Logic II
(LGIC 320 / MATH 571 / PHIL 412)
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EXERCISES

Lect. 11–15: First Order Intuitionistic Logic

Exercise 1. For each of the following formulae, find out whether it is provable in FO-Int. If yes, provide a proof (possibly, using Deduction Theorem). You can use a proof assistant to speed your work up a bit. If no, construct a Kripke model that falsifies the formula.

1. $\neg\exists x\, P(x) \to \forall x\, \neg P(x)$
2. $\forall x\, \neg P(x) \to \neg\exists x\, P(x)$
3. $\neg\neg\forall x\, P(x) \to \forall x\, \neg\neg P(x)$
4. $\neg\neg\exists x\, P(x) \to \exists x\, \neg\neg P(x)$
5. $\exists x\, \neg\neg P(x) \to \neg\neg\exists x\, P(x)$
6. $\forall x\, P(x) \vee \exists x\, \neg P(x)$
7. $\exists x\, P(x) \vee \forall x\, \neg P(x)$
8. $\exists x\, \forall y\, Q(x, y) \to \forall y\, \exists x\, Q(x, y)$
9. $\forall y\, \exists x\, Q(x, y) \to \exists x\, \forall y\, Q(x, y)$
10. $\forall x\, P(x) \to \exists x\, P(x)$
11. $\exists x\, P(x) \to \forall x\, P(x)$
12. $\neg\neg\forall x\, (P(x) \vee \neg P(x))$
13. $\forall x\, \neg\neg(P(x) \vee \neg P(x))$
14. $\neg(\forall x\, \neg\neg P(x) \wedge \neg\forall x\, P(x))$

Exercise 2. Show that Glivenko's theorem is false for FO-Int, i.e., construct a formula φ (without free variables) such that $\vdash_{\text{FO-CL}} \varphi$, but $\nvdash_{\text{FO-Int}} \neg\neg\varphi$.
Hint: use one of the formulae of the previous exercise, and remember that for formulae of the form $\varphi = \neg\psi$ the double negation principle ($\neg\neg\varphi \to \varphi$) is intuitionistically derivable.

Exercise 3. Does adding the principle $\forall x\, (\psi \vee \varphi(x)) \to \psi \vee \forall x\, \varphi(x)$ (where x is not a free variable of ψ) to FO-Int yield FO-CL?

Exercise 4*. Prove the *constructive property* of $\exists$ in intuitionistic logic: if $\vdash_{\text{FO-Int}} \exists x\, \varphi(x)$, then $\vdash_{\text{FO-Int}} \varphi(t)$ for some term t.