qA)^*$$

(see [1, 2]).

For any smooth $F\colon S^{q+1}\times\mathbb{R}\to M$ and $A\subset\Lambda^*(M)$ we construct a natural d-continuation $\hat F$ of the map $F^*\colon A\to\Lambda^*(S^{q+1}\times\mathbb{R})$

$$\hat F\colon C_qA\to\Lambda^*(S^{q+1}\times\mathbb{R}). \tag{3}$$

Theorem 1. *Any element $z\in C_qA$, $dz=0$ determines the homotopy invariant integral (4) correctly defined for a class $[z]\in H^{q+1}(C_qA)=\pi_{q+1}(A)^*$*

$$H(f_t,z)=\int_{S^{q+1}\times t}\hat F z,\quad d(\hat F z)=\hat F\,dz=0,\quad dH/dt=0. \tag{4}$$

Proof of the H-property $dH/dt=0$ is obvious as above, by the Stoke's formula. For the constructing of $\hat F$ we observe that there is a natural filtration on the set of new free generators $v_{j\alpha}\in C_qA$ such that

$$\begin{aligned}&A_0=A\subset A_1\subset A_2\subset\dots\subset C_qA\\&A_p=A_{p-1}[\dots,v_{j\alpha},\dots],\quad dv_{j\alpha}\in A_{p-1},\ v_{j\alpha}\in A_p.\end{aligned} \tag{5}$$

Beginning from $A_0=F^*\Lambda^*(M)\subset\Lambda^*(S^{q+1}\times\mathbb{R})$ we construct the d-map $\hat F$ by induction on the number p because the operation d^{-1} is well-defined on the spaces $\Lambda^j(S^{q+1}\times\mathbb{R})$ for $j\leqslant q$. The isomorphism (6) for $A=\Lambda^*(M)$ may be deduced from the results of H. Cartan, J.-P. Serre and D. Sullivan:

$$\pi_{q+1}(A)\otimes K=\pi_{q+1}(M)\otimes K. \tag{6}$$

The construction (4) of the homotopy integrals is absolutely elementary (as the E. Cartan–de Rham's definition of the homology groups). It uses only the simplest properties of forms and may be useful in the field theory.

2. Minimal d-Algebras and Their Deformations

Definition 2 ([3])**.** The algebra A is minimal iff A is free and for any free generator $x_{j\alpha} \in A_j$ we have

$$dx_{j\alpha} = P_{j\alpha}(\dots, x_{q\beta}, \dots), \quad q = \dim x_{q\beta} < j, \quad A_0 = K, \quad A_1 = 0. \tag{7}$$

For any minimal algebra A we shall construct its universal deformations. Consider the free algebra B with set of free generators $(x_{j\alpha}, w_{j\alpha}, v_{j-1,\alpha})$ such that: $dv_{j-1,\alpha} = w_{j\alpha}$.

The universal (infinitesimal) deformation f_λ of subalgebra $A \subset B$ has the next properties

a) $f_0 = 1$, $A \to A$

b) $f_\lambda(x_{j\alpha}) = x_{j\alpha} + \lambda\, dv_{j-1,\alpha}$ if $dx_{j\alpha} = 0$

c) suppose $f_\lambda x_{j\alpha} = x_{j\alpha}(\lambda)$ is well-defined for all $j < p$:

$$x_{j\alpha}(\lambda) = x_{j\alpha} + \lambda y_{j\alpha} + O(\lambda^2).$$

Lemma 1.

$$P_{p\beta}(\dots, x_{j\alpha}(\lambda), \dots) = P_{p\beta}(\dots, x_{j\alpha}, \dots) + \lambda\, dQ_{p\beta}(\dots, x_{j\alpha}, w_{j\alpha}, v_{j-1,\alpha}, \dots) + O(\lambda^2)$$

for some polynomials $Q_{p\beta}$.

Definition 3. The universal deformation of minimal algebra is given by the formula (8)

$$f_\lambda(x_{j\alpha}) = x_{j\alpha} + \lambda Q_{j\alpha} + \lambda\, dv_{j-1,\alpha} + O(\lambda^2). \tag{8}$$

The deformation (infinitesimal) of any concrete map $g\colon A \to C$ is some continuation $G\colon B \to C$ and completely determines by the elements $G(v_{j-1,\alpha}) \in C$.

For the minimal algebra A we have the obvious isomorphism

$$\pi_{q+1}(A)^* = A_{q+1}/(\text{decomposable elements}).$$

Any free generator $x_{q+1,\alpha} \in A_{q+1}$ determines the unique element $z_{q+1,\alpha} \in H^{q+1}(C_q A) = \pi_{q+1}(A)^*$, represented by the cycle $\tilde{z}$:

$$\tilde{z}_{q+1,\alpha} = x_{q+1,\alpha} + \dots \in C_{q+1}A, \quad d\tilde{z}_{q+1,\alpha} = 0 \tag{9}$$

(the difference $\tilde{z}_{q+1,\alpha} - x_{q+1,\alpha}$ is decomposable element).

Lemma 2. *The deformation* (8) *of minimal algebra A determines the extended deformation of the algebras $C_q A$ for all $q \geqslant 1$ such that*

$$\tilde{z}_{q+1,\alpha}(\lambda) = \tilde{z}_{q+1,\alpha} + \lambda\, du_{j\alpha} + O(\lambda^2) \in C_q B. \tag{10}$$

Example 1. $x_2 = \omega \in \Lambda^2(S^2 \times \mathbb{R}^N)$, $\omega^2 = dx_3$,

$$A = Q[x_2, x_3], \quad C_2 A = A[v_1], \quad dv_1 = x_2 = \omega.$$

$$z_3 = x_3 - v_1 x_2 \in \pi_3^*(A) = k = \mathbb{Q}, \mathbb{R}, \mathbb{C}.$$

For the deformation f_λ we have

$$x_2(\lambda) = x_2 + \lambda\, d\tau_1 = \omega + \lambda\, d\tau_1$$

$$x_3(\lambda) = x_3 + 2\lambda\omega\tau_1 + \lambda\, d\tau_2 + O(\lambda^2)$$

$$v_1(\lambda) = v_1 + \lambda\tau_1$$

$$\begin{aligned}\tilde{z}_3(\lambda) &= x_3(\lambda) - v_1(\lambda)\omega(\lambda) = z_3 + \lambda[2\omega\tau_1 - \omega\tau_1 - v_1\, d\tau_1] + \lambda\, d\tau_2 + O(\lambda^2) \\ &= z_3 + \lambda\, d(v_1\tau_1 + \tau_2) + O(\lambda^2)\end{aligned}$$

Example 2. $M = \mathbb{R}^3 \setminus \bigcup_{\alpha=1}^{m} (*_\alpha)$, $K = M \times \mathbb{R}^N$

$$A = Q[x_{2,\alpha}; x_{3,(\alpha\beta)}; x_{4,(\alpha\beta\gamma)}; \dots],$$
$$d_{2,\alpha} = 0, \quad dx_{3,(\alpha\beta)} = x_{2\alpha} x_{2\beta}, \quad dx_{4,(\alpha\beta\gamma)} = \cdots .$$

a) $C_2 A = A[v_{11}, v_{12}]$, $dv_{1\alpha} = x_{2\alpha}$,

$$k \otimes \pi_3 M = S^2 k^m, \quad k = \mathbb{Q}, \mathbb{R}, \mathbb{C},$$
$$z_{3(\alpha\beta)} = z_{3(\beta\alpha)} = x_{3(\alpha\beta)} - [v_{1\alpha} x_{\alpha\beta}]. \tag{9'}$$

b) $C_3 A = C_2 A[\dots, v_{2(\alpha\beta)}, \dots]$, $d_{2(\alpha\beta)} = z_{3(\alpha\beta)}$

$$z_{4(\alpha\beta\gamma)} = x_{4(\alpha\beta\gamma)} = x_{4(\alpha\beta\gamma)} + \cdots \pm [v_{1\alpha} v_{1\beta} x_{2\gamma}]. \tag{9''}$$

There is a natural map $\bar{\Psi} \colon A \to \Lambda^*(M)$ such that

$$\bar{\Psi}(x_{3(\alpha\beta)}) = 0, \quad \bar{\Psi}(x_{4(\alpha\beta\gamma)}) = 0, \quad \dots$$

For any map $F \colon S^{q+1} \times \mathbb{R} \to M$, $q + 1 = 3, 4$ the formula for $\hat{F} z$ includes only the last term in the brackets $[\dots]$ from $(9')$ and $(9'')$. There is a relation

$$z_{4(\alpha\beta\gamma)} + z_{4(\beta\gamma\alpha)} - z_{4(\alpha\gamma\beta)} = 0$$
$$z_{3(\alpha\beta)} = z_{3(\beta\alpha)}.$$

3. Minimal Models in $\Lambda^*(M)$. Moduli Space. R-Property. Complex Structures

The Whitney's ring of all semilinear forms for any simplicial complex K contains all SL-forms over the field $k = \mathbb{R}, \mathbb{C}$. The SL-form ω is by definition the collection of K-forms ω on each simplex $\sigma_\alpha \subset K$ such that their restrictions are completible. The d-algebra $\Lambda^*_{SL}(K, k)$ determines real homotopy type (k-type). The Sullivan's subring $\Lambda^*_{SL}(K, \mathbb{Q}) \subset \Lambda^*_{SL}(K, \mathbb{R})$ contains all SL-forms ω such that all ω_α have the polynomial coefficients over $\mathbb{Q}$ in the standard coordinates of σ_α. For the smooth triangulated manifold M we have

$$\Lambda^*_{SL}(M, \mathbb{Q}) \overset{i_{\mathbb{Q}}}{\subset} \Lambda^*_{SL}(M, \mathbb{R}) \supset \Lambda^*(M, \mathbb{R}). \tag{11}$$

Consider the "stable" manifold $K = M \times \mathbb{R}^N$ for $N \to \infty$. There is "N-embedding" of the minimal algebra $\varphi \colon A \to \Lambda^*_{SL}(K, \mathbb{Q})$ such that

a) φ is the monomorphism up to $\dim N/2$;

b) $\varphi_* \colon H^*(A) \to H^*(K, \mathbb{Q})$ is isomorphism.

There is a deformation Ψ_t of the map $i_{\mathbb{Q}} \varphi = \Psi_0$ such that $\Psi_1(A) \subset \Lambda^*(K, \mathbb{R}) \subset \Lambda^*_{SL}(K, \mathbb{R})$. As a result we obtain the next

Theorem 2 (D.V. Millionšikov)**.** *There is N-embedding $\Psi_1 \colon A \to \Lambda^*(M \times \mathbb{R}^N, \mathbb{R})$ of the rational minimal model A over $\mathbb{Q}$ in the ring of ordinary C^∞-forms.*

Corollary. *For any smooth manifold M there is a homomorphism $\Psi \colon A \to \Lambda^*(M, \mathbb{Q})$ of the minimal $\mathbb{Q}$-model A in the algebra of the ordinary C^∞-forms $\Lambda^*(M)$ such that*

$$\Psi^* \colon H^*(A, \mathbb{Q}) \to H^*(M, \mathbb{Q}), \quad \Psi(x_{n\alpha}) = 0, \ n \geqslant \dim M, \tag{12}$$

is isomorphism (see discussion in [2]).

The construction of Ψ_1 and Ψ (above) is non-effective. More natural construction [2] leads to nontrivial obstructions. Suppose by induction that the N-embedded $\mathbb{Q}$-subring $\Psi^{(p)}\colon A^{(p)} \subset \Lambda^*(M \times \mathbb{R}^N, \mathbb{R})$, $N \to \infty$ is constructed, $A^{(p)}$ contains all free generators $x_{j\alpha}$ for $j \leqslant p-1$, $H^j(A^{(p)}, \mathbb{Q}) = H^j(M, \mathbb{Q})$, $j \leqslant p-1$. The ring $A^{(p)}$ is minimal and free. Consider the homological embedding in the dimensions p and $p+1$:

$$\Psi^{(p)}_{*,p}\colon H^p(A^{(p)}, k) \to H^p(M, \mathbb{C}), \tag{13}$$

$$\Psi^{(p)}_{*,p+1}\colon H^{p+1}(A^{(p)}, k) \to H^{p+1}(M, \mathbb{C}), \quad k = \mathbb{Q}, \mathbb{R}, \mathbb{C}. \tag{14}$$

Has the image $\operatorname{Im} \Psi^{(p)}_{*,p} \cap H^{(p)}(M, k) \subset H^p(M, \mathbb{C})$ the same k-rank as the $\mathbb{C}$-rank of $\operatorname{Im}(\Psi^{(p)}_{*,p} \otimes \mathbb{C})$?

Has the kernel $\operatorname{Ker} \Psi^{(p)}_{*,p+1}$ the same k-dimension as $\mathbb{C}$-dimension of $\operatorname{Ker}(\Psi^{(p)}_{*,p+1} \otimes \mathbb{C})$ or $\mathbb{R}$-dimension of $\operatorname{Ker}(\Psi^{(p)}_{*,p+1} \otimes \mathbb{R})$?

If so, the p-obstruction to the k-continuation of N-embedding of the minimal model is zero ($k = \mathbb{Q}$). Suppose the p-obstruction is zero.

Choose the basis $\bar{x}_{p,1}, \dots, \bar{x}_{p,\alpha_p} \in \operatorname{Coker} \Psi^{(p)}_{*,p} \cap H^p(M, k)$ and the basis $y_{p,1}, \dots, y_{p,\beta_p} \in \operatorname{Ker} \Psi^{(p)}_{*,p+1} \subset H^{p+1}(A^{(p)}, k)$. The representatives of the classes $\bar{x}_{p\alpha}$ are the C^∞-forms—elements $\bar{x}_{p\alpha}$ ($\alpha = 1, \dots, \alpha_p$) in $\Lambda^p(M \times \mathbb{R}^N)$. The representatives of classes $y_{p,\tau}$ are the polynomials ("structural polynomials")

$$P_{p,\alpha_p+\tau}(\dots, x_{j\alpha}, \dots), \quad x_{j\alpha} \in A^{(p)},\ j < p.$$

By definition the algebra $A^{(p+1)} \supset A^{(p)}$ has the new set of free generators $(x_{p,1}, \dots, x_{p,\alpha_p}, x_{p,\alpha_{p+1}}, \dots, x_{p,\alpha_p+\beta_p})$ and extended map

$$\Psi^{(p+1)}\colon A^{(p+1)} \to \Lambda^*(M \times \mathbb{R}^N, \mathbb{C})$$

such that $\Psi^{(p+1)} = \Psi^{(p)}$ on $A^{(p)} \subset A^{(p+1)}$ and

$$\begin{cases} \Psi^{(p+1)}(x_{p\alpha}) = \tilde{x}_{p\alpha}, \quad \alpha = 1, \dots, \alpha_p, \\ d\Psi^{(p+1)}(x_{p,\alpha_p+\tau}) = d\tilde{x}_{p,\alpha_p+\tau} = P_{p,\alpha_p+\tau}(\dots, \tilde{x}_{j\alpha}, \dots). \end{cases} \tag{15}$$

The construction (15) is always possible for $k = \mathbb{R}, \mathbb{C}$. The resulting ring $\bigcup_p A^{(p)} = A$ is by definition the minimal k-model.

For $p = 2, 3$ the construction (15) is obviously possible also for $k = \mathbb{Q}$ ($\pi_1 = 0$) . But the continuation of $\Psi^{(p)}$ to higher dimensions is not unique:

$$\tilde{x}'_{p\alpha} = \tilde{x}_{p\alpha} + u_{p\alpha}, \quad du_{p\alpha} = 0. \tag{16}$$

For the closed free generators we have a natural "integrality requirement": the class $[\tilde{x}_{p\alpha}]$ should be integral

$$\tilde{x}_{p\alpha} \in H^p(M, \mathbb{Z}), \quad \alpha = 1, \dots, \alpha_p.$$

For the nonclosed $\tilde{x}_{p\alpha}$ ($\alpha > \alpha_p$) all elements (16) are "a priori" equivalent. From the Theorem 1 and its Corollary (above) we deduce the next

Proposition. *There exist* (*noneffectively*?) *such collection of elements* ($x_{p\alpha}$) *for all* $p \geqslant 3$ *and* $k = \mathbb{Q}$ *such that the construction* (15) *does not meet any obstructions* (*above*).

Problem. How to find the nonclosed $\mathbb{Q}$-generators effectively without any triangulation?

Consider now the infinitesimal variations of the minimal k-subring $\Psi(A) \subset \Lambda^*(M \times \mathbb{R}^N, \mathbb{C})$ constructed by the procedure (15) and such that the cohomology classes of all *closed* generators $\tilde{x}_{p\alpha}$ $(\alpha = 1, \dots, \alpha_p)$ are fixed.

$$\delta[\tilde{x}_{p\alpha}] = 0, \quad \alpha = 1, \dots, \alpha_p, \quad d\tilde{x}_{p\alpha} = 0. \tag{17}$$

For the variation of nonclosed generators $\tilde{x}_{p\alpha}$ we shall use (16). As a result we obtain

Lemma 3. *There is an embedding of the set of homotopy classes* (8) *of all variations* $\delta\Psi$ *fixed on the set of closed generators* (17) *in the* (*pre*)*-moduli space* $\kappa = \bigcup_p \kappa_p$ *of minimal k-models for* $k = \mathbb{R}, \mathbb{C}$:

$$\kappa_p = \sum_{j \geqslant 3}^{p-1} \operatorname{Ker} h_j^* \otimes H^j(M, k).$$

The proof may be easily deduced from (16), (17) and (8) because in the formula (16) we use only $x_{p\alpha}$ dual to $\operatorname{Ker} h_p$ and $[u_{p\alpha}] \in H^p(M)$. Here $h_j : \pi_j(M) \to H_j(M)$ is the Hurewitch homomorphism.

Corollary (R-property)**.** *Suppose* $\kappa_p = 0$*. In that case all homotopy integrals are rigid for* $p = q + 1$*. They don't depend on the map* Ψ *of the minimal* $\mathbb{Q}$*-model* A *in the algebra of* C^∞*-forms, fixed on the set of the closed free generators.*

Remark. For the N-embedding Ψ_1 of the Theorem 2 (above) all homotopy integrals are rational numbers. The proof may be easily deduced from the properties of ring $\Lambda^*_{SL}(K, \mathbb{Q}) \subset \Lambda^*_{SL}(K, \mathbb{C})$ and deformation Ψ_t.

In fact, tangent space to the moduli space of all homotopy classes of minimal k-models with fixed set of closed free generators is isomorphic to some factor-space of the space $\kappa = \sum \operatorname{Ker} h_p^* \otimes H^p$: $\operatorname{Mod}_k(M) = \kappa(M)/V(M)$. The space $\operatorname{Mod}_k(M)$ contains some "Sullivan's" point A_0 and subspace $\operatorname{Mod}_k^0(M) \subset \operatorname{Mod}_k(M)$ of the minimal models, isomorphic to $A_0 \otimes k$, $\operatorname{Mod}_k(M) = \kappa/V(M)$. The student of author proved the next theorem.

Theorem 3 (D.V. Millionšikov)**.** *There is a spectral sequence* $E_r^{p,q}$*,* $r \geqslant 1$*, such that*

a) $d_r : E_r^{p,q} \to E_r^{p+r, q+r+1}$
b) $E_1^{p,q} = \pi_p^*(M) \otimes H_q(M, k)$ $(\pi_0 = \pi_1 = 0,\ H^0 = k,\ H^1 = 0)$
c) $\operatorname{Im} d_r \cap E_r^{p,q} \subset \operatorname{Ker} h_p^* \otimes H^p(M, k) = \kappa(M)$
d) $\sum_{p \geqslant 3} E_\infty^{p,p} \cap \kappa = \operatorname{Mod}_k^0(M) \subset \kappa/V(M)$
e) $\sum_{q-p=m} E_\infty^{p,q} = H^m(D)$

Here the $\mathbb{Z}$*-graded differential Lie superalgebra* $(D, \tilde{d})$ *contains all infinitesimal automorphisms of* $A_0 \otimes k$*,* $D = \sum_{p \in \mathbb{Z}} D_p$*,* $\tilde{d}t(x) = dt(x) \pm td(x)$*,* $t \in D$*,* $\dim t = \dim tx - \dim x = p$*.*

Algebra D has the filtration $i \geqslant 2$; $\dots \supset D^i \supset D^{i+1} \supset \dots$ such that $t \in D^i$ iff $tx = 0$ for all x, $\dim x \leqslant i$.

Example. Suppose $a = [M^n] \in H^n(M^n, \mathbb{Z})$ and $\mu \in H^n(M^n, \mathbb{Z})$—the fundamental classes $(\mu, a) = 1$. For $n = 4$, $\pi_1 = 0$, $b_2 \neq 0$ we have

a) $\pi_4(M^4) = \operatorname{Ker} h_4$;
b) $\operatorname{Ker} h_4^* \otimes H^4(M^4) \cong \pi_4^*(M^4) \otimes \mathbb{R} = \kappa$;

c) $\mathrm{Mod}_k(M^4) = \mathrm{Mod}_k^0(M^4) = 0$ because:

$$d_1 \colon E_1^{3,2} = \pi_3^* \otimes H^2 \to E_1^{4,4} = \mathrm{Ker}\, h_4^* \otimes H^4 = \mathrm{Im}\, d_1 = \pi_4^* \otimes H^4.$$

Conjecture. *For any closed simply connected manifold M with some nontrivial Betti numbers $b_j \neq 0$ for $1 < j < n$ all group $\mathrm{Ker}\, h_n \otimes H^n(M^n) = \pi_n \otimes \mathbb{R} = E_1^{n,n}$ is covered by the image of differentials d_r and $E_\infty^{n,n} = 0$. The moduli space $\mathrm{Mod}_k(M^n)$ may be reduced to κ_n:*

$$\mathrm{Mod}_k(M^n) = \left(\sum_{p=3}^{n-1} \mathrm{Ker}\, h_p^* \otimes H^p \right) / V'(M).$$

Consider now closed simply connected complex manifold M^{2n}, $n = \dim_{\mathbb{C}} M$. There are the standard operators $\partial, \bar{\partial}$ and their real combinations

$$d = \partial + \bar{\partial}, \quad d_c = i(\partial - \bar{\partial}), \quad i^2 = -1.$$

We shall use the subring $B = \mathrm{Ker}\, d_c \subset \Lambda^*(M^{2n}, \mathbb{C})$ as d-algebra. For the Kählerian manifold the subalgebra (B, d) is homotopy equivalent to Λ^* by dd_c-Lemma ([5]):

$$H^*(B, d) = H^*(B, d_c) = H^*(M, \mathbb{C}).$$

The natural projection

$$B = \mathrm{Ker}\, d_c \to H^*(B, d_c) = H^*(M, \mathbb{C})$$

is the main map of ([5]) which leads to formality of the R-homotopy type $\{M^{2n}\}$. There is a formula ([5])

$$(\mathrm{Ker}\, d_c \cap \mathrm{Ker}\, d) / \mathrm{Im}\, dd_c = H^*(M). \tag{18}$$

The important for us corollary from (18) in the next

Proposition. $(\mathrm{Im}\, d_c \cap \mathrm{Ker}\, d) / \mathrm{Im}\, dd_c = 0$. *By the standard procedure* (13)–(15) *for $k = \mathbb{R}, \mathbb{C}$ we construct the map of minimal k-model*

$$\Psi \colon A \to (B, d) = (\mathrm{Ker}\, d_c, d) \subset \Lambda^*(M^{2n}).$$

Lemma 4. *There is a canonical construction Ψ whose homotopy class is determined completely by the complex structure on M^{2n}.*

Proof. By dd_c-lemma we construct canonically $d^{-1}(y) \in \mathrm{Im}\, d_c$ in the subalgebra $B = \mathrm{Ker}\, d_c$, $dy = 0$. The transformation (16) will have special form:

$$\begin{gathered} x' = x + d_c u, \quad dx' = dx = y \\ d(d_c u) = 0 \end{gathered} \tag{16'}$$

so the class of $[d_c u] \in H^*(M)$ by Proposition (above) is zero. □

Theorem 4. *There is a canonical homomorphism of the tangent space T to the moduli space of all complex (Kählerian) structures in the moduli space $\mathrm{Mod}_{\mathbb{C}}(M)$ of the minimal $\mathbb{R}$-models*:

$$\Phi \colon T \to \mathrm{Mod}_{\mathbb{C}}(M) = \kappa / V(M)$$
$$\kappa = \sum_{p=3}^{2n-1} \mathrm{Ker}\, h_p^* \otimes H^p(M^{2n}, \mathbb{R}).$$

Example. Consider the case $n = 3$ and Ricci-flat manifolds (Kummer–Calaby–Yau type). We have $\pi_1 = 0$, $H^1 = H^5 = 0$:

$$b_2 = a, \quad b_3 = h^{2,0} + h^{1,1} + h^{0,2} = 2 + h^{1,1}$$

for $a > 1$ the group $\operatorname{Ker} h_3$ is nonzero

$$\operatorname{Ker} h_3^* \otimes H^3(M^6, \mathbb{R}) \neq 0.$$

Problem. Calculate the deformation Φ of the minimal R-models in the moduli space $\mathrm{Mod}_{\mathbb{R}}$ the corresponding deformation of the homotopy integrals.

Conjecture. *The minimal model constructed* (*above*) *by the ring* $B = (\operatorname{Ker} d_c, d) \subset \Lambda^*(M)$ *represents Sullivan's rational minimal model for the complex algebraic manifolds* M *over the field* $\mathbb{Q}$.

References

[1] Novikov S.P. Usp. Mat. Nauk (Russian Math. Surveys), 1984, **39**, N 5, 113–124.
[2] Novikov S.P. Dokl. Akad. Nauk SSSR (Soviet Math. Doklady), 1985, **283**, 1088–1091.
[3] Sullivan D. Publ. Math., Inst. Hautes Etud. Sci, 1977, **4**, 269–331.
[4] Chen K.-T. Bull. Amer. Math. Soc., **83** (1977), 831–879.
[5] . Deligne P., Griffits Ph., Morgan J., Sullivan D. Invent Math., **29**:3 (1975) , 245–274.
[6] Novikov S.P. Dokl. Akad. Nauk SSSR **260** (1981), 31–35, (transl) Soviet Math. Dokl. **24** (1981), 222–226.

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