Non-Commutative Linear Logic with Subexponentials

(based on work by M. Kanovich, S. Kuznetsov, V. Nigam, A. Scedrov)

Logic II, University of Pennsylvania, Spring 2017

The Lambek Calculus

$$\frac{}{A \to A}\ (\mathrm{id})$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, (A \backslash B), \Delta \to C}\ (\backslash \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B}\ (\to \backslash)$$

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- A fragment of intuitionistic **non-commutative** linear logic.

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- A fragment of intuitionistic **non-commutative** linear logic.
- No Lambek's restriction: antecedents are allowed to be empty (as in linear logic).

Substructural Modalities

Introduction rules:

$$\frac{\Delta_1, A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (! \to) \qquad \frac{!A_1, \ldots, !A_n \to B}{!A_1, \ldots, !A_n \to !B}\ (\to !)$$

Structural rules:

1. exchange:

$$\frac{\Delta_1, \Delta_2, !A, \Delta_3 \to C}{\Delta_1, !A, \Delta_2, \Delta_3 \to C}\ (\mathrm{ex}_1) \qquad \frac{\Delta_1, !A, \Delta_2, \Delta_3 \to C}{\Delta_1, \Delta_2, !A, \Delta_3 \to C}\ (\mathrm{ex}_2)$$

2. weakening:

$$\frac{\Delta_1, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\mathrm{weak})$$

3. (local) contraction:

$$\frac{\Delta_1, !A, !A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\mathrm{contr})$$

Non-Local Contraction

$$\frac{\Delta_1, !A, \Delta_2, !A, \Delta_3 \to C}{\Delta_1, !A, \Delta_2, \Delta_3 \to C}\ (\mathrm{ncontr}_1) \qquad \frac{\Delta_1, !A, \Delta_2, !A, \Delta_3 \to C}{\Delta_1, \Delta_2, !A, \Delta_3 \to C}\ (\mathrm{ncontr}_2)$$

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Cut vs. Contraction

For the system with only local contraction (contr), cut elimination **fails.**

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- Derivable with cut.
- Not derivable without cut in the system with (contr) and (optionally) (weak).
- Enjoys a cut-free derivation with (ncontr).

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$$\frac{\dfrac{A \rightarrow A \quad B \rightarrow B}{A, A \backslash_1 B \rightarrow B}}{A \backslash_1 B \rightarrow A \backslash_2 B}$$

- **but:** $!_1 A \leftrightarrow !_2 A$, even if they obey the same rules.

Multi-Subexponential System SLC_Σ

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Subexponential signature:

$$\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle,$$

where

- $\mathcal{I} = \{s_1, \ldots, s_n\}$ is a set of subexponential labels;
- $\preceq$ is a preorder on $\mathcal{I}$;
- if $s_1 \in \mathcal{W}$ and $s_1 \preceq s_2$, then $s_2 \in \mathcal{W}$; the same for $\mathcal{C}$ and $\mathcal{E}$;
- $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

Multi-Subexponential System SLC_Σ

Introduction rules:

$$\frac{!^{s_1}F_1, \ldots, !^{s_n}F_n \to F}{!^{s_1}F_1, \ldots, !^{s_n}F_n \to !^{s}F}\ (\to !^s) \qquad \frac{\Gamma, F, \Delta \to C}{\Gamma, !^{s}F, \Delta \to C}\ (!^s \to)$$

provided $s \preceq s_i$ for $1 \leq i \leq n$.

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provided $s \preceq s_i$ for $1 \le i \le n$.

Structural rules:

- if $s \in \mathcal{W}$, add weakening for $!^s$;
- if $s \in \mathcal{C}$, add **non-local** contraction for $!^s$;
- if $s \in \mathcal{E}$, add exchange for $!^s$.

Since (weak) and (ncontr) yield (ex), we require $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

Cut Elimination in SLC_Σ

The cut rule is admissible in SLC_Σ.

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Strategy 1: mix elimination

$$\frac{\Pi \to !^s A \quad \Delta_0, !^s A, \Delta_1, !^s A, \ldots, \Delta_{i-1}, !^s A, \Delta_i, \ldots, !^s A, \Delta_n \to C}{\Delta_0, \Delta_1, \ldots, \Delta_{i-1}, \Pi, \Delta_i, \ldots, \Delta_n \to C}\ (\mathrm{mix})$$

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- We eliminate cut and mix by joint induction (cf. [Lincoln et al 1992] for the commutative case with one exponential).
- Induction parameter: $\omega \cdot c + d$, where c is $|A|$ for cut and $|A| + 1$ for mix and d is the total number of applications of rules in the cut-free derivations of the premises.

Cut Elimination in SLC_{Σ}

Strategy 2: deep cut elimination [Bräuner and de Paiva 1996]

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Induction on three parameters: the number of cuts, the complexity of cut formula, the depth of the proof trees.

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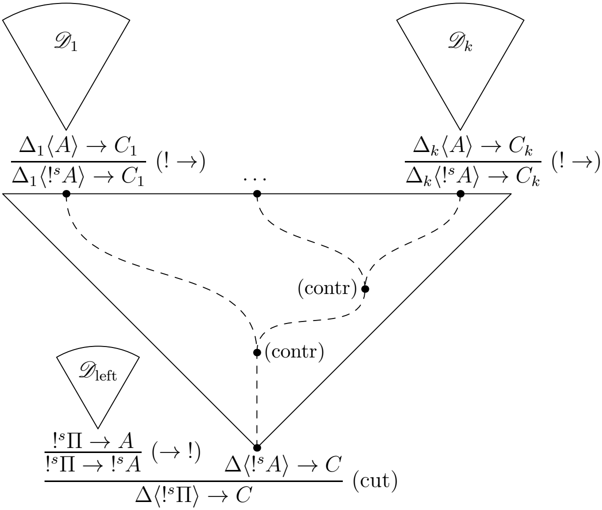

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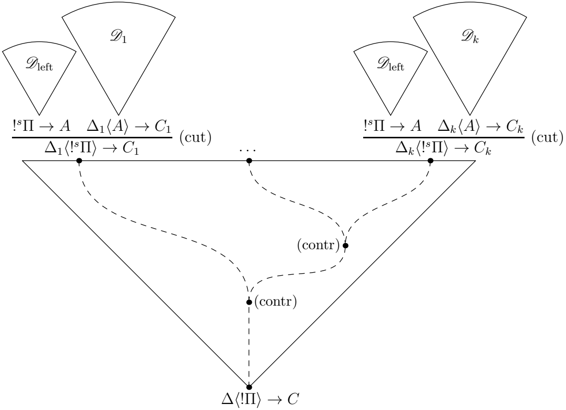

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  - **NP**, if $!^s$ for all $s \in \mathcal{C}$ is applied only to variables.

Undecidability Proof Sketch

- Semi-Thue systems: $\eta u_1 \ldots u_k \theta \Rightarrow \eta v_1 \ldots v_m \theta$, if $\langle u_1 \ldots u_k, v_1 \ldots v_m \rangle \in P$ (P is a finite set of *productions*).

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- Encoding the semi-Thue system using a ! equipped with contraction: $\mathcal{B} = \{(u_1 \cdot \ldots \cdot u_k) / (v_1 \cdot \ldots \cdot v_m) \mid \langle u_1 \ldots u_k, v_1 \ldots v_m \rangle \in P\} = \{B_1, \ldots, B_n\}$.

  $!^s(\mathbf{1} \,/\, !^s B_1), !^s B_1, \ldots, !^s(\mathbf{1} \,/\, !^s B_n), !^s B_n, b_1, \ldots, b_k \rightarrow a_1 \cdot \ldots \cdot a_m$

  is derivable ($s \in \mathcal{C}$) iff $a_1 \ldots a_m \Rightarrow^* b_1 \ldots b_k$.

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  is derivable ($s \in \mathcal{C}$) iff $a_1 \dots a_m \Rightarrow^* b_1 \dots b_k$.
- The $\mathbf{1}$ constant can be eliminated.