BLOCH HOMOLOGY. CRITICAL POINTS OF FUNCTIONS AND CLOSED 1-FORMS

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Suppose given a regular covering space over a closed manifold, with discrete group Γ:

$$\pi\colon \hat{M} \to M^n, \quad \gamma\colon \hat{M} \to \hat{M}, \quad \gamma \in \Gamma. \tag{1}$$

The Generalized Morse Problem (Problem 1). Find effective lower bounds on the number of critical points of a smooth function f on M^n, taking account of the group Γ (see [1], where the question of existence of such bounds is examined, but not effectively).

Problem 2 (the author [2, 3])**.** Find lower bounds on the number of critical points of a closed 1-form ω on M^n.

Of particular interest in Problem 2 is the case $\Gamma = \mathbb{Z}^k$, where k is the number of integrals of ω, over 1-cycles in M^n, that are linearly independent over $\mathbb{Z}$. The problem was solved by the author in [2] and [3] for the case $k = 1$, and it was shown by Farber in [4] that the bounds in [2] are sharp. Here we shall examine only those bounds that can theoretically be obtained from the properties of elliptic operators on the manifold, by analogy with [5]. Every representation

$$\rho\colon \Gamma \to GL(N, C) \tag{2}$$

determines a complex of forms Λ^*_ρ, where

$$\Omega \in \Lambda^*_\rho \subset \Lambda^*(\hat{M}, \mathbb{C}^N), \quad \gamma^*\Omega = \rho(\gamma)\Omega, \quad \gamma \in \Gamma. \tag{3}$$

The cohomology of the complex (Λ^*_ρ, d) coincides with the cohomology $H^*_\rho(M^n, \mathbb{C}^N)$ with local coefficients (see [6], p. 134). We have the obvious

Proposition 1. *For any Morse function f on M^n,*

$$m_j(f) \geqslant \max_\rho [b^\rho_j(M^n)/N], \tag{4}$$

where $b^\rho_j(M^n)$ is the rank $H^j_\rho(M^n, \mathbb{C}^N)$.

We denote by $R(\Gamma, N)$ the space of all representations (2), given by certain algebraic equations (the relations of the group Γ) on the elements of the matrices that are images of the generators of Γ. Obviously $R(\Gamma, N)$ is a complex algebraic subvariety of $GL(N, C)^s$. We denote by $R_U(\Gamma, N)$ the subspace of unitary representations. In the unitary case we have always $b^\rho_j(M^n) = b^\rho_{n-j}(M^n)$.

Date: Received 28/NOV/85.

1991 *Mathematics Subject Classification.* Primary 58E05; Secondary 57R70, 58E35, 58F05, 58F09. UDC 513.835.

Translated by J. A. Zilber.

Problem. Determine the cases where the numbers $[b_j^\rho(M^n)/N]$, $N=\infty$, are defined and yield inequalities of the type (4). For unitary ρ, this probably can happen at least if the image of the group ring generates residually finite von Neumann factors (Π_1, Π_∞).

Proposition 2. *There exist at most countably many complex algebraic subvarieties $W_{\alpha j} \subset R(\Gamma, N)$, $N<\infty$, of positive codimension that satisfy the following conditions*:

(a) *$b_j^\rho(M^n)$ is constant on $R(\Gamma,N)$ outside all the $W_{\alpha j}$.*

(b) *$b_j^\rho(M^n)$ is constant on each $W_{\alpha j}$ outside all the intersections $W_{\alpha j}\cap W_{\beta j}$; more generally, $b_j^\rho(M^n)$ is constant on each intersection $\bigcap_{q=1}^m W_{\alpha_q j}$ outside all the intersections with the remaining $W_{\alpha j}$, $\alpha\neq\alpha_1,\dots,\alpha_m$.*

(c) *On each new intersection the rank can increase.*

(d) *If the ring $C[\Gamma]$ is noetherian, then the number of subvarieties $W_{\alpha j}$ is finite.*

Proposition 2 is proved in a straightforward fashion by regarding the complex of chains on the covering space $\hat{M}$ as a free finite complex of $C[\Gamma]$-modules and analyzing the degeneracies that can arise under a representation ρ.

Important Example. $\Gamma=\mathbb{Z}^k$ and $N=1$. Here we have $R(\Gamma,C)\subset C^k$, where

$$\rho(t_q)=\mu_q=\exp(p_q)\neq 0,\quad \rho=(\mu_1,\dots,\mu_k)\in C^k. \tag{5}$$

The subspace of unitary representations $R_U\subset R$ is then given by

$$R_U(\mathbb{Z}^k,C)=T^k\subset C^k,\quad |\mu_q|=1,\ q=1,\dots,k. \tag{6}$$

The finite number of singular varieties $W_{\alpha j}$, $\alpha=1,\dots,m_j$, is given by equations with integer coefficients and includes the following set:

$$j=0,1,n,n-1,\quad W_{1j}=\{1=\mu_1\},\quad W_{2j}=\{1=\mu_2\},\ \dots,\ W_{kj}=\{1=\mu_k\}. \tag{7}$$

Thus, we have always $m_j\geqslant k$. From the exterior product

$$\Lambda_\rho^*\wedge\Lambda_{\rho'}^*\subset\Lambda_{\rho\otimes\rho'}^*, \tag{8}$$

and it follows, for $\Gamma=\mathbb{Z}^k$, $N=1$, that

$$\begin{gathered}\Lambda_\rho^*\wedge\Lambda_{\rho'}^*\subset\Lambda_{\rho\rho'}^*,\quad \bar{\Lambda}_\rho^*=\Lambda_{\bar\rho}^*,\\ \rho\rho'=(\mu_1\mu_1',\dots,\mu_k\mu_k'),\quad \bar\rho=(\bar\mu_1,\dots,\bar\mu_k).\end{gathered} \tag{9}$$

We obtain the inner product

$$\langle\Omega_\rho,\Omega_{\rho'}\rangle=\int_{M^n}\Omega_\rho\wedge*\bar\Omega_{\rho'}, \tag{10}$$

where $\rho\bar\rho'=(1,\dots,1)$, $\Omega_\rho\in\Lambda_\rho^*$, and $\Omega_{\rho'}\in\Lambda_{\rho'}^*$. Thus,

$$b_j^\rho(M^n)=b_{n-j}^{\rho'}(M^n),\quad \rho\bar\rho'=(1,\dots,1).$$

Definition 1. For $\Gamma=\mathbb{Z}^k$, the Λ_ϕ^*, are called *Bloch complexes*, and their cohomology *Bloch cohomology* (by analogy with the theory of linear operators with periodic coefficients).

Let $\omega_1,\dots,\omega_k$ be a set of closed 1-forms in M^n forming a basis of $H^1(M^n,\mathbb{Z})$, i.e., there is a set of basis cycles $a_1,\dots,a_k\in H_1(M^n,\mathbb{Z})$ such that

$$\oint_{a_l}\omega_q=\delta_{ql},\quad k=b_1(M^n). \tag{11}$$

Consider the following operator d_ω for forms on M^n:

$$d_\omega \Omega = d\Omega + \omega \wedge \Omega, \quad d_\omega^2 = 0, \quad \omega = \sum p_q \omega_q + df. \tag{12}$$

The cohomology of the complex $\Lambda^*(M^n)$ with the operator (12) is canonically isomorphic to the cohomology $H^*_\rho(M^n, C)$, where $\rho = (\mu_1, \dots, \mu_k)$ and $p_q = \ln \mu_q$.

We now define a sequence of "Massey operations": for a 1-form class $[\omega]$ and a class $[a] \in H^q(M^n, C)$,

$$\begin{aligned} &\{\omega, a\}_0 = [\omega \wedge a], \\ &\{\omega, a\}_1 = \{\omega \wedge v_1\}, \quad dv_1 = \omega \wedge a, \quad \dots, \\ &\{\omega, a\}_l = [\omega \wedge v_l], \quad dv_l = \{\omega, a\}_{l-1}. \end{aligned} \tag{13}$$

The Massey operation of index l is defined for any pair of elements $[w], [a]$ in the kernel of every Massey operation of order $s < l$, and is multiple-valued for $l \geqslant 1$:

$$\{w, a\}_l \in \bigcap_{s<l} \operatorname{Ker}\{\omega, \cdot\}_s / \bigcup_s \operatorname{Im}\{\omega, \cdot\}_s.$$

Theorem 1. *The cohomology of the operator $d_{\varepsilon\omega}$ is isomorphic, except for certain complex "root" values of ε, to the following linear space*:

$$H^*_{\rho(\varepsilon)}(M^n, C) = \bigcap_l \operatorname{Ker}\{\omega, \cdot\} / \bigcup_l \operatorname{Im}\{\omega, \cdot\}. \tag{14}$$

The number of root values of ε is finite for rational $[\omega]$, and for all $[\omega] \in H^1(M^n, C)$ is finite in any compact region in $C \setminus 0$.

Theorem 2. *Let ρ and ρ_1 be two representations of the group $\Gamma = \pi^1(M^n)$, where ρ is unitary and ρ_1 real and one-dimensional, with*

$$\ln \rho_1(\gamma) = \oint_\gamma \omega, \quad d\omega = 0.$$

Then for the real closed Morse form ω we have the inequalities

$$\begin{aligned} m_q(\omega) &\geqslant \varinjlim b_q^{\rho(\varepsilon)}/N, \quad \varepsilon \to \infty, \\ \rho(\varepsilon) &= \rho\rho_1^\varepsilon \colon \Gamma \to U(N) \cdot R. \end{aligned} \tag{15}$$

The proof of Theorem 2 follows from the arguments in [2] with some simple additions: although a level surface of $\pi^*\omega$ on the covering space $\hat{M}$ may not be compact, any compact set moving "downwards" along the gradient either is snagged at a critical point or passes through all values of g, where $dg = \pi^*\omega$, and its size increases no faster than linearly.

Idea of the proof of Theorem 1. Consider a form $a(\varepsilon) = a_0 + \varepsilon a_1 + \dots + \varepsilon^m a_m + \cdots$ such that $d_{\varepsilon\omega} a(\varepsilon) = 0$. It is easily verified that $da_0 = 0$, $da_1 = \omega \wedge a_0$, ..., $da_m = \omega \wedge a_{m-1}$. This implies that the left-hand side of (14) is at most equal to the right. The opposite direction is harder. Working with the nonuniqueness of the choice of the a_i in the cohomology classes, one must construct $a(\varepsilon)$ as a convergent series for small $\varepsilon \to 0$. The root values of ε are the intersections of the complex curve $\mu_q(\varepsilon) = \exp(\varepsilon p_q)$ with the subvarieties $W_{\alpha j}$ of Proposition 2. For rational classes $(p_1 : \dots : p_k) \in O$ this curve is algebraic.

Remark. For $\Gamma = \mathbb{Z}$, $k = 1$ and $\rho = 1$, it has been pointed out to the author by Pazhitnov that inequality (15) can be proved by the procedure in [5], and also that the rank $b_q^{\rho(\varepsilon)}$ coincides for large $\varepsilon \to \infty$ with the number $b_q(M^n, [\omega])^q$ introduced by the author in [2].[1]

Proposition 3. *For $k = 1$, there exists a singular point $\varepsilon = \varepsilon_{0j} \in C \setminus 0$ for which the rank $b_j^{\rho(\varepsilon_{0j})}$ has the form of a "Morse–Smale number over K":*

$$b_j^{\rho(\varepsilon_{0j})} = b_j(\hat{M}, K) + q_j(\hat{M}, K) + q_{j-1}(\hat{M}, K),$$

where b_j is the dimension of the K-free part of the module $H_j(\hat{M}, K)$ and q_j is the number of K-generators in the K-periodic part. Here K is the principal ideal ring $C[t, t^{-1}]$. It can happen that

$$|\rho(\varepsilon_{0j})| \neq 1 \qquad p(\varepsilon_{0j}) \notin R_U(\mathbb{Z}, C).$$

For unitary representations ρ the proofs of the generalized inequalities of Morse and of the author (Proposition 1 and Theorem 2) are easily derived from the procedure in [5], applied to the operator $d + \pi^*\omega \cdot \varepsilon$ on the complexes Λ_ρ^*, where $\omega = df$ for the single-valued case.

The operators $d_\omega = d + \omega$, δ_ω (the Hermitian adjoint), and $\Delta_w = (d_\omega + \delta_\omega)^2$ can be defined also for nonclosed forms ω. Let $\omega = \omega^0 + i\omega^1$. Then $\Delta_\omega = \tilde{\nabla}_j \tilde{\nabla}^j + (\omega^0, \omega^0) + C$; here the connection operators $\tilde{\nabla}$ are defined on vector fields by the local formula $\tilde{\nabla}_j = \partial_j + \Gamma_{kj}^s + i\delta_k^s \omega_j^1$, where Γ_{kj}^s is the usual Riemannian connection. The operator C is of order 0 and depends linearly on ω^0. In the Euclidean metric $g_{ij} = \delta_{ij}$ we have $C = C_{kj}(a^{*k} - a^k)(a^{*j} + a^j)$, where $\omega^0 = C_{kj} x^j \, dx^k$, $C_{kj} = \text{const} \in R$, the operators a^{*k} are multiplication by dx^k, and the a^k are their adjoints:

$$a^{*k} a^j + a^j a^{*k} = g^{kj}, \quad a^{*j} a^{*k} + a^{*k} a^{*j} = 0.$$

Suppose $\det C_{kj} \neq 0$. Denote by $m_\pm(C_{kq})$ the number of zero modes of the operator $\Delta_{\varepsilon\omega}$, $\omega = C_{kq} x^q \, dx^k$, in the Euclidean metric on R^n, on the spaces $\Lambda^\pm$ of even and odd forms, for $\varepsilon \to +\infty$. Suppose given a real 1-form Ω with nondegenerate critical points x_j, $\Omega(x_j) = 0$, $\partial_k \Omega_q(x_j) = C_{kq}^{(j)}$, and a unitary bundle ρ of zero curvature. Then

$$m_\pm(\Omega) = \sum_j m_\pm(C_{kq}^{(j)}) \geqslant \overrightarrow{\lim}\, b_\pm^\rho(\varepsilon\Omega)/N, \quad \varepsilon \to \infty. \tag{16}$$

Here N is the dimension of ρ and $b_\pm^\rho(\Omega)$ is the kernel (the number of zero modes) of the operator $(d_\Omega + \delta_\Omega)$ on $\Lambda_\rho^\pm$. The right-hand side $b_\pm^\rho$ is topologically invariant for closed Ω. We have always $m_+(C_{kq}) - m_-(C_{kq}) = \operatorname{sgn} \det C$. Almost always, $m_\pm(C_{kq}) = 1$ or 0. The most important classes of matrices are (a) $C_{kq} = C_{qk}$, (b) $C_{kq} = -C_{qk}$ for isometries, and (c) $[S, \Lambda] = 0$, where S and Λ are the symmetric and skewsymmetric parts of the tensor C_{kq}. This corresponds to holomorphic vector fields on Kählerian manifolds. Here the numbers $m_\pm(C_{kq})$ are easily computed.

Problem. Compute the number of zero modes $m_\pm(C_{kq})$. Let $M_\pm(\varepsilon\Omega)$ be the spaces of quasiclassical zero modes, for $\varepsilon \to \infty$, of the operator $\Delta_{\varepsilon\Omega}$. Describe geometrically, in the language of the dynamical system $\dot{x}^i = \Omega^i = g^{ij}\Omega_j$, the operator $T = d_{\varepsilon\Omega} + \delta_{\varepsilon\Omega} \colon M_+(\Omega) \mapsto M_-(\Omega)$ that picks out in order the "true" zero modes from the quasiclassical ones as $\varepsilon^{-1} \to 0$ (for $\Omega_j = \partial_j f$ we obtain the cell

[1] *Translator's note.* More precisely, in [3].

complex of the function f). The bounds (16) then take the form of the "Morse inequalities for dynamical systems", $T \sim e^{-\varepsilon}$.

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