Logic II
(LGIC 320 / MATH 571 / PHIL 412)
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Lectures 11–15: First Order Intuitionistic Logic

1. Hilbert-Style Calculus

First order terms and formulae over a signature Ω are defined exactly as in the classical case (refer to LOGIC I). We denote first order formulae by small Greek letters ($\varphi, \psi, \ldots$) in order to aviod confusion with propositional formulae.

The axioms of FO-Int, the first order intuitionistic calculus, are as follows.

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \xi)) \to ((\varphi \to \psi) \to (\varphi \to \xi))$
3. $(\varphi \land \psi) \to \varphi$
4. $(\varphi \land \psi) \to \psi$
5. $\varphi \to (\psi \to (\varphi \land \psi))$
6. $\varphi \to (\varphi \lor \psi)$
7. $\psi \to (\varphi \lor \psi)$
8. $(\varphi \to \xi) \to ((\psi \to \xi) \to ((\varphi \lor \psi) \to \xi))$
9. $\bot \to \varphi$
10. $\forall x\, \varphi(x) \to \varphi(t)$, if the substitution of t for x is allowed (free)
11. $\varphi(t) \to \exists x\, \varphi(x)$, if the substitution of t for x is allowed (free)
12. $\forall x\, (\psi \to \varphi(x)) \to (\psi \to \forall x\, \varphi(x))$, if x is not a free variable of ψ
13. $\forall x\, (\varphi(x) \to \psi) \to ((\exists x\, \varphi(x)) \to \psi)$, if x is not a free variable of ψ

The first 9 axioms are actually propositional principles, but it is allowed to substitute formulae with quantifiers for φ, ψ, and ξ. For example, $\forall x\, P(x) \to ((\exists x\, \forall y\, Q(x,y)) \to \forall x\, P(x))$ is an instance of Axiom 1.

The calculus is equipped with two rules of inference, modus ponens and generalization:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathrm{MP}) \qquad\qquad \frac{\varphi(x)}{\forall y\, \varphi(y)}\ (\mathrm{Gen})$$

The (Gen) rule corresponds to the reasoning strategy of the following type: in order to prove $\forall y\, \varphi(y)$, take an arbitrary x and prove $\varphi(x)$.

Generalization makes Deduction Theorem formally invalid, since $P(x) \vdash \forall y\, P(y)$, by (Gen), but $\not\vdash P(x) \to \forall y\, P(y)$. Deduction Theorem is still valid, if we use it only for a formula without free variables, or, more generally, do not apply (Gen) to the free variables of φ in the derivation $\Gamma, \varphi \vdash \psi$; then it is safe to state that $\Gamma \to \varphi \to \psi$.

Example 1. Formula $\exists x\, \neg P(x) \to \neg \forall y\, P(y)$ is derivable. First, replace $\neg\varphi$ with $\varphi \to \bot$ (by definition). We get $\exists x\, (P(x) \to \bot) \to (\forall y\, P(y) \to \bot)$. This can be derived by modus ponens from $\forall x\, ((P(x) \to \bot) \to (\forall y\, P(y) \to \bot)) \to (\exists x\, (P(x) \to \bot) \to (\forall y\, P(y) \to \bot))$ and $\forall x\, ((P(x) \to \bot) \to (\forall y\, P(y) \to \bot))$. The first formula is an instance of Axiom 13. To derive the second one, use (Gen). Now we have to establish $(P(x) \to \bot) \to (\forall y\, P(y) \to \bot)$. Apply Deduction Theorem twice. Now our goal is $P(x) \to \bot, \forall y\, P(y) \to \bot$, and we are not allowed to apply (Gen) to x in the derivation. By modus ponens with axiom 10, we get $P(x)$ from $\forall y\, P(y)$. Then the goal formula $\bot$ is obtained by modus ponens from $P(x)$ and $P(x) \to \bot$.

2. Kripke Completeness

A first order intuitionistic *Kripke model* is a structure $\mathcal{M} = \langle W, R, \mathcal{D}, \alpha \rangle$. Here R is a preorder relation on a non-empty set W, $\mathcal{D}$ is a function that maps each world $w \in W$ to a non-empty *support* set D_w, and α maps each world w to an interpretation of the signature Ω on D_w.

For simplicity we consider signatures without functional symbols: only predicate symbols and constants. For each predicate symbol P and $w \in W$,

$$\alpha(w)(P)\colon \underbrace{D_w \times \ldots \times D_w}_{v(P) \text{ times}} \to \{0, 1\},$$

where $v(P)$ is the arity of P. For a constant c, $\alpha(w)(c)$ is a designated element of D_w.

Also for simplicity (to avoid using Zorn lemma or equivalent techniques) we consider only finite and countable first order signatures (thus, the number of formulae is countable).

Every Kripke model $\mathcal{M}$ should satisfy the following **monotonicity conditions:**

1. if wRu, then $D_w \subseteq D_u$ (the set of known objects increases along R);

2. if wRu and c is a constant, then $\alpha(u)(c) = \alpha(w)(c)$ (constants don't change their values);

3. if wRu, P is a predicate symbol, $a_1, \ldots, a_{v(P)} \in D_w$, and $\alpha(w)(P)(a_1, \ldots, a_{v(P)}) = 1$, then $\alpha(u)(P)(a_1, \ldots, a_{v(P)}) = 1$ (once a predicate is declared true, it'll never become false).

In order to define **forcing** of closed (without free variables) formulae in Kripke worlds, we use formulae in a richer language. By $\Omega + S$ we denote the signature Ω enchanced by a set S of new constants. We recursively define the following relation: $w \Vdash \varphi$ ("formula φ is true in world w"), where φ is a closed formula in the $\Omega + D_w$ signature ($\varphi \in \mathrm{CFm}_{\Omega + D_w}$). Note that the signature depends on the world in which we consider the formula. The interpretation $\alpha(w)$ is extended naturally: if $c \in D_w$ is a new constant, then $\alpha(w)(c)$ is just c itself. The recursive definition is a follows. The only two non-classical cases are $\to$ and $\forall$.

1. for atomic formulae: $w \Vdash P(c_1, \ldots, c_{v(P)})$ iff $\alpha(w)(P)(\alpha(w)(c_1), \ldots, \alpha(w)(c_{v(P)})) = 1$;

2. for falsity: $w \not\Vdash \bot$;

3. for conjunction: $w \Vdash \varphi_1 \wedge \varphi_2$ iff $w \Vdash \varphi_1$ and $w \Vdash \varphi_2$;

4. for disjunction: $w \Vdash \varphi_1 \vee \varphi_2$ iff $w \Vdash \varphi_1$ or $w \Vdash \varphi_2$;

5. for implication: $w \Vdash \varphi_1 \to \varphi_2$ iff for any $u \in R(w)$ either $u \nVdash \varphi_1$ or $u \Vdash \varphi_2$;

6. for the existential quantifier: $w \Vdash \exists x\, \psi(x)$ iff $w \Vdash \psi(a)$ for some $a \in D_w$;

7. for the universal quantifier: $w \Vdash \forall x\, \psi(x)$ iff for any $u \in R(w)$ and for any $a \in D_u$ we have $u \Vdash \psi(a)$.

As in propositional case, this definition is designed to preserve monotonicity: if $w \Vdash \varphi$ and wRu, then $u \Vdash \varphi$.

One can easily check **correctness:** if a closed formula is derivable, it is true in all worlds of all Kripke models. We'll prove the converse **(completeness):**

Theorem 1. *If φ is true in all worlds of all Kripke models, then it is derivable in* FO-Int.

In order to prove completeness, we construct the **canonical model** $\mathcal{M}_0$.

Let S_0 be a countable set of *possible new constants.* For simplicity, let there be no constants in Ω itself, only predicate symbols. We consider *bi-theories* of the form (S, Γ, Δ), where $S \subset S_0$ and $\Gamma, \Delta \subset \mathrm{CFm}_{\Omega+S}$. Such a bi-theory is

- *consistent*, if there are no such finite $\Gamma_0 \subset \Gamma$ and $\Delta_0 \subset \Delta$ that $\vdash \bigwedge \Gamma_0 \to \bigvee \Delta_0$ (the empty conjunction is $\top$, the empty disjunction is $\bot$);
- *complete*, if $\Gamma \cup \Delta = \mathrm{CFm}_{\Omega+S}$;
- $\exists$-*complete*, if for any formula $\exists x\, \psi(x) \in \Gamma$ there exists such $a \in S$ that $\psi(a) \in \Gamma$;
- *small*, if $S_0 - S$ is infinite.

As in the propositional case, consistent complete bi-theories have good properties:

Lemma 2. *Let (S, Γ, Δ) be a consistent complete bi-theory. Then:*

- *if $\Gamma \vdash \varphi$, then φ is in Γ (deductive closure);*
- *$(\varphi \wedge \psi) \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$;*
- *$(\varphi \vee \psi) \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$ (disjunctive property);*
- *if $\forall x\, \psi(x) \in \Gamma$, then $\psi(a) \in \Gamma$ for any $a \in S$.*

Now the canonical model $\mathcal{M}_0$ is the structure $\langle W_0, R_0, \mathcal{D}_0, \alpha_0 \rangle$, where

- W_0 is the set of all small consistent complete $\exists$-complete bi-theories;
- $(S_1, \Gamma_1, \Delta_1) R_0 (S_2, \Gamma_2, \Delta_2)$ iff $S_1 \subseteq S_2$ and $\Gamma_1 \subseteq \Gamma_2$;
- for each world $D_{(S,\Gamma,\Delta)} = S$;
- for each predicate symbol P and $a_1, \ldots, a_{v(P)} \in S$ let $\alpha((S, \Gamma, \Delta))(P)(a_1, \ldots, a_{v(P)}) = 1$ iff $P(a_1, \ldots, a_{v(P)}) \in \Gamma$.

To proceed further, we prove two key lemmas:

Lemma 3 (Saturation Lemma)**.** *If (S, Γ, Δ) is a small consistent bi-theory, then there exists a small consistent complete $\exists$-complete bi-theory (S', Γ', Δ'), such that $S \subseteq S'$, $\Gamma \subseteq \Gamma'$, and $\Delta \subseteq \Delta'$.*

Proof. First let $S' \subset S_0$ be a set of constants such that $S \subset S'$ and both $S_0 - S'$ and $S' - S$ are infinite. (For example, one could enumerate $S_0 - S$ and add to S only the elements of $S_0 - S$ with even numbers.)

Let us enumerate all closed formulae: $\mathrm{CFm}_{\Omega+S'} = \{\varphi_1, \varphi_2, \varphi_3, \ldots\}$. Now we inductively construct a sequence of consistent bi-theories (S', Γ_k, Δ_k). $\Gamma_0 = \Gamma$, $\Delta_0 = \Delta$. For the step from k to $k+1$ consider two cases:

Case 1. φ_{k+1} is not of the form $\exists x\, \psi(x)$. At least one of two bi-theories $(S', \Gamma_k \cup \{\varphi_{k+1}\}, \Delta_k)$ and $(S', \Gamma_k, \Delta_k \cup \{\varphi_{k+1}\})$ is consistent (the argument is the same as for propositional case). Take this bi-theory for $(S', \Gamma_{k+1}, \Delta_{k+1})$.

Case 2. $\varphi_{k+1} = \exists x\, \psi(x)$. Again, if $(S', \Gamma_k, \Delta_k \cup \{\exists x\, \psi(x)\})$ is consistent, take it for $(S', \Gamma_{k+1}, \Delta_{k+1})$. In the other case, take a constant $a \in S'$ not yet used in Γ_k and Δ_k (such a constant exists, since we've added a finite number of formulae and therefore used a finite number of constants from $S' - S$) and let $(S', \Gamma_{k+1}, \Delta_{k+1}) = (S', \Gamma_k \cup \{\exists x\, \psi(x), \psi(a)\}, \Delta_k)$.

We need to show that this bi-theory is consistent, given that $(S', \Gamma_k \cup \{\exists x\, \psi(x)\}, \Delta_k)$ is consistent (otherwise we'd have just added $\exists x\, \psi(x)$ to Δ_k). Suppose, $\vdash G \wedge (\exists x\, \psi(x)) \wedge \psi(a) \to D$, where G is a conjunction of formulae from Γ_k and D is a disjunction of formulae from Δ_k. Now we use the **fresh constant argument:** all occurrences of a in the derivation can be replaced by a variable y, and, since a doesn't occur in Γ, Δ, or $\psi(x)$, this yields $\vdash G \wedge (\exists x\, \psi(x)) \wedge \psi(y) \to D$, and, by generalization, $\vdash \forall y \big(G \wedge (\exists x\, \psi(x)) \wedge \psi(y) \to D\big)$. Applying axioms and rules of FO-Int, we get $\vdash G \wedge (\exists x\, \psi(x)) \wedge (\exists y\, \psi(y)) \to D$ (the $\forall$ quantifier changes to $\exists$ when moved to the left side of the implication). This is equivalent to $\vdash G \wedge (\exists x\, \psi(x)) \to D$, which means that $(S', \Gamma_k \cup \{\exists x\, \psi(x)\}, \Delta_k)$ is inconsistent. Contradiction.

Now let $\Gamma' = \bigcup_{k=0}^{\infty} \Gamma_k$, $\Delta' = \bigcup_{k=0}^{\infty} \Delta_k$. It is easy to see that (S', Γ', Δ') is the required bi-theory. □

Lemma 4 (Main Semantic Lemma)**.** *In the canonical model, $(S, \Gamma, \Delta) \Vdash \varphi$ iff $\varphi \in \Gamma$.*

Proof. Induction on the structure of φ.

1. The atomic case is by definition.

2. The $\bot$ constant is never true, and, on the other hand, can never belong to Γ, otherwise the bi-theory is inconsistent due to the ex falso principle.

3. The $\vee$ and $\wedge$ cases come immediately from Lemma 2 and the definition of forcing.

4. The $\to$ case is considered exactly as in the propositional case. If $\varphi = \varphi_1 \to \varphi_2 \in \Gamma$, then for any world $(S', \Gamma', \Delta') \in R((S, \Gamma, \Delta))$ if $\varphi_1 \in \Gamma'$, then, since also $\varphi_1 \to \varphi_2 \in \Gamma'$ by monotonicity, $\varphi_2 \in \Gamma'$ by deductive closure.

   If $\varphi = \varphi_1 \to \varphi_2$ is not in Γ, then the bi-theory $(S, \Gamma \cup \{\varphi_1\}, \{\varphi_2\})$ is consistent (otherwise $\Gamma \vdash \varphi_1 \to \varphi_2$ by Deduction Theorem). By saturating it, we obtain a canonical model world $(S', \Gamma', \Delta') \in R((S, \Gamma, \Delta))$, such that $(S', \Gamma', \Delta') \Vdash \varphi_1$ and $(S', \Gamma', \Delta') \nVdash \varphi_2$. Hence, $(S, \Gamma, \Delta) \nVdash \varphi_1 \to \varphi_2$.

5. The $\exists$ case follows from $\exists$-completeness of (S, Γ, Δ): $(S, \Gamma, \Delta) \Vdash \exists x\, \psi(x)$ iff $(S, \Gamma, \Delta) \Vdash \psi(a)$ for some $a \in S$ iff $\psi(a) \in \Gamma$ for some $a \in S$ iff $\exists x\, \psi(x) \in \Gamma$.

6. Finally, the $\forall$ case is considered as follows. If $\varphi = \forall x\, \psi(x)$ is in Γ, the it is in Γ' for any $(S', \Gamma', \Delta') \in R((S, \Gamma, \Delta))$. Take an arbitrary $a \in S'$. By deductive closure, $\psi(a) \in \Gamma'$, and by induction hypothesis $(S', \Gamma', \Delta') \Vdash \psi(a)$. Therefore, by definition of forcing, $(S, \Gamma, \Delta) \Vdash \forall x\, \psi(x)$.

   Now let $\forall x\, \psi(x)$ be in Δ. Let a be a new constant from $S_0 - S$ (non-empty, since our bi-theory is small). By the fresh constant argument (see above), the bi-theory $(S \cup \{a\}, \Gamma, \{\psi(a)\})$ is consistent. Saturate it. We obtain a world $(S', \Gamma', \Delta') \in R((S, \Gamma, \Delta))$ that falsifies $\psi(a)$. Therefore, $\forall x\, \psi(x)$ is false in (S, Γ, Δ).

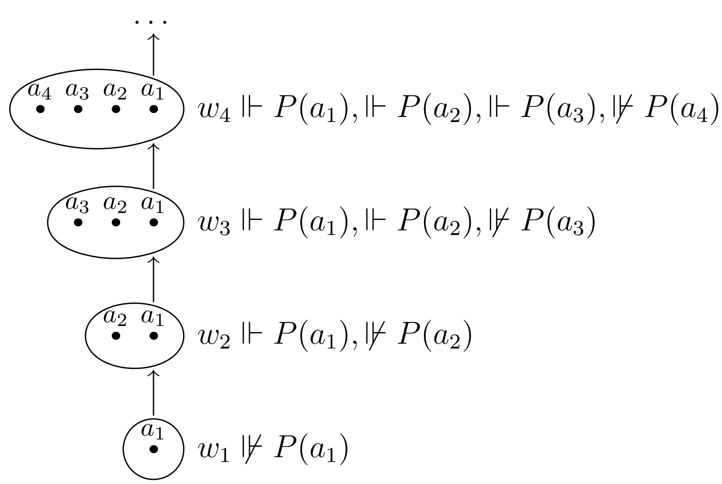
□

Now we're ready to prove completeness theorem by contraposition. Let φ be not derivable in FO-Int. Then the bi-theory $(\varnothing, \varnothing, \{\varphi\})$ is consistent. Saturate it. We obtain a world $w = (S', \Gamma', \Delta')$ in $\mathcal{M}_0$ with φ in Δ'. Therefore, $w \nVdash \varphi$, and φ is not universally true.

3. Notes on Kripke Models

In this section we consider two examples for better understanding of some nuances of the first order Kripke semantics in comparison with the propositional case.

Example 2. Consider the formula $\forall x\, \neg\neg P(x) \to \neg\neg\forall x\, P(x)$ (called *double negation shift, DNS).* This formula is not derivable in FO-Int, since it is falsified on the following counter-model:

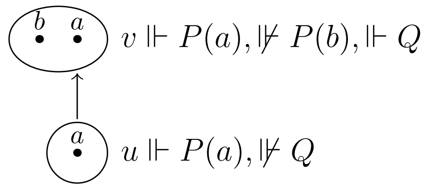


Since $\neg\neg\varphi$ in world w means "for every world $u \in R(w)$ there exists a world $v \in R(u)$ such that $v \Vdash \varphi$", in each world w in this model for each $a_i \in D_w$ we have $w \Vdash \neg\neg P(a_i)$ (since $P(a_i)$ eventually becomes true), and so $w \Vdash \forall x\, \neg\neg P(x)$.

On the other hand, $\forall x\, P(x)$ is never true, and therefore so is $\neg\neg\forall x\, P(x)$. Thus, the implication $\forall x\, \neg\neg P(x) \to \neg\neg\forall x\, P(x)$ is false in all worlds in this model.

However, the DNS formula is true in all models with a finite W (exercise). Thus, as opposed to the propositional case, **FO-Int doesn't enjoy the finite model property.**

Example 3. Consider the formula $\forall x\,(Q \vee P(x)) \to (Q \vee \forall x\,P(x))$ (here P is a unary predicate symbol and Q is a 0-ary predicate symbol). This formula is also not derivable, since it is falsified by the following Kripke model:



In this model, $\forall x\,(Q \vee P(x))$ is true both in u and v, but neither $\forall x\,P(x)$, nor Q is true in u. Therefore, the implication fails.

However, this formula is true in all Kripke models in which D_w is the same for all $w \in W$. Indeed, if $\forall x\,(Q \vee P(x))$ is true in some world w, then either $w \Vdash Q$ (and then also $Q \vee \forall x\,P(x)$)), or for every $a \in D_w$ we have $w \Vdash P(a)$. But, since $D_u = D_w$ for any $u \in R(w)$, we also have $u \Vdash P(a)$ for every $a \in D_u = D_w$ by monotonicity. Therefore, $w \Vdash \forall x\,P(x)$. Essentially, in this case forcing for the $\forall$ quantifier becomes classical ($w \Vdash \forall x\,\psi(x)$ iff $w \Vdash \psi(a)$ for all $a \in D_w$).

Being true in all models with a constant $\mathcal{D}$, the formula $\forall x\,(Q \vee P(x)) \to (Q \vee \forall x\,P(x))$ is called the *constant domain principle, CD.*

CD also shows up some problems with the informal BHK semantics of FO-Int. Its premise, $\forall x\,(Q \vee P(x))$, is BHK-witnessed by a function f that takes an arbitrary a and produces either $\langle 1, \text{witness for } Q\rangle$ or $\langle 2, \text{witness for } P(a)\rangle$. On the other side, a witness for the conclusion, $Q \vee \forall x\,P(x)$, is either $\langle 1, \text{witness for } Q\rangle$ or $\langle 2, g\rangle$ for a function $g\colon a \mapsto \text{witness for } P(a)$. In order to justify the implication CD, one needs to construct a function

$$h\colon \text{witness for the premise} \mapsto \text{witness for the conclusion.}$$

And, indeed, such a function exists! Namely,

$$h(f) = \begin{cases} \langle 1, u\rangle, & \text{if } f(a) = \langle 1, u\rangle \text{ for some } a, \\ \langle 2, \pi_2 f\rangle, & \text{if } f(a) \text{ is always of the form } \langle 2, v\rangle. \end{cases}$$

(Here π_2 means the second projection: if $f(a) = \langle i, v\rangle$, then $(\pi_2 f)(a) = v$.)

This shows that the naïve, purely "set-theoretic" understanding of BHK leads to a logic different from FO-Int.

If we add some "constructivity" to our BHK understanding, this justification for CD fails. Indeed, if, say, f is given by an algorithm that, for given a, either yields a witness for Q or a witness for $P(a)$, one cannot, by Uspensky – Rice theorem, algorithmically find out whether f is going to choose the first option at least for one a or not, and this is crucial for computing h.