Logic II
(LGIC 320 / MATH 571 / PHIL 412)
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EXERCISES

Lect. 16–18: Sequent Calculi & Cut Elimination

Exercise 1. Prove the cut elimination theorem for the Gentzen-style sequent calculus for classical first-order logic:

$$\overline{A \Rightarrow A} \qquad \overline{\bot \Rightarrow A}$$

$$\frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta}\ (\wedge L) \qquad \frac{\Gamma \Rightarrow \Delta, A_1 \quad \Gamma \Rightarrow \Delta, A_2}{\Gamma \Rightarrow \Delta, A_1 \wedge A_2}\ (\wedge R)$$

$$\frac{\Gamma, A_1 \Rightarrow C \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta}\ (\vee L) \qquad \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ (\vee R)$$

$$\frac{\Gamma_1 \Rightarrow A \quad \Gamma_2, B \Rightarrow \Delta}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta}\ (\to L) \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ (\to R)$$

$$\frac{\Gamma, A(t) \Rightarrow \Delta}{\Gamma, \forall x\, A(x) \Rightarrow \Delta}\ (\forall L) \qquad \frac{\Gamma \Rightarrow \Delta, A(y)}{\Gamma \Rightarrow \Delta, \forall x\, A(x)}\ (\forall R) \qquad \frac{\Gamma, A(y) \Rightarrow \Delta}{\Gamma, \exists x\, A(x) \Rightarrow \Delta}\ (\exists L) \qquad \frac{\Gamma \Rightarrow \Delta, A(t)}{\Gamma \Rightarrow \Delta, \exists x\, A(x)}\ (\exists R)$$

(Constraints: $y \notin FV(\Gamma, \Delta)$; the substitution of t for x is free.)

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (WL) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (WR) \qquad \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (CL) \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ (CR)$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \quad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (cut)$$

Exercise 2. Using syntactic properties of FO-Int, show that the following formulae are not derivable in FO-Int:

1. $\neg(\neg A \wedge \neg B) \to (A \vee B)$;
2. $(A \to B) \to (\neg A \vee B)$;
3. $\neg \forall x\, \neg A(x) \to \exists x\, A(x)$;
4. $\neg \forall x\, A(x) \to \exists x\, \neg A(x)$;
5. $\neg(A \wedge B) \to (\neg A \vee \neg B)$;
6. $\forall x\, \neg\neg A(x) \to \neg\neg \forall x\, A(x)$.

Exercise 3. *Kuroda's translation.* If φ is a first-order formula, let φ^K be obtained from φ by putting $\neg\neg$ in front of the whole formula and after each $\forall$ quantifier. Prove that φ is derivable in FO-CL iff φ^K is derivable in FO-Int.

Exercise 4. Show that the $(\vee L)$, $(\wedge R)$, and $(\to R)$ rules in the Gentzen-style sequent calculus for FO-Int (see on slides) and FO-CL (Exercise 1) are invertible, i.e., if the goal sequent is derivable, then the premise, or both premises, are also derivable.