ALGEBRAS OF VIRASORO TYPE, RIEMANN SURFACES AND STRUCTURES OF THE THEORY OF SOLITONS

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Introduction

The role of the algebra of complex-valued vector fields on the circle $\mathcal{L}(S^1)$ (Witt algebra) and its central extension $\mathcal{L}^c$ in the theory of a free boson quantum string, especially in dimension $d = 26$, is well known. These algebras contain Z-graded subalgebras $L \subset \mathcal{L}(S^1)$ and $L^c \subset \mathcal{L}^c$, where L^c is a central extension of L and generated by a basis (e_i, t) in which one has the relations

$$[e_i, e_j] = (j - i)e_{i+j} + t\,\frac{i^3 - i}{12}\,\delta_{i,-j}. \tag{1}$$

Here, $e_i = z^{i+1}(\partial/\partial z)$, t is a generator of the center. The algebra L^c is called the "Virasoro algebra," and $\mathcal{L}^c$ the "Gel'fand–Fuks algebra."

The algebras L and L^c have the decomposition

$$L^c = L_+ + L_0^c + L_-, \tag{2}$$

where $L_0^c = (e_0, t)$, and the subalgebras $L_\pm$ are generated by the elements e_i, $i \geqslant 1$ and $i \leqslant -1$, respectively.

The central extensions L^c, $\mathcal{L}^c$ of the algebras L, $\mathcal{L}$ are given by the Gel'fand–Fuks cocycle

$$\gamma(f, g) = \frac{1}{24\pi i}\int_0^{2\pi} f^m g\, d\varphi, \qquad z = e^{i\varphi}. \tag{3}$$

Here the fields have the form $f(\varphi)\partial_\varphi$, $g(\varphi)\partial_\varphi$.

The most fundamental class of representations of the algebra L^c for the theory of a free string, the "Verma modules," is given by a generating vector Φ_0 with the conditions

$$L_+\Phi_0 = 0, \quad e_0\Phi_0 = h\Phi_0, \quad t\Phi_0 = c\Phi_0 \tag{4}$$

and is realized by vectors of the form

$$e^{n_1,\dots,n_k}_{i_1,\dots,i_k}\Phi_0, \qquad i_1 < i_2 < \cdots < i_k < 0. \tag{5}$$

In particular, the vacuum vector in the Fok representation is an example of a vector Φ_0, although in quantum theory there arises a quite complicated algebraic aggregate composed of different Verma modules (cf. [1, 2]).

The geometric approach of Polyakov et al., to the introduction of interactions in the theory of a string necessarily leads to complicated problems of the algebraic geometry of Riemann surfaces [3, 4]. However, the role of the Virasoro algebra in this approach is absolutely not apparent. The goal of the present paper is

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the construction, as we hope, of regular analogs of Virasoro algebras and Verma modules, connected with nontrivial Riemann surfaces of genus $g > 0$. It is not surprising that the vacuum should be reconstructed and naive Verma modules be replaced by more complicated objects. Green and Shwartz [5] point to this in the single loop case also.

Although the goal cited is basic, we also consider briefly another physically important example of algebras, "the current algebras" $G(S^1)$, composed of functions on the circle with values in a semisimple Lie algebra G with the natural commutator, and its central extension. In it there also lies a Z-graded subalgebra (KacMoody algebra), consisting of trigonometric polynomials. For such algebras there is also a decomposition of type (2), where $L_0 = G + C$, and also a theory of Verma-type modules.

The starting point of our paper is the observation that the nontrivial Riemann surfaces generate, in the algebras of vector fields $\mathcal{L}(S^1)$, the current algebras $G(S^1)$ and their central extensions, dense subalgebras more complicated than Virasoro and KacMoody. These subalgebras are not Z-graded (cf. Sec. 1). Starting from the properties of these subalgebras there arises naturally an important concept of "generalized-graded" or "k-graded" algebras and modules.

Definition 1. The algebra $G = \sum_{i=-\infty}^{\infty} G_i$ is said to be k-graded if for all G_j, G_i we have

$$G_i G_j \subset \sum_{s=i+j-k}^{i+j+k} G_s. \tag{6}$$

For $k = 0$, we get ordinary Z-graded algebras.

Analogously, one introduces the concept of N-graded modules M over k-graded algebras

$$G_i M_j = \sum_{s=-k-N}^{s=k+N} M_{i+j-s}. \tag{7}$$

Trivial Example. Let $k = 0$ and G be the algebra of Laurent polynomial fields $G = L$ on S^2 (or on S^1, $|z| = 1$). Let the module $M = \sum M_i$ consist of functions of the form

$$P(z, z^{-1}) \exp\left(\sum_{j=-N}^{N} x_j z^{-j}\right), \qquad M_i = \left(\lambda z^i \exp\left(\sum_{j=-N}^{N} x_j z^{-j}\right)\right). \tag{8}$$

Here P is a Laurent polynomial. We have

$$L_i M_j \subset \sum_{s=-N}^{N} M_{i+j-s}. \tag{9}$$

Thus, the module M is N-graded although the algebra itself is 0-graded. This example shows the naturality of the class of N-graded modules in algebrogeometric constructions of the theory of solitons of the type of "Baker–Akhiezer functions" (cf. Sec. 6).

The concepts introduced here are easily generalized to gradings with values in any Abelian group (continuous ones included), while the examples of greatest interest for us are the groups Z, Z^k, R, R^k (Sec. 6).

In Sec. 1 we introduce important subalgebras in the algebras of vector fields on the circle, depending on a Riemann surface Γ, we prove that they have k-gradings, decompositions of type (2), where L_+, L_- are Lie subalgebras and the dimension of L_0 depends on the genus g. This decomposition lets us introduce analogs of Verma modules

$$L_+\Phi_0 = 0, \qquad t\Phi_0 = c\Phi_0.$$

A realization of these modules, naturally generalizing the realization of Feigin–Fuks for $g = 0$ (cf. [6]), is given in Sec. 4. For any algebraic curves of genus $g > 0$ we establish the density of these subalgebras in the algebra of vector fields on S^1 and an important formula for the central charge in terms of the tensor weight of the modules on which the representation

$$c = -2(6\lambda^2 - 6\lambda + 1) \tag{10}$$

is realized. The polynomial (10) appeared in Mumford [7] for the Chern class of bundles over the space of moduli, but its connection with algebras of Virasoro type is important.

We note that just as naturally as the generalizatoin of Virasoro and KacMoody algebras there arises in our considerations a generalization of the Heisenberg algebra. These results are given in Secs. 3 and 4. The concluding section of the paper is devoted to connections of this theory with the theory of solitons.

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1. Algebras of Vector Fields on Algebraic Curves

Let Γ be a nonsingular algebraic curve of genus g with two distinguished points $P_\pm$ in general position. We denote by L^Γ the algebra of meromorphic vector fields on Γ, holomorphic outisde $P_\pm$. (For $g = 0$, if as $P_\pm$ one chooses the points $z = 0$ and $z = \infty$ in the extended complex plane, then the algebra L^Γ coincides with the ordinary algebra of fields which are Laurent polynomials.)

It follows from the Riemann–Roch theorem that for $g \geqslant 2$ one can introduce a basis e_i of fields in L^Γ, which are determined uniquely up to proportionality by the following conditions: e_i has a zero of multiplicity $i - g_0 + 1$ at the point P_+ and a pole of multiplicity $i + g_0 - 1$ at the point P_-. Here $g_0 = 3g/2$, for even g the index i runs through all integers, $i = \dots, -1, 0, 1, \dots$; for odd g, i runs through all half-integral values $i = \dots, -3/2, -1/2, 1/2, 3/2, \dots$ If we fix local parameters $z_\pm(Q)$, $z_\pm(P_\pm) = 0$ in neighborhoods of the points $P_\pm$, then e_i, in neighborhoods of $P_\pm$, will have the form

$$e_i = a_i^\pm z_\pm^{\pm i - g_0 + 1}(1 + O(z_\pm))\,\frac{\partial}{\partial z_\pm}. \tag{1.1}$$

Let us agree to normalize e_i uniquely so that the constant $a_i^+ = 1$.

Remark. If $g = 1$, the normalization conditions for e_i for $|i| > 1/2$ are the same as in the general case and slightly different for $|i| = 1/2$. We return to this in more detail in Sec. 5, where explicit formulas will be given for the e_i in the elliptic case.

Lemma 1. *With respect to the basis e_i the algebra L^Γ is k-graded where $k = g_0$:*

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} c_{ij}^s e_{i+j-s}. \tag{1.2}$$

[*The summation in* (1.2) *is over integral s for even g and over half-integral s for odd g.*]

The proof of the lemma is almost obvious and follows from simple calculation of the multiplicities of zeros and poles of $[e_i, e_j]$ in $P_\pm$.

Remark. Comparison of the principal parts at $P_\pm$ of the expansions of the right and left sides of (1.2) gives

$$c_{ij}^{g_0} = (j-i), \qquad c_{ij}^{-g_0} = (i-j)\,\frac{a_i^- a_j^-}{a_{i+j-g_0}^-}. \tag{1.3}$$

We denote by $L_\pm^{(s)}$ the subspaces of L^Γ, generated by the vector fields e_i with indices $\pm i \geqslant g_0 + s$, $s \in Z$. As follows from (1.2), the subspaces $L_\pm^{(s)}$ with $s \geqslant -1$ are subalgebras of L^Γ. In particular, $L_\pm^{(-1)}$ are the subalgebras of vector fields from L^Γ which, at the points $P_\pm$ (respectively), are holomorphic.

On Γ we define a one-parameter family C_τ of contours. Let dp be a differential of the third kind on Γ with poles of the first order at the points $P_\pm$ with residues ± 1, respectively.

It can be normalized uniquely by requiring that its periods over all cycles be imaginary. Then on Γ there is a well-defined harmonic function $\operatorname{Re} p(Q)$, where $P(Q) = \int_{Q_0}^{Q} dp$ and Q_0 is an arbitrary initial point. The contours C_τ are level lines of this function $C_\tau = \{Q\colon \operatorname{Re} p(Q) = \tau\}$. For $\tau \to \pm\infty$ the contours C_τ are small circles enveloping the points $P_\mp$.

Restriction of any vector field from L^Γ to C_τ defines a homomorphism of the algebra L^Γ into the algebra of smooth vector fields on $C_\tau - \mathcal{L}(C_\tau)$, which, for sufficiently large τ, is ismorphic with the algebra of smooth vector fields on the circle $\mathcal{L}(S^1)$.

Theorem 1. *The image $i_\tau(L^\Gamma)$ is everywhere dense in $\mathcal{L}(C_\tau)$.*

Proof. We consider the vector field e_0. Outside the points $P_\pm$ it has exactly g zeros, $\gamma_1, \dots, \gamma_g$, which, generically, one can assume different. We denote by $\psi_n(Q)$ a meromorphic function on Γ having, outside $P_\pm$, simple poles at $\gamma_1, \dots, \gamma_g$, and the form (1.4) in neighborhoods of $P_\pm$:

$$\psi_n = b_n^\pm z_\pm^{\pm n}(1 + O(z_\pm)), \qquad z_\pm = z_\pm(Q), \quad b_n^+ = 1 \tag{1.4}$$

(such functions were introduced in [8] for the construction of commuting difference operators). Moreover, we need the "dual" collection of functions $\psi_n^+(Q)$, defined as follows.

Let $d\omega$ be the unique differential of the third kind with poles at the points $P_\pm$ with residues ± 1, vanishing at the points $\gamma_1, \dots, \gamma_g$. Besides these, it has g additional zeros $\gamma_1^+, \dots, \gamma_g^+$. We denote by $\psi_n^+(Q)$ a meromorphic function on Γ, having, outside $P_\pm$, poles at the points $\gamma_1^+, \dots . \gamma_g^+$, and the form (1.5) in neighborhoods of the $P_\pm$:

$$\psi_n^+ = (b_n^\pm)^{-1} z_\pm^{\mp n}(1 + O(z_\pm)), \qquad z_\pm = z_\pm(Q). \tag{1.5}$$

The existence and uniqueness of ψ_n and ψ_n^+ are simple consequences of the Riemann–Roch theorem.

Lemma 2. *For any continuously differentiable function $F(t)$ on $C_\tau \ni t$ one has*

$$F(t) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \psi_n(t) \left[\oint_{C_\tau} \frac{F(t')\psi_n^+(t')}{\langle \psi_n(t')\psi_n^+(t')\rangle} \, dp(t') \right], \tag{1.6}$$

where $\langle \psi_n \psi_n^+ \rangle$ denotes the mean over n (this mean exists, since $\psi_n\psi_n^+$ quasiperiodic function of n).

We note that on the whole the conditions on convergence and dependence of the rate of decrease of the coefficients of ψ_n on the smoothness of F are the same as for the ordinary Fourier transform.

The *proof* of the lemma repeats, to a considerable degree, the course of the proof for ordinary Fourier series. Analogously to the proof of (30) of [9] one can show that

$$d\omega = \frac{dp}{\langle \psi_n \psi_n^+ \rangle}. \tag{1.7}$$

Hence

$$\frac{1}{2\pi i} \oint_{C_\tau} \psi_n^+(t') \, \frac{dp(t')}{\langle \psi_n(t')\psi_n^+(t')\rangle} = \frac{1}{2\pi i} \oint_{C_\tau} \psi_n^+(t') \, d\omega = \delta_{n,0}. \tag{1.8}$$

The last equation is valid since by the definition of ψ_n^+ and $d\omega$ the differential $\psi_n^+ \, d\omega$ holomorphic outside $P_\pm$ and the integral in (1.8) is equal to the residue of this differential at $P_\pm$, or the residue at P_- with opposite sign.

We denote by S_N the partial sum of (1.6) in which n varies from $-N$ to N. Then it follows from (1.8) that

$$S_N - F(t) = \frac{1}{2\pi i} \oint_{C_\tau} \sum_{n=-N}^{N} \frac{\psi_n(t)\psi_N^+(t')}{\langle \psi_n(t')\psi_n^+(t')\rangle} \, (F(t') - F(t)) \, dp(t'). \tag{1.9}$$

We denote by $\lambda(Q)$ a function on Γ having a simple zero at the point P_+ and a pole of order $g+1$ at the point P_-. Then a special case of an assertion of [8] is the equation

$$\lambda(Q)\psi_n(Q) = \sum_{i=1}^{g+1} h_n^i \psi_{n+i}(Q). \tag{1.10}$$

Analogously to the case of differential operators [10] one proves that ψ_n^+ satisfies the adjoint equation

$$\lambda(Q')\psi_n^+(Q') = \sum_{i=1}^{g+1} h_{n-i}^n \psi_{n-i}^+. \tag{1.11}$$

From this

$$\sum_{n=-N}^{N} (\lambda(Q) - \lambda(Q'))\psi_n(Q)\psi_n^+(Q') = \sum_{i=1}^{g+1} \left(\sum_{n=N-i}^{N} h_n^i \psi_{n+i}\psi_n^+ - \sum_{n=-N-i}^{-N} h_n^i \psi_{n+i}\psi_n^+ \right) \tag{1.12}$$

The standard expressions for the ψ_n in terms of the Riemann theta-functions [8, 14] have the form

$$\psi_n(Q) = e^{np(Q)}\varphi_n(Q), \qquad \psi_n^+(Q) = e^{-np(Q)}\varphi_n^+(Q). \tag{1.13}$$

We do not need the exact expressions for φ_n and φ_n^+ in terms of theta-functions. It is sufficient that they are quasiperiodic in n and uniformly bounded on C_τ, if C_τ does not pass through the points γ_s, γ_k^+ [(1.6) is also valid when C_τ passes through γ_s, γ_k^+, only it is necessary to change the normalization of $\psi_n(Q)$].

It follows from this and from (1.12) that

$$S_N - F(t) = \frac{1}{2\pi i} \oint_{C_\tau} dp \, \frac{F(t') - F(t)}{\lambda(t') - \lambda(t)} \big[\Phi_n(t,t') e^{N(p(t)-p(t'))} - \Phi_{-N}(t,t') e^{-N(p(t)-p(t'))}\big], \quad (1.14)$$

where the functions $\Phi_\pm$ are uniformly bounded on C_τ. Since on C_τ, $\operatorname{Re}(p(t) - p(t')) = 0$ in (1.14) one has the integral of a bounded, rapidly oscillating function. Integrating once by parts in (1.14), we get $S_N - F(t) \to 0$ and the lemma is proved.

The assertion of the theorem follows directly from the lemma. Let $E \subset \mathcal{L}(C_\tau)$ be any smooth field on C_τ. Then $F(t) = E/e_0$ is a smooth function on C_τ. From this, if S_N is a partial sum of the series (1.6), constructed from F, then $S_N e_0 - E \to 0$. By virtue of the choice of the points γ_S, the vector field $\psi_n e_0$ for any ψ_n is holomorphic away from $P_\pm$, and hence belongs to L^Γ. From this, $S_N e_0$ belongs to L^Γ and the theorem is proved. □

Remark. It follows from the proof of the theorem that for any contour C, not containing the points $P_\pm$, the restriction of L^Γ to C is dense in the subalgebra of vector fields which can be extended holomorphically to the "annulus" between the two closest contours C_{τ_1}, C_{τ_2}, including C between them.

The proposition proved establishes a connection of the theory of representations of algebras of meromorphic vector fields with the theory of representations of $\mathcal{L}(S^1)$.

To conclude the section, we give an interpretation of the subspace $\tilde{L}_0 \subset L^\Gamma$, generated (for $g \geqslant 2$) by the field e_i, $|i| \leqslant g_0 - 2$. Let D be the group of diffeomorphisms of a circle. It acts on the manifold of moduli of curves of genus g with a distinguished Jordan parametrized contour C. To define its action it suffices to reglue Γ along C with the help of any diffeomorphism. One gets a new algebraic curve Γ' with contour $C' = C$. We denote by $D_\pm$ the subgroups of D formed by diffeomorphisms which can be extended holomorphically to $\Gamma^\pm$, respectively.

The two-sided cosets of D with respect to $D_\pm$ are points of the manifold of moduli of curves of genus g with distinguished contour. When C is a small contour encircling the point P_+, this construction has been discussed by the authors A. I. Bondal, A. A. Beilinson, and M. L. Kontsevich.

It is interesting that this same category is the geometric foundation of the method of integration of $(2+1)$-system of type KP.

The subalgebras $L_\pm^{(-1)}$ are the Lie algebras of the subgroups $D_\pm$. Hence the subspace $\tilde{L}_0$, which for $g \geqslant 2$ has dimension $3g - 3$, can be identified naturally with the tangent space to the manifold of moduli of curves of genus g.

2. Generalized-Graded Modules over L^Γ

The algebra L^Γ acts naturally on the space of meromorphic forms of weight λ on Γ, holomorphic away from the points $P_\pm$. This space will be denoted in what follows by $\mathcal{F}_\lambda^\Gamma = \mathcal{F}_\lambda^\Gamma(P_\pm)$ ($P_\pm$ being points in general position).

Remark. In the present paper we restrict ourselves to the case of integral λ. All the constructions also carry over to the case of arbitrary λ, if one considers piecewisemeromorphic forms, analogously to the way that will be done in Lemma 4.

By the Riemann–Roch theorem there exist unique forms $f_i \in \mathcal{F}_\lambda^\Gamma$ (here $\lambda \neq 0$; the important case $\lambda = 0$ is considered separately in Sec. 3), which in neighborhoods of $P_\pm$ have the form

$$f_j = \varphi_j^\pm z_\pm^{\pm j - S(\lambda)}(1 + O(z_\pm))\,(dz_\pm)^\lambda, \qquad \varphi_j^+ = 1. \tag{2.1}$$

Here j, as before, runs through integral or half-integral values, depending on the parity of g. The quantity $S(\lambda)$ is equal to

$$S(\lambda) = \frac{g}{2} - \lambda(g+1). \tag{2.2}$$

Lemma 3. *The modules $\mathcal{F}_\lambda^\Gamma$ are generalized-graded*:

$$e_i f_i = \sum_{s=-g_0}^{g_0} r_{ij}^s f_{i+j-s}. \tag{2.3}$$

The proof of the lemma is standard. We note that it follows from (1.1) and (2.1) that

$$r_{ij}^{\pm g_0} = (\pm j - S(\lambda) + \lambda(\pm i - g_0 + 1))\left(\frac{\varphi_j^\pm a_i^\pm}{\varphi_{i+j\mp g_0}^\pm}\right). \tag{2.4}$$

Now we describe more general modules over L^Γ. We fix a collection of numbers $(x_{-N}, \dots, x_N) = x$ and some Jordan curve σ joining $P_\pm$.

Lemma 4. *There exists a unique form $f_j(x)$ of weight λ on Γ, which is holomorphic on Γ away from the points $P_\pm$ and the slit σ. It can be extended continuously to σ, where its boundary values satisfy the condition*

$$f_j^+ = e^{2\pi i x_0} f_j^-. \tag{2.5}$$

In neighborhoods of the points $P_\pm$ the form f_j can be represented in the form

$$f_j = \varphi^\pm(x) z_\pm^{\pm j \pm x_0 - S(\lambda)} \exp\left(\sum_{k=1}^N x_{\pm k} z_\pm^{-k}\right)(1 + O(z_\pm))\,(dz_\pm)^\lambda, \quad \varphi_j^+ \equiv 1. \tag{2.6}$$

The space $\mathcal{F}_\lambda^{\Gamma,N}(x)$, generated by the forms $f_j(x)$ of the type described, has a natural L^Γ-model structure.

Lemma 5. *The action of e_i on $f_j(x)$ has the form*

$$e_i f_j = \sum_{s=-g_0-N}^{g_0+N} R_{ij}^s(x) f_{i+j-s}. \tag{2.7}$$

If $N \geqslant 1$,

$$R_{ij}^{\pm g_0 \pm N} = -(N x_{\pm N})\,\frac{\varphi_j^\pm a_i^\pm}{\varphi_{i+j\mp(g_0+N)}^\pm}. \tag{2.8}$$

If $N = 0$, then

$$R_{ij}^{\pm g_0} = (\pm j \pm x_0 - S(\lambda) + \lambda(\pm i - g_0 + 1))\,\frac{\varphi_j^\pm a_i^\pm}{\varphi_{i+j\mp g_0}^\pm}. \tag{2.9}$$

Remark. For $g=0$, $N=0$ the modules $\mathcal{F}_\lambda^{\Gamma,0}(x_0)$ coincide with $\mathcal{F}_{\lambda,x_0}$ the basic modules over the Witt algebra introduced in [6].

For $\lambda=0$, the functions f_j of the type described are a special case of the so-called functions of Gordan–Clebsch–Baker–Akhiezer type in the theory of finite-zone integration (one can find a survey in [11]–[16]). For $\lambda=1$ the forms defined in Lemma 3 are a special case of the forms introduced in [17] for the construction of asymptotically finite-zone solutions of equations of Kadomtsev–Petviashvili type.

We shall call the modules $\mathcal{F}_\lambda^{\Gamma,N}(x)$ modules of Clebsch–Gordan–Baker–Akhiezer (CGBA) type, $x=(x_{-N},\dots,x_N)$.

The proofs of Lemmas 3 and 4 reduce easily to an assertion about the existence and uniqueness of functions of Baker–Akhiezer type. We do not give them in detail since now they have become absolutely standard. The outline is the following. It follows from the theory of boundary problems that there exists a unique form $\tilde f_j(x_0)$, satisfying the hypotheses of Lemma 3 for $x_{\pm k}=0$, $k\neq 0$. This form, away from $P_\pm$, has exactly g zeros $\gamma_1^{j,x_0},\dots,\gamma_g^{j,x_0}$. As is known, there exists a unique Baker–Akhiezer function $\psi_{j,x_0}(\tilde x,P)$ having poles at the points $\gamma_1^{j,x_0},\dots,\gamma_g^{j,x_0}$ and the form

$$\psi_{j,x_0}=\varphi_{j,x_0}^{\pm}\exp\left(\sum_{k=1}^{n}x_{\pm k}z_\pm^{-k}\right)(1+O(z_\pm)),\qquad \tilde x=(x_{\pm k}),\tag{2.10}$$

in neighborhoods of $P_\pm$. From this,

$$f_j(x)=\psi_{j,x_0}(\tilde x,Q)\,\tilde f_j(x_0).\tag{2.11}$$

3. Singular case $\lambda=0$, $x_0=0$. Additional Structures on the Modules $\mathcal{F}_\lambda^\Gamma(x_0)$

In what follows we shall denote the space of $\mathcal{F}_0$-meromorphic functions on Γ having poles only at the points $P_\pm$ by $\mathcal{A}^\Gamma$. It has a natural ring structure. One can define an additive basis for it as follows.

Let A_j, $|j|\geqslant g/2+1$ be the unique functions $A_j\in\mathcal{A}^\Gamma$, which, in neighborhoods of $P_\pm$ have the form

$$A_j=\alpha_j^{\pm}z_\pm^{\pm j-g/2}(1+O(z_\pm)),\qquad \alpha_j^+=1\tag{3.1}$$

(as before, j is integral or half-integral, depending on the parity of g). For $j=-g/2,\dots,g/2-1$ we denote by $A_j\in\mathcal{A}^\Gamma$ a function which, in neighborhoods of $P_\pm$, has the form

$$A_j=\alpha_j^{\pm}z_\pm^{\pm j-g/2\pm 1/2-\varepsilon}(1+O(z_\pm)),\qquad \alpha_j^+=1,\quad \varepsilon=1/2.\tag{3.2}$$

These conditions define A_j uniquely up to addition of a constant, which we denote by $A_{g/2}\equiv 1$.

The structure of $\mathcal{A}^\Gamma$ as a module over L^Γ is somewhat more complicated than in the general case $\lambda\neq 0$.

If $|i+j+g_0+n|>g/2$, then

$$e_iA_j=\sum_{s=-g_0-n}^{g_0}\tilde r_{ij}^sA_{i+j-s},\tag{3.3}$$

where n [here and in (3.4)] is equal to 0 if $|j|>g/2$, and 1 if $|j|\leqslant g/2$.

In those cases when $|i+j+g_0+n| \leqslant g/2$, we have

$$e_iA_j = \sum_{s=-g_0-n+1}^{g_0} \tilde{r}^s_{ij}A_{i+j-s} + \tilde{r}_{ij}A_{g/2}. \tag{3.4}$$

We note further that $e_iA_{g/2} = 0$ for all i.

The multiplicative structure of the commutative ring $\mathcal{A}^\Gamma$ has analogous form.

For $|i+j+g/2+m| > g/2$ [where m, here and in (3.5) and (3.6), is equal to 0 if both numbers $|i|, |j| > g/2$; $m = 1, 2$, if one or, respectively, two of these numbers is $\leqslant g/2$] one has

$$A_iA_j = \sum_{s=-g/2-m}^{g/2} \alpha^s_{ij}A_{i+j-s}. \tag{3.5}$$

For $|i+j+g/2+m| \leqslant g/2$,

$$A_iA_j = \sum_{s=-g/2-m+1}^{g/2} \tilde{\alpha}^s_{ij}A_{i+j-s} + \tilde{\alpha}_{ij}A_{g/2}. \tag{3.6}$$

The ring $\mathcal{A}^\Gamma$ is, from the point of view of the definition given above, g_0-graded, although essentially the degree of "diffusion" of the grading is equal to $g/2$ for almost all i and j for it. In particular, the subrings $\mathcal{A}^\Gamma_\pm$, generated by A_j with $\pm j > g/2$ are $g/2$-graded. The subrings $\mathcal{A}^\Gamma_\pm$ together with the $(g+1)$-dimensional subspace $\mathcal{A}^\Gamma_0$, generated by A_j with $|j| \leqslant g/2$, define a decomposition of $\mathcal{A}^\Gamma$ into a direct sum

$$\mathcal{A}^\Gamma = \mathcal{A}^\Gamma_+ + \mathcal{A}^\Gamma_0 + cA^\Gamma_-, \tag{3.7}$$

analogous to (2).

The multiplicative structure of $\mathcal{A}^\Gamma$ lets us define, for any semisimple Lie algebra G, the algebra

$$G^\Gamma = G \otimes \mathcal{A}^\Gamma, \tag{3.8}$$

which is a generalization to the case of arbitrary Riemann surfaces of genus $g > 0$, of the KacMoody algebras. The elements of this algebra are meromorphic functions on Γ, holomorphic away from $P_\pm$, and assuming values in G. The connection of G^Γ with the current algebra $G(S^1)$ is given by Theorem 2.

Theorem 2. *For any contour C_τ, the restriction of G^Γ to C_τ defines a dense subalgebra of the algebra of smooth functions on C_τ with values in G:*

$$G(C_\tau) \approx G(S^1).$$

Here the contours C_τ are the same as in Sec. 1. One can get a proof of the theorem completely analogous to the proof of Lemma 2. We note that ψ_n goes into A_n as the divisor $\gamma_1, \dots, \gamma_g$ tends to $P_\pm$.

The spaces $\mathcal{F}^{\Gamma,N}_\lambda(x)$ are modules over $\mathcal{A}$. Multiplication of $\mathcal{A}^\Gamma$ by f_i can be represented in the form

$$A_if_j = \sum_{s=-g/2}^{g/2} \beta^s_{ij}f_{i+j-s}, \qquad \text{for } |i| > g/2, \tag{3.9}$$

$$A_if_j = \sum_{s=-g/2-1}^{g/2} \beta^s_{ij}f_{i+j-s}, \qquad \text{for } |i| \leqslant g/2. \tag{3.10}$$

Separately we have $A_{g/2}f_j = f_j$.

The spaces $\mathcal{F}_\lambda^{\Gamma,N}$ are modules over the ring of differentiable operators in the variables $x_{\pm k}$. The action of the generators has the form

$$\frac{\partial}{\partial x_{\pm k}} f_j = \sum_{s=0}^{k} F_{kj}^s(x) f_{j\pm s}, \qquad k = 1, \dots, N. \tag{3.11}$$

We return to this in more detail in the concluding section.

4. "Local" Central Extension of the Algebra L^Γ and Analogs of the Verma Modules

As follows from the assertion of Lemma 5 (for $N = 0$), the action of the operators e_i on f_j defines a homomorphism of the algebra L^Γ into the algebra of difference operators $\mathfrak{Gl}^\infty$ of finite order. The latter algebra can be represented in the form of the algebra of infinite matrices having only a finite number of nonzero diagonals. The subalgebras of matrices from $\mathfrak{Gl}^\infty$, having nonzero elements only over or under the main diagonal, are denoted by $\mathfrak{Gl}_+^\infty$ or $\mathfrak{Gl}_-^\infty$, respectively.

It follows from (2.7) that the images of the subalgebras $L_\pm^\Gamma$, generated by e_i with $\pm i > g_0$, belong to $\mathfrak{Gl}_\pm^\infty$. The decomposition

$$L^\Gamma = L_+^\Gamma + L_0^\Gamma + L_-^\Gamma, \qquad L_0^\Gamma = \{e_{-g_0}, \dots, e_{g_0}\}, \tag{4.1}$$

is the analog of the decomposition 2 for the case of algebras generated by Riemann surfaces of arbitrary genus $g > 0$.

The algebra $\mathfrak{Gl}^\infty$ has a unique central extension $\widehat{\mathfrak{Gl}}^\infty$. One can construct a representation of this extension starting from the space of semiinfinite forms over the modules $\mathcal{F}_\lambda^\Gamma(x_0)$, $N = 0$. A basis in this space $H_\lambda^\Gamma(x_0)$ is formed from expressions of the form

$$f_{i_0} \wedge f_{i_1} \wedge \dots \wedge f_{i_{m-1}} \wedge f_m \wedge f_{m+1} \wedge f_{m+2} \wedge \cdots, \qquad f_j \in \mathcal{F}_\lambda^\Gamma(x_0), \tag{4.2}$$

where $i_0 < i_1 < \dots < i_{m-1} < m$ (cf. [6] for the case $g = 0$).

For any operator D from $\mathfrak{Gl}_\pm^\infty$, in particular $e_i \in L_\pm^\Gamma$, there is a well-defined action of this operator on $H_\lambda^\Gamma(x_0)$. The action of $e_i \in L_\pm^\Gamma$ on the generators (4.2) is defined by the Leibniz formula. Since in (4.2), starting from some place all indices stand in succession, as a result of the action of e_i on $|i| > g_0$ one gets a finite sum of expressions of the form (4.2). The natural action of $\mathfrak{Gl}_\pm^\infty$ on $H_\lambda^\Gamma(x_0)$ extends to a representation of the central extension $\widehat{\mathfrak{Gl}}^\infty$. Thus, the homomorphism $L^\Gamma \to \mathfrak{Gl}^\infty$ induces a representation of some central extension $\hat{L}^\Gamma$ of the algebra L^Γ on $H_\lambda^\Gamma(x_0)$.

On Γ we define a "projective structure," where admissible systems of local coordinates differ by a projective substitution. If $f(z)(\partial/\partial z)$ and $g(z)(\partial/\partial z)$ are representations of two vector fields in an admissible coordinate system, then the form $\tilde{\chi}(f, g) = f''' g\, dz$ is well defined. Any closed contour C on Γ, not passing through $P_\pm$, defines a two-dimensional cocycle on the algebra L^Γ:

$$\chi_C(e_i, e_j) = \frac{1}{24\pi i} \oint_C \tilde{\chi}(e_i, e_j). \tag{4.3}$$

Central extensions of L^Γ, defined by the cocycles (4.3), are the algebras generated by the elements e_i and a central element t with the following commutation relations:

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} c_{ij}^s e_{i+j-s} + t\chi_C(e_i, e_j), \qquad [e_i, t] = 0. \tag{4.4}$$

Standard calculations of the two-dimensional cohomology of algebras let one prove that (4.3) and (4.4) give all central extensions of L^Γ.

Lemma 6. *There exists a unique "local" central extension of L^Γ, having the property*

$$\chi_0(e_i, e_j) = 0, \qquad |i + j| > 3g. \tag{4.5}$$

This extension corresponds to a unique homology class of a non-self-intersecting contour, dividing Γ into $\Gamma^\pm$ so that $P_\pm \subset \Gamma^\pm$. It preserves the property of g_0-gradedness.

In what follows, this "local" extension will be denoted by $\hat{L}^\Gamma$.

Proof. Let us assume that the cocycle χ_C satisfies (4.5) and is gotten by integrating $\tilde{\chi}$ along a cycle C which is not homologous to zero.

Let $\gamma_1, \dots, \gamma_g$ (as in the proof of Lemma 2) be the zeros of e_0 away from the points $P_\pm$. Then $e_i = \psi_n e_0 = e^{np}\varphi_n e_0$, where $\varphi_n(Q)$ is a quasiperiodic function of n. The integral

$$\chi_C(e_n, e_0) = \oint_C \psi_n(Q)\tilde{\chi}(e_0, e_0)$$

can be calculated for large n by the saddle-point method. We get that it follows from (4.5) that one of the zeros of ψ_n tends exponentially to a point Q_0, at which the function $p(Q)$ has a maximum on C. Sibnce the equivalence class of divisors of zeros of ψ_n is uniformly distributed on the Jacobian of Γ, this is impossible. This argument fails only when C is homologous to one of the components of the contour C_τ, which is not homologous to zero, for some value of τ. Let C' be the complement of C to C_τ, $C_\tau = C \cup C'$. Then it follows from (4.5) and the course of the proof of Lemma 2 that the function $F(t)$, equal to $\tilde{\chi}(e_0, e_0)/d\omega$ on C and to zero on C', splits into a finite sum of functions ψ_k^+, $|k| < 3g$. But this is impossible, because any finite sum of such functions is meromorphic on Γ and cannot be identically equal to zero on C'. The lemma is proved. □

Theorem 3. *The action of the subalgebras $L_\pm^\Gamma$ on $H_\lambda^\Gamma(x_0)$ extends to a representation of the central extension $\hat{L}^\Gamma$. Here the vector*

$$\Phi_0 = f_\varepsilon \wedge f_{\varepsilon+1} \wedge f_{\varepsilon+2} \wedge \dots, \qquad \begin{cases} \varepsilon = 0, & g \equiv 0 \pmod 2, \\ \varepsilon = 1/2, & g \equiv 1 \pmod 2, \end{cases} \tag{4.6}$$

is singular for the subalgebra L_+,

$$L_+\Phi_0 = 0. \tag{4.7}$$

Moreover, one has

$$e_{g_0}\Phi_0 = h\Phi_0, \quad t\Phi_0 = c\Phi_0. \tag{4.8}$$

Here

$$c(\lambda) = -2(6\lambda^2 - 6\lambda + 1), \quad h = \frac{1}{2}(x_0 - S(\lambda))(S(\lambda) + 1 - x_0 - 2\lambda). \tag{4.9}$$

Equations (4.7) and (4.8) define analogs of Verma modules, h is called the "highest weight" and c the "central charge."

We give the proof of the theorem. The first part of the.assertions follows from the fact that by the locality property the extension $\hat{L}^\Gamma$ is induced by the extension $\widehat{\mathfrak{Gl}}^\infty$ under a homomorphism of L^Γ into $\mathfrak{Gl}^\infty$. Hence restriction of a representation of $\widehat{\mathfrak{Gl}}^\infty$ in $H^\Gamma_\lambda(x_0)$ to $\hat{L}^\Gamma$ defines an extension of the action of $L^\Gamma_\pm$. The uniqueness assertions which are needed in the proof are the formulas of (4.9), expressing the highest weight and central charge of the representation in terms of the tensor weight λ and the parameter x_0.

It follows from the form (1.1) of the fields e_i and e_{-i+2g_0} neighborhoods of $P_\pm$ that

$$\chi_0(e_i, e_{-i+2g_0}) = \frac{1}{12}((i-g_0)^3 - (i-g_0)). \tag{4.10}$$

We apply the operator $[e_i, e_{-i+2g_0}]$ to the singular vector Φ_0, $i > g_0$. We get

$$[e_i, e_{-i+2g_0}]\Phi_0 = e_i e_{-i+2g_0}\Phi_0 = \sum_{j=\varepsilon}^{i-g_0+\varepsilon-1} (R^{g_0}_{-i+2g_0,j} R^{g_0}_{i,j-i+g_0})\Phi_0 = \tilde{c}\Phi_0, \tag{4.11}$$

where $\varepsilon = 0$ or $1/2$, depending on the parity of g. It follows from (2.9) that

$$\tilde{c} = -[(i-g_0)^3 - (i-g_0)](2\lambda^2 - 2\lambda + 1) - (i-g_0)(x_0 - S)(S + 1 - 2\lambda - x_0). \tag{4.12}$$

Since $L_+\Phi_0 = 0$, it follows from (4.4) that

$$[e_i, e_{-i+2g_0}]\Phi_0 = c^{g_0}_{i,-i+2g_0} e_{g_0}\Phi_0 + \chi_0(e_i, e_{-i+2g_0})c\Phi_0 = \tilde{c}\Phi_0 \tag{4.13}$$

and the equations of (4.9) are proved, and with them the theorem.

We consider all linearly independent vectors of the form

$$\Phi^{n_1,\dots,n_k}_{i_1,\dots,i_k} = e^{n_1}_{i_1} \dots e^{n_k}_{i_k}\Phi_0, \qquad -\infty < i_s < g_0, \quad i_1 < i_2 < \dots < i_k. \tag{4.14}$$

Lemma 7. *Equations* (4.7) *and* (4.8)*, together with the condition*

$$t\Phi^{n_1\dots n_k}_{i_1\dots i_k} = c\Phi^{n_1\dots n_k}_{i_1\dots i_k}$$

uniquely and consistently define a representation $U^\Gamma_{h,c}$ of the algebra $\hat{L}^\Gamma$ with central charge c.

The *proof* of Lemma 7 is based on the "filtration" of elements $n = -\sum n_j(i_j - g_0)$, where the filtration of Φ_0 is equal to 0. Here one makes use of (1.2), (1.3), and also (4.5). The proof does not differ essentially from the corresponding elementary argument for ordinary Verma modules over the Virasoro algebra. If for all $m < n$ the action of $\hat{L}^\Gamma$ and e_{g_0} is constructed on elements of filtration m, then it is necessary to extend it consistently to elements of filtration n. This is done starting from the commutator relations, letting one restrict oneself to monomials such that $n_1 = \dots = n_k = 1$. We set $\bar{e}_i = e_{i-g_0}$. Then we have, according to (1.2) and (4.5),

$$[\bar{e}_i, \bar{e}_j] = (j-i)\bar{e}_{i+j} + \beta(i,j), \tag{4.15}$$

where the filtration of the element $\beta(i,j)$ does not exceed $i+j-1$ (the filtration of the element t is equal to zero). There are no other relations. Hence, the collection of elements $\Phi^{n_1\dots n_k}_{i_1\dots i_k}$ is not subject to factorization. The fact that the collection of elements $(\bar{e}_i)$, $i < 0$ does not form a subalgebra plays no role. Lemma 7 is proved.

Thus, the module constructed above, spanned by the vector Φ_0 from the space $H^\Gamma_\lambda(x_0)$, is a quotient-module (homomorphic image) of the universal "Verma module" $U^\Gamma_{h,c}$.

We consider the associated Z-graded algebra $\bar{L}^\Gamma$ with respect to the filtration indicated in the proof of Lemma 7 with basis $\bar{e}_i$. The commutator in this algebra, according to (1.3), coincides with the commutator in the Virasoro algebra. Hence, $\bar{L}^\Gamma$ simply coincides with the Virasoro algebra.

Analogously, starting from the basis $\bar{\bar{e}}_i = e_{-g_0-i}$, we get a second filtration, decreasing on the opposite side. The commutator in the algebra associated with this filtration $\bar{\bar{L}}^\Gamma$ has the form

$$[\bar{\bar{e}}_i, \bar{\bar{e}}_j] = (j-i)\,\frac{b_i b_j}{b_{i+j}}\,\bar{\bar{e}}_{i+j}, \tag{4.16}$$

where $b_i = a^{-g_0}_{-g_0-i}$. The substitution $\bar{\bar{e}}_i = b_i \bar{e}_i$ shows that $\bar{\bar{L}}^\Gamma$ is isomorphic with the Virasoro algebra.

As already stated in the preceding section, on the modules $\mathcal{F}^\Gamma_\lambda(x_0)$ the commutative algebra $\mathcal{A}^\Gamma$ also acts. Its central extension can be described with the help of cocycles of the form

$$\gamma(A_i, A_j) = \frac{1}{2\pi i}\oint_C A_i\, dA_j,$$

where C is a closed contour on Γ, not passing through the points $P_\pm$. Here there is a representation (3.9), (3.10) of the algebra $\mathcal{A}^\Gamma$ on the algebra of difference operators $\mathfrak{Gl}^\infty$. The action of the subalgebras $\mathcal{A}^\Gamma_\pm$ is well defined on the spaces $H^\Gamma_\lambda(x_0)$ and extends to an action of the central extension $\hat{\mathcal{A}}^\Gamma$ of the whole algebra $\mathcal{A}^\Gamma$. This central extension is generated by A_i and t with the commutation relations

$$[A_i, A_j] = \gamma_0(A_i, A_j)t, \qquad [A_i, t] = 0. \tag{4.17}$$

Here γ_0, as in the case of the algebra L^Γ, is the unique "local" cocycle

$$\gamma_0(A_i, A_j) = 0, \qquad |i+j| > g, \tag{4.18}$$

corresponding to the contour C_0 (more correctly its homology class), which divides Γ into two parts $\Gamma^\pm$, such that $P_\pm \subset \Gamma^\pm$. The "central charge" in this case is equal for all λ and x_0, to one, i.e.,

$$t\Phi_0 = \Phi_0. \tag{4.19}$$

For genus $g = 0$, the algebra $\hat{\mathcal{A}}^\Gamma$ coincides with the Heisenberg algebra p_i, q_j, t:

$$[p_i, p_j] = [q_i, q_j] = 0, \qquad [p_i, q_j] = i\delta_{i,j} t, \tag{4.20}$$

where $p_i = A_i$, $q_i = A_{-i}$.

5. Case of Elliptic Curves ($g = 1$)

Let Γ be an elliptic curve with periods 2ω and $2\omega'$. All information from the theory of elliptic functions which is needed for what follows can be found in [18], whose notation we adhere to in detail.

On Γ the vector field $\partial/\partial z$ has no zeros or poles. Hence L^Γ and $\mathcal{A}^\Gamma$, as linear spaces, are isomorphic. Their bases e_i and A_i are connected as follows:

$$e_i = A_i(z)\,\frac{\partial}{\partial z}. \tag{5.1}$$

For all half-integral i except $i = -1/2$, the basis functions $A_i \in \mathcal{A}^\Gamma$ are defined by

$$A_i(z) = \frac{\sigma^{i-1/2}(z-z_0)\,\sigma(z+2iz_0)}{\sigma^{i+1/2}(z+z_0)\,\sigma((2i+1)z_0)}\,\sigma^{i+1/2}(2z_0). \tag{5.2}$$

Here $\sigma(z)$ is the Weierstrass σ-function. The function $A_{-1/2}$, completing (5.2) to a basis in $\mathcal{A}^\Gamma$ can be chosen in the form

$$A_{-1/2} = \frac{\sigma^2(z)\,\sigma(2z_0)}{\sigma(z+z_0)\,\sigma(z-z_0)\,\sigma^2(z_0)}. \tag{5.3}$$

The commutation relations in L^Γ have a form slightly different from the general case $g \neq 1$, and due to (5.1), recall to a considerable degree the formulas (3.3) and (3.4) in structure.

For $|i| \neq 1/2$, $|j| \neq 1/2$, and such that $i+j \neq -2$,

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} c_{ij}^s e_{i+j-s}, \qquad g_0 = 3/2. \tag{5.4}$$

For $|i| \neq 1/2$, $|j| \neq 1/2$, and $i+j = -2$,

$$[e_i, e_j] = \sum_{s=-1/2}^{g_0} c_{ij}^s e_{-2-s} + \tilde{c}_{ij} e_{1/2} \tag{5.5}$$

The commutators e_i with $e_{1/2} = \partial/\partial z$ have the form

$$[e_{1/2}, e_i] = \sum_{s=-1/2}^{g_0} c_{1/2,i}^s e_{i+1/2-s}, \qquad i \neq 3/2, -1/2, \tag{5.6}$$

$$[e_{1/2}, e_{-3/2}] = \sum_{s=1/2}^{g_0} c_{1/2,-3/2}^s e_{-1-s} + \tilde{c}_{1/2,-3/2} e_{1/2}, \tag{5.7}$$

$$[e_{1/2}, e_{-1/2}] = \sum_{s=-g_0}^{g_0} c_{1/2,-1/2}^s e_s. \tag{5.8}$$

Finally,

$$[e_{-1/2}, e_i] = \sum_{s=-5/2}^{g_0} c_{-1/2,i}^s e_{i-1/2-s}, \qquad i \neq -5/2, \tag{5.9}$$

$$[e_{-1/2}, e_{-5/2}] = \sum_{s=-g_0}^{g_0} c_{-1/2,-5/2}^s e_{-3-s} + \tilde{c}_{-1/2,-3/2} e_{1/2}. \tag{5.10}$$

First we find the coefficients c_{ij}^s in the general case (5.4). For this, as also in the case of all the other formulas (5.5)–(5.10), it suffices to find expressions for the coefficients $a_i = \alpha_i^-$ and $\xi_i^\pm$ in the decompositions of e_i, $|i| \neq 1/2$, in neighborhoods of the points $z = \pm z_0$:

$$e_i = \alpha_i^\pm z_\pm^{\pm i-1/2}(1 + \xi_i^\pm z_\pm + O(z_\pm^2))\,\frac{\partial}{\partial z_\pm}, \qquad z_\pm = z_\pm z_0, \tag{5.11}$$

and the coefficients $\xi_{-1/2}^\pm$ in the analogous decomposition

$$e_{-1/2} = \pm z_\pm^{-1}(1 + \xi_{-1/2}^\pm z_\pm + O(z_\pm^2))\,\frac{\partial}{\partial z_\pm}. \tag{5.12}$$

It follows from (5.2) that

$$a_i = (-1)^{i-1/2}\,\frac{\sigma^{2i}(2z_0)\,\sigma((2i-1)z_0)}{\sigma((2i+1)z_0)}, \tag{5.13}$$

$$\xi_i^+ = \zeta((2i+1)z_0) - (i+1/2)\,\zeta(2z_0), \tag{5.14}$$

$$\xi_i^- = \zeta((2i-1)z_0) - (i-1/2)\,\zeta(2z_0), \tag{5.15}$$

For $i=-1/2$, we have

$$\xi^{\pm}_{-1/2} = 2\zeta(z_0) - \zeta(2z_0). \tag{5.16}$$

The coefficients $c_{ij}^{3/2}$ in all the formulas are equal, as also in the general case,

$$c_{ij}^{3/2} = (j-i). \tag{5.17}$$

In order to find c_{ij}^s in (5.4), it is necessary to substitute the decomposition (5.11) in (5.4) and equate the coefficients of $z_\pm^{\pm i\pm j-2}$ and $z_1^{\pm i\pm j-1}$ on both sides of the equations. We get

$$c_{ij}^{-3/2} = (i-j)\,\frac{a_i a_j}{a_{i+j+3/2}}, \tag{5.18}$$

where the a_i are given by (5.13). Moreover,

$$c_{ij}^{1/2} = (i-j)\xi^+_{i+j-3/2} + (j-1/2)\xi_j^+ - (i-1/2)\xi_i^+.$$

For $i+j\neq 1$, substituting (5.14), we get

$$c_{ij}^{1/2} = (i-j)(\zeta((2i+2j-2)z_0) + \zeta(2z_0)) + (j-1/2)\,\zeta((2j+1)z_0) - (i-1/2)\,\zeta((2i+1)z_0). \tag{5.19}$$

Analogously, for $c_{ij}^{-1/2}$ for $|i+j+3/2|\neq -1/2$

$$c_{ij}^{-1/2} = \frac{a_i a_j}{a_{i+j+1/2}}\big[(j-i)(\zeta((2i+2j+2)z_0) - \zeta(2z_0)) - (j+1/2)\,\zeta((2j-1)z_0) + (i+1/2)\,\zeta((2i-1)z_0)\big]. \tag{5.20}$$

If $i+j=1$, $|i|\neq 1/2$, $|j|\neq 1/2$, we have, using the fact that $\xi^{\pm}_{1/2}=0$,

$$c_{i,1-i}^{1/2} = (4i-2)\,\zeta(z_0) + (1/2-i)\,\zeta((3-2i)z_0) - (i-1/2)\,\zeta((2i-1)z_0), \tag{5.21}$$

$$c_{i,1-i}^{-1/2} = -a_i a_{1-i}\big[(4i+2)\,\zeta(z_0) - (1/2+i)\,\zeta((3+2i)z_0) + (i+1/2)\,\zeta((2i+1)z_0)\big], \tag{5.22}$$

One finds the coefficients in (5.5)–(5.8) analogously. We do not give them here only in order to conserve space.

There is a difference in (5.9) and (5.10). They contain on the right side not four, as in the other cases, but five summands. The coefficients $c^s_{-1/2,i}$ for $s = 3/2, 1/2, -3/2, -5/2$ in (5.9) and $c^s_{-1/2,-5/2}$ for $s=3/2,1/2,-3/2$, and $\tilde{c}_{-1/2,-5/2}$ in (5.10) can be expressed as before in terms of the corresponding a_i and $\xi_i^{\pm}$. For finding $c^{-1/2}_{-1/2,i}$ one could find an expression in terms of the following coefficient of the decomposition of e_i in neighborhoods of $\pm z_0$, but one can avoid this if one makes use of the fact that $e_{-1/2}$ has a double zero at the point $z=0$. Hence, the left side of (5.9) vanishes at $z = 0$ and $c^{-1/2}_{-1/2,i}$ for (5.9) can be found from the supplementary equation

$$\sum_{s=-5/2}^{3/2} c^s_{-1/2,i} A_i(0) = 0. \tag{5.23}$$

Analgously, one finds $c^{-1/2}_{-1/2,-5/2}$ in the case of (5.10).

6. Structures of the Theory of Solitons

A generalization of the construction of Sec. 4 is the realization of the representations of $\hat{L}^\Gamma$ on the space $H^{\Gamma,N}_\lambda(x)$ of semiinfinite forms of the form (4.2), where $f_j(x) \in \mathcal{F}^{\Gamma,N}_\lambda(x)$. Here and below, $x = (x_{-N}, \dots, x_N)$.

According to Lemma 5, the action of e_i on $f_j(x)$ defines, for each x, a homomorphism of L^Γ into the algebra of difference operators. As follows from (2.7), on $H^{\Gamma,N}_\lambda(x)$ there is a well-defined action of the subalgebras $L^{(N+1)}_\pm \subset L^\Gamma$, generated (in correspondence with the definition of Sec. 1) by the elements e_i with $\pm i \geqslant g_0 + N + 1$. As in the case $N = 0$, this action extends to a representation of the algebra $\hat{L}^\Gamma$ such that

$$L^{(N+1)}_+ \Phi_0 = 0, \tag{6.1}$$

$$t\Phi_0 = c\Phi_0, \qquad e_{g_0+N}\Phi_0 = h\Phi_0. \tag{6.2}$$

The indicated family of representations of $\hat{L}^\Gamma$ (in which the central charge and analog of the "highest weight" h can depend on x) is needed in the more detailed analysis to which we expect to return in the future.

We consider the space $\mathcal{F}^{\Gamma,N}_{-1}(x)$. By definition, this is the space of vector fields on Γ, holomornhic away from $P_\pm$, and having, on the line σ, joining $P_\pm$, the jump (2.5) and an exponential singularity at $P_\pm$. The direct integral of such spaces has a natural Lie algebra structure and will be denoted by $L^{\Gamma,N}$.

A basis in this space is formed by the vector fields $e_i(x)$ of the form (2.6).

Lemma 8. *The commutator of two basic fields has the form*

$$[e_i(x), e_j(x)] = \sum_{s=-g_0-N}^{g_0+N} c^s_{ij}(x,y) e_{i+j-s}(x+y). \tag{6.3}$$

The proof of (6.3) is standard and uses only the uniqueness of $e_i(x)$, having the analytic properties listed in the assertion of Lemma 4.

The equations of (6.3) show that $L^{\Gamma,N}$ is a "generalized-multigraded" algebra. The (i,x) appear in the role of "multiindices," while the "diffusion" of the grading occurs only with respect to the index i. There is exactly a grading with respect to the continuous vector index x. In the algebra $L^{\Gamma,N}$ one can single out the subalgebras $L^{\Gamma,N}_\pm(S)$, generated by the fields $e_i(x^\pm)$ with $\pm i \geqslant g_0+N+s$, $s \geqslant -1$. Here $x^\pm$ are vectors of the form $x^+ = (0, \dots, 0, x_0, x_1, \dots, x_N)$, $x^- = (x_{-N}, \dots, x_0, 0, \dots)$.

The spaces $\mathcal{F}^{\Gamma,N}_\lambda$, the direct integrals of the $\mathcal{F}^{\Gamma,N}_\lambda(x)$, are "generalized-multigraded" modules over $L^{\Gamma,N}$:

$$e_i(x) f_j(y) = \sum_{s=-g_0-N}^{g_0+N} R^s_{ij}(x,y) f_{i+j-s}(x+y). \tag{6.4}$$

Up to now we have used representations of L^Γ on the algebra of difference operators [for $L^{\Gamma,N}$ on the algebra of "generalized-difference operators" (6.4)]. With this L^Γ also admits representations in the form of differential operators.

Lemma 9. *Let $f_j \in \mathcal{F}_\lambda^{\Gamma,N}$ be the form of weight λ, defined in Lemma 4. Then for any $e_i \in L^\Gamma$ there exists a unique operator in the variables x_{-1} and x_1:*

$$D_i^j = \sum_{s=0}^{+N-i+g_0} u_{is}^j(x)\left(\frac{\partial}{\partial x_1}\right)^s + \sum_{s=1}^{N+i+g_0} v_{is}^j(x)\left(\frac{\partial}{\partial x_{-1}}\right)^s \tag{6.5}$$

such that

$$e_i f_j = D_i^j f_j \tag{6.6}$$

[*there is no summation over j in* (6.6)].

By (2.11) and the results of [19], we have that f_j satisfies the two-dimensional Schrödinger equation

$$\hat{H}_j f_j = 0, \qquad \hat{H}_j = \frac{\partial^2}{\partial x_1 \partial x_{-1}} + v_j(x)\frac{\partial}{\partial x_-} + u_j(x) \tag{6.7}$$

and the ideal generated by $\hat{H}_j$ in the ring of differential operators in the variables x_1, x_{-1} coincides with the ideal of operators annihilating f_j. It follows from (6.5) and (6.7) that the operators D_i^j realize the algebra L^Γ on the space of solutions of the equation $\hat{H}_j\psi = 0$. It was proved in [20] that there is also an analogous representation for the commutative ring $\mathcal{A}^\Gamma$.

Now we consider the forms $f_j^+ = f_j(x^+)$, $x^+ = (0, 0, \ldots, x_1, \ldots, x_N)$ which are forms having exponential singularities only at one point $P_\pm$.

It was proved in [20] that each Baker–Akhiezer function generates a homomorphism of the ring $\mathcal{A}_-^\Gamma \in \mathcal{A}^\Gamma$ of functions on Γ, having poles at the point P_+, into the ring of ordinary differential operators. It follows from (2.11) and this assertion that for any A_i with $-i \geqslant g/2 + 1$, there exist unique operators

$$M_i^j = \sum_{s=0}^{g/2-i} w_{is}^j(x^+)\left(\frac{\partial}{\partial x_1}\right)^s \tag{6.8}$$

such that $M_i^j f_j^+ = A_i f_j^+$.

In the case of the forms f_j^+ considered, the action of e_i on f_j^+ is also equivalent, for $i \leqslant -g_0$ to the action of an ordinary differential operator

$$D_i^{j+} f_j^+ = e_i f_j^+, \qquad D_i^{j+} = \sum_{s=0}^{-i+N+g_0} u_{is}^j(x^+)\left(\frac{\partial}{\partial x_1}\right)^s \tag{6.9}$$

Proposition. *The operators D_i^{j+}, $i \leqslant -g_0$ define an extension of the commutative rings of ordinary differential operators M_i^j. This extension generates a representation of the Z_2-graded Lie algebra $W^\Gamma = W_0^\Gamma + W_1^\Gamma$, where W_0^Γ is the algebra $L_-^{\Gamma,(-1)} \subset L^\Gamma$, and $W_1^\Gamma = \mathcal{A}_-^\Gamma$ is the commutative algebra of functions on Γ with pole at $P_\pm$. The product $[W_0^\Gamma, W_1^\Gamma]$ is the natural one.*

Remark. In [21, 22] there are constructed representations of the algebra of smooth vector fields on S^1 on the algebra of symmetries of nonlinear equations admitting a representation of curvature zero. To fields from $\mathcal{L}(S^1)$, which are restrictions of the algebra L^Γ to the contour, correspond symmetries leaving invariant the manifold of finite-zone solutions, corresponding to the curve Γ (by virtue of the results of our paper).

Example. Let Γ be an elliptic curve and f_n a Baker–Akhiezer function of the form

$$f_n(z) = z_+^n e^{z_+^{-1}x}(1 + O(z_+))$$

in a neighborhood of z_0 and such that $f_n = O(z_-^{-n})$ in a neighborhood of $-z_0$.

It is well known that the operators

$$L_n = \frac{\partial^2}{\partial x^2} - 2\wp(x + 2nz_0), \quad \tilde{A}_n = -2\frac{\partial^3}{\partial x^3} + 6\wp(x + 2nz_0) + 3\wp'(z + 2nz_0) \tag{6.10}$$

satisfy the condition

$$L_n f_n = \wp(z - z_0) f_n, \qquad \tilde{A}_n = \wp'(z - z_0) f_n \tag{6.11}$$

and thus generate a commutative subring of the ring of ordinary differential operators. The action of $e_{1/2}$ on f_n is equal to

$$e_{1/2} f_n = D_{1/2} f_n = \left(-x\left(\frac{\partial^2}{\partial x^2} - 2\wp(x + 2nz_0)\right) - \frac{\partial}{\partial x} - \zeta(x + 2nz_0) f_n. \tag{6.12}$$

The commutative relations between L_n and $\tilde{A}_n$, $D_{1/2}$ are easy to find:

$$[L_n, D_{1/2}] = +\tilde{A}_n, \qquad [D_{1/2}, \tilde{A}_n] = -6L_n^2 + \frac{1}{2}\,g_2.$$

In conclusion, we briefly formulate results showing that consideration of hhe form of an arbitrary weight λ on Γ lets us extend the classes of exact solutions of spatially-two-dimensional equations of Kadomtsev–Petviashvili type.

On Γ we fix an arbitrary collection of points in general position $\gamma_1, \dots, \gamma_M$. If $M \geqslant 2S(\lambda)$, then the dimension of the space of forms of weight λ on Γ, meromorphic away from P_+, where they have poles at $\gamma_1, \dots, \gamma_M$, and having the form

$$\Psi = \exp(z^{-1}x + z^{-2}y + z^{-3}t)\left(\sum_{s=0}^{\infty} \xi_s z^s\right)(dz)^\lambda \tag{6.13}$$

in a neighborhood of $P_\pm$, is equal to $M - 2S + 1$. The proof of this fact, analogously to the proof of Lemma 4, reduces to a similar assertion for the Baker–Akhiezer functions [13] [here, as before, $2S = g - 2\lambda(g-1)$].

Let $\sigma_1, \dots, \sigma_{M-2S}$ be an arbitrary collection of contours on Γ. On them we define $M - 2S$ forms h_k of weight $1 - \lambda$; then one can define a form Ψ from the conditions

$$\oint_{\sigma_k} \Psi h_k = 0, \qquad k = 1, \dots, M - 2S, \tag{6.14}$$

and ξ_0 in (6.13) is equal to one: $\xi_0 \equiv 1$.

Theorem 4. *There exist unique operators*

$$L = \frac{\partial^2}{\partial x^2} + u(x,y,t), \qquad A = \frac{\partial^3}{\partial x^3} + \frac{3}{2}\,u\,\frac{\partial}{\partial x} + w(x,y,t) \tag{6.15}$$

such that

$$(\partial_y - L)\Psi = (\partial_t - A)\Psi = 0. \tag{6.16}$$

The coefficient $u(x,y,t)$, which due to (6.16) is a solution of the Kadomtsev–Petviashvili equation, is an "asymptotically finite zone" solution.

For the case $\lambda = 1$ the assertion of this theorem is found in [17], where one can find the precise meaning of the term "asymptotically finite-zone." The proof of the theorem for all λ is practically no different.

In a similar way one can use forms of Baker–Akhiezer type for the construction of solutions of general equations admitting commutation representation, not containing a spectral parameter explicitly. These are equations of KP type or equations having L, A, B-triples.

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