**Logic II**
(LGIC 320 / MATH 571 / PHIL 412)
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# Gentzen-Style Sequent Calculi

## 1. The Language

We define Gentzen-style sequential calculi for first-order classical and intuitionistic logics. Formulae are defined in a standard way (see previous lecture notes) and are denoted by small Greek letters. Capital Greek letters denote finite (possibly empty) *multisets* of formulae. For classical logic, sequents are of the form $\Gamma \Rightarrow \Delta$, for intuitionistic—of the form $\Gamma \Rightarrow \varphi$.

The left- and right-hand sides of the sequent are called antecedent and succedent respectively.

The system LK and LJ formulated below are equivalent to Hilbert-style calculi for FO-CL and FO-Int respectively: $\varphi$ is derivable in a Hilbert-style calculus iff $\Rightarrow \varphi$ is derivable in the corresponding sequent calculus. Moreover, $\Gamma \Rightarrow \Delta$ is derivable in the sequent calculus iff $\bigwedge \Gamma \to \bigvee \Delta$ is derivable in the Hilbert-style calculus. Finally, in the Hilbert-style calculus we have $\Gamma \vdash \varphi$ for a finite $\Gamma$ (if $\Gamma$ is infinite, it can be truncated, since only a finite number of formulae can be used in the derivation) iff $\Gamma \Rightarrow \varphi$ is derivable in the corresponding sequent calculus.

## 2. Axioms and Rules for Classical Logic (LK)

I. Axioms

$$\frac{}{\varphi \Rightarrow \varphi} \qquad \frac{}{\bot \Rightarrow}$$

II. Propositional rules

$$\frac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge L) \qquad \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2, \Delta}\ (\wedge R)$$

$$\frac{\Gamma, \varphi_1 \Rightarrow \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta}\ (\vee L) \qquad \frac{\Gamma \Rightarrow \varphi_i, \Delta}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2, \Delta}\ (\vee R)$$

$$\frac{\Gamma_1 \Rightarrow \varphi, \Gamma_2 \quad \Delta_1, \psi \Rightarrow \Delta_2}{\Delta_1, \Gamma_1, \varphi \to \psi \Rightarrow \Delta_2, \Gamma_2}\ (\to L) \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\to R)$$

III. Quantifier rules

$$\frac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x\, \varphi(x) \Rightarrow \Delta}\ (\forall L) \qquad \frac{\Gamma \Rightarrow \varphi(y), \Delta}{\Gamma \Rightarrow \forall x\, \varphi(x), \Delta}\ (\forall R)$$

$$\frac{\Gamma, \varphi(y) \Rightarrow \Delta}{\Gamma, \exists x\, \varphi(x) \Rightarrow \Delta}\ (\exists L) \qquad \frac{\Gamma \Rightarrow \varphi(t), \Delta}{\Gamma \Rightarrow \exists x\, \varphi(x), \Delta}\ (\exists R)$$

Constraints: $y \notin FV(\Gamma, \Delta)$; the substitution of $t$ for $x$ is free.

IV. Structural rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ (weak - L) \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ (contr - L)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ (weak - R) \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ (contr - R)$$

V. Cut

$$\frac{\Gamma_1 \Rightarrow \varphi, \Gamma_2 \quad \varphi, \Delta_1 \Rightarrow \Delta_2}{\Gamma_1, \Delta_1 \Rightarrow \Gamma_2, \Delta_2}\ (cut)$$

## 3. Axioms and Rules for Intuitionistic Logic (LJ)

$$\overline{\varphi \Rightarrow \varphi} \qquad \overline{\bot \Rightarrow \varphi}$$

II. Propositional rules

$$\frac{\Gamma, \varphi_i \Rightarrow \zeta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \zeta}\ (\wedge L) \qquad \frac{\Gamma \Rightarrow \varphi_1 \quad \Gamma \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2}\ (\wedge R)$$

$$\frac{\Gamma, \varphi_1 \Rightarrow C \quad \Gamma, \varphi_2 \Rightarrow \zeta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \zeta}\ (\vee L) \qquad \frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2}\ (\vee R)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Delta, \psi \Rightarrow \zeta}{\Delta, \Gamma, \varphi \to \psi \Rightarrow \zeta}\ (\to L) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\to R)$$

III. Quantifier rules

$$\frac{\Gamma, \varphi(t) \Rightarrow \zeta}{\Gamma, \forall x\, \varphi(x) \Rightarrow \zeta}\ (\forall L) \qquad \frac{\Gamma \Rightarrow \varphi(y)}{\Gamma \Rightarrow \forall x\, \varphi(x)}\ (\forall R)$$

$$\frac{\Gamma, \varphi(y) \Rightarrow \zeta}{\Gamma, \exists x\, \varphi(x) \Rightarrow \zeta}\ (\exists L) \qquad \frac{\Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x\, \varphi(x)}\ (\exists R)$$

Constraints: $y \notin FV(\Gamma, \zeta)$; the substitution of $t$ for $x$ is free.

IV. Structural rules

$$\frac{\Gamma \Rightarrow \zeta}{\Gamma, \varphi \Rightarrow \zeta}\ (weak) \qquad \frac{\Gamma, \varphi, \varphi \Rightarrow \zeta}{\Gamma, \varphi \Rightarrow \zeta}\ (contr)$$

V. Cut

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Delta \Rightarrow \zeta}{\Gamma, \Delta \Rightarrow \zeta}\ (cut)$$

## 4. Cut Elimination

**Theorem 1.** *Both for LK and LJ, any sequent that can be derived, can be derived without using (cut).*

To prove this theorem, first replace cut with a more general rule called *mix.*

For LK:

$$\frac{\Gamma_1 \Rightarrow \varphi^k, \Gamma_2 \quad \varphi^n, \Delta_1 \Rightarrow \Delta_2}{\Gamma_1, \Delta_1 \Rightarrow \Gamma_2, \Delta_2}\ (mix)$$

For LJ:

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi^n, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ (mix)$$

Here $\varphi^n$ means $\varphi, \varphi, \ldots, \varphi$ ($n$ times).

Then proceed by triple nested induction on the following parameters:

1. the total number of mixes in a derivation (we start by eliminating the topmost ones);
2. the complexity of the formula $\varphi$ in a mix;
3. the summary height of the (mix-free) derivations of the premises of mix.

## 5. Harrop's Theorems

Cut-free proofs are extremely easy to analyse. For example, proving disjunctive property of intuitionistic logic becomes a triviality: if $\vdash_{\mathrm{Int}} \varphi_1 \vee \varphi_2$, then the sequent $\Rightarrow \varphi_1 \vee \varphi_2$ can be proved in LJ without cut. Let's look at the last rule in this derivation. It couldn't be a structural rule (since the left-hand side is empty), therefore it is $(\vee R)$ decomposing the disjunction. Its premise is exactly $\Rightarrow \varphi_i$ ($i = 1$ or 2). Hence, $\vdash_{\mathrm{Int}} \varphi_1$ or $\vdash_{\mathrm{Int}} \varphi_2$.

The same works for the existential quantifier (where semantic proof is quite subtle): if $\Rightarrow \exists x\, \varphi(x)$ is derivable in LJ, then the last rule could only be $(\exists R)$, yielding $\Rightarrow \varphi(t)$ for some term $t$.

Harrop's theorems generalize disjunctive and existential properties of intuitionistic logic.

**Definition.** The *strictly positive position (s.p.p.)* of a subformula occurrence is defined recursively:

1. a formula is in s.p.p. in itself;
2. a subformula is in s.p.p. in $\varphi_1 \vee \varphi_2$ or $\varphi_1 \wedge \varphi_2$, if it is in s.p.p. in $\varphi_1$ or $\varphi_2$;
3. a subformula is in s.p.p. in $\varphi \to \psi$, if it is in s.p.p. in $\psi$.

In other words, a subformula is in s.p.p. iff it never goes into the left-hand side of an implication.

**Theorem 2** (Harrop)**.** *1. If $\Gamma$ has no subformula of the form $\xi_1 \vee \xi_2$ in s.p.p. and $\Gamma \Rightarrow \varphi_1 \vee \varphi_2$ is derivable in* LJ*, then $\Gamma \Rightarrow \varphi_i$ is derivable in* LJ *for some $i$.*

*2. If $\Gamma$ has no subformula of the form $\xi_1 \vee \xi_2$ or $\exists z\, \xi(z)$ in s.p.p. and $\Gamma \Rightarrow \exists x\, \psi(x)$ is derivable in* LJ*, then $\Gamma \Rightarrow \psi(t)$ is derivable in* LJ *for some term $t$.*

The constraints are inevitable due to the trivial examples: $\varphi_1 \vee \varphi_2 \Rightarrow \varphi_1 \vee \varphi_2$, $\exists x\, \psi(x) \Rightarrow \exists x\, \psi(x)$, and $\psi(t_1) \vee \psi(t_2) \Rightarrow \exists x\, \psi(x)$, that violate this theorem.

*Proof.* 1. Starting from the root of the proof tree (the goal sequent), we draw the *main track:* it goes upwards through one-premise rules, and turns right on $(\to L)$; on $(\vee L)$ and $(\wedge R)$ it branches. However, $(\vee L)$ can be applied only if there is a $\vee$ in s.p.p. in the antecedent, which is not allowed. The other rule, $(\wedge R)$, cannot be applied, until $\varphi_1 \vee \varphi_2$ gets decomposed, since the main connective in the succedent is not $\wedge$. Therefore, the main track doesn't branch until we decompose $\varphi_1 \vee \varphi_2$:

$$\dfrac{\dfrac{\dfrac{\widetilde{\Gamma} \Rightarrow \varphi_i}{\widetilde{\Gamma} \Rightarrow \varphi_1 \vee \varphi_2}\ (\vee R)}{\vdots}}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2}$$

Then we can remove this rule and replace $\varphi_1 \vee \varphi_2$ with $\varphi_i$ along the main track. This yields $\Gamma \Rightarrow \varphi_i$. All the rules remain valid.

2. For the $\exists$ case, we proceed in the same way. We go along the main track until we reach $(\exists R)$:

$$\dfrac{\dfrac{\widetilde{\Gamma} \Rightarrow \psi(t)}{\widetilde{\Gamma} \Rightarrow \exists x\, \psi(x)}\ (\vee R)}{\dfrac{\vdots}{\Gamma \Rightarrow \exists x\, \psi(x)}}$$

and replace $\exists x\, \psi(x)$ with $\psi(t)$. There is, however, a tricky place here. The global constraint $y \notin \mathrm{FV}(\Gamma, \varphi)$ in the $(\exists L)$ and $(\forall R)$ rules can be violated. Indeed, when we replace

$$\dfrac{\Gamma, \xi(y) \Rightarrow \exists x\, \psi(x)}{\Gamma, \exists z\, \xi(z) \Rightarrow \exists x\, \psi(x)}\ (\exists L)$$

with

$$\dfrac{\Gamma, \xi(y) \Rightarrow \psi(t)}{\Gamma, \exists z\, \xi(z) \Rightarrow \psi(t)}\ (\exists L)^?$$

the "fresh" variable $y$ could accidentally occur in $t$ (e.g., $t = y$). This makes this rule invalid and ruins our derivation. Luckily, $(\exists L)$ on the main track is explicitly prohibited (no $\exists$ in s.p.p. in $\Gamma$), and $(\forall R)$ cannot appear before we decompose $\exists x\, \psi(x)$. All other rules are OK. $\square$

## 6. Herbrand's Theorem

Harrop's theorems don't work for classical logic. The first one is violated by the law of excluded middle: one can derive the disjunction $p \vee \neg p$, but neither of its disjuncts is derivable. For the second one, one could construct something like $\exists x\, ((x = 0 \wedge P) \vee (x \neq 0 \wedge \neg P))$, which is the same as $p \vee \neg p$, but with $\exists$ instead of $\vee$. However, a weaker version of the second theorem still holds.

**Definition.** The *strictly negative position (s.n.p.)* of a subformula is defined recursively:

1. a subformula is in s.n.p. in $\varphi \to \psi$, if it is in s.p.p. in $\varphi$ or in s.n.p. in $\psi$;
2. a subformula is in s.n.p. in $\varphi_1 \vee \varphi_2$ or $\varphi_1 \wedge \varphi_2$ if it is in s.n.p. in $\varphi_1$ or $\varphi_2$;
3. a formula is not in s.n.p. in itself.

In other words, the subformula is under implication of depth exactly 1.

**Theorem 3** (Herbrand)**.** *If $\Gamma$ has no subformulae of the form $\exists z\, \xi(z)$ in s.p.p. and $\psi(x)$ has no subformulae of the form $\forall z\, \xi(z)$ in s.p.p. and no subformulae of the form $\exists z\, \xi(z)$ in s.n.p., and $\Gamma \Rightarrow \exists x\, \psi(x)$ is derivable in* LK*, then there exist a finite number of terms $t_1, \ldots, t_k$ such that $\Gamma \Rightarrow \psi(t_1) \vee \ldots \vee \psi(t_k)$ is derivable in* LK*.*

*Proof.* Consider the main track of the succedent formula. In contrast to Harrop's case, now it can branch on two-premise rules ($(\vee L)$ and $(\wedge R)$) and also on $(contr - R)$: in the latter case, two tracks go along the same sequent. We trace back along these branches up to the rules that introduces $\exists x\, \psi(x)$. There is a finite number of $(\exists R)$ rules:

$$\dfrac{\Phi_1 \Rightarrow \psi(t_1)}{\Phi_1 \Rightarrow \exists x\, \psi(x), \Psi_1}\ (\exists R) \qquad \ldots \qquad \dfrac{\Phi_k \Rightarrow \psi(t_k)}{\Phi_k \Rightarrow \exists x\, \psi(x), \Psi_k}\ (\exists R)$$

Then we replace $\exists x\, \psi(x)$ with $\psi(t_1) \vee \ldots \vee \psi(t_k)$ along all branches of the main track. These $(\exists R)$ rules get replaced by series of $(\vee R)$ rules. The formula $\exists x\, \psi(x)$ could also appear from an axiom or weakening. The former case is prohibited by the constraints of the theorem. The latter one is possible, then after replacement weakening is still valid. (If there were no $(\exists R)$ applications at all, only weakenings, one can take arbitrary $t_1, \ldots, t_k$.)

The restrictions of the theorem prohibit usage of $(\exists L)$ and $(\forall R)$, which cause problems as in the 2nd Harrop's theorem (see above). Other rules are OK. $\square$

A non-trivial example that shows importance of the constraints in Herbrand's theorem even for an empty $\Gamma$ is the so-called *drinker's formula,* $\exists x\, (P(x) \to \forall y\, P(y))$. The informal reading is "in a bar, there always exists a guy such that if he drinks, everybody drinks." This is classically (but not intuitionistically) valid: indeed, if everyone drinks, we can take an arbitrary drinker for $x$; if some $x_0$ doesn't drink, then $P(x_0) \to \forall y\, P(y)$ is true *ex falso.* This can be transformed into an LK proof (exercise!). Herbrand's theorem and, of course, existential property here is invalid (therefore, by the way, drinker's formula is not derivable intutionistically).