Logic II
(LGIC 320 / MATH 571 / PHIL 412)
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EXERCISES

Lect. 6–8: λ-calculus, Natural Deduction, and the Curry – Howard Correspondence

Exercise 1. For each of the following formulae give a natural deduction proof (if possible, using only intuitionstic rule for $\perp$) and construct a λ-term that encodes this proof. You can use a proof assistant to speed your work up a bit.

1. $(p \vee q) \to (q \vee p)$
2. $(p \vee q) \to (q \wedge p)$
3. $(p \to q) \to (\neg q \to \neg p)$
4. $(\neg q \to \neg p) \to (p \to q)$
5. $p \to \neg\neg p$
6. $\neg\neg p \to p$
7. $\neg\neg\neg p \to \neg p$
8. $(p \to q) \to (\neg p \vee q)$
9. $(\neg p \vee q) \to (p \to q)$
10. $(p \to q) \vee (q \to r) \vee (r \to p)$
11. $(\neg p \vee \neg q) \to \neg(p \wedge q)$
12. $\neg(p \wedge q) \to (\neg p \vee \neg q)$
13. $(\neg p \wedge \neg q) \to \neg(p \vee q)$
14. $\neg(p \vee q) \to (\neg p \wedge \neg q)$
15. $((p \to q) \to q) \to p$
16. $((((p \to q) \to p) \to p) \to q) \to q$

Exercise 2. Is the following substitution free (e.g., allowed to be performed without violating the meaning of the λ-term)? If yes, what is the result of the substitution?

1. y for x in $((\lambda x.(zx))y)x)$;
2. (xz) for y in $\lambda z.(y(\lambda x.z))$;
3. (xz) for y in $\lambda y.(y(\lambda z.z))$.

Exercise 3. Does there exist a closed simply typed λ-term of type $((p \to q) \to q) \to p$?

Exercise 4. Construct a combinatory term of type:

1. $(p \to q) \to ((q \to r) \to (p \to r))$;
2. $(p \to r) \to (p \to (q \to r))$;
3. $(p \to (q \to r)) \to ((p \to (r \to s)) \to (p \to (q \to s)))$.

Hint: first construct a λ-term (without free variables) of the desired type, then simulate λ as λ^* using the $\mathbb{K}$ and $\mathbb{S}$ combinators.

Exercise 5. Construct such λ-terms u_1 and u_2 that u_1 is β-reducible to u_2 and u_2 is typable but u_1 is not typable.

Exercise 6. Construct such λ-terms u_1 and u_2 without free variables and such types A_1 and A_2 that u_1 is β-reducible to u_2, u_2 is of types both A_1 and A_2, but u_1 is only of type A_1, but not A_2.

Exercise 7*. Let a propositional formula A contain only variables $p_1, \ldots, p_k$. Prove that

$$\vdash_{\mathrm{CL}} A \qquad \text{iff} \qquad \vdash_{\mathrm{Int}} (p_1 \vee \neg p_1) \to (p_2 \vee \neg p_2) \to \ldots \to (p_k \vee \neg p_k) \to A.$$

Exercise 8*. Prove Kripke completeness of the fragment of Int with only one connective, implication ($\to$). *Hint:* construct a canonical model consisting of deductively closed theories. (Note that disjunctivity is not needed here, since we don't have disjunction.)
Prove that adding Peirce's law, $((p \to q) \to q) \to p$ to this calculus yields the implicational fragment of CL.