Logic II
(LGIC 320 / MATH 571 / PHIL 412)
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EXERCISES

Lect. 1 & 2: Propositional Intuitionstic Logic

Exercise 1. For each of the following formulae establish whether the formula is derivable if Int (if "yes", derive it, maybe using Deduction Theorem; if "no", construct a Kripke model that falsifies it):

1. $(p \vee q) \to (q \vee p)$
2. $(p \vee q) \to (q \wedge p)$
3. $(p \to q) \to (\neg q \to \neg p)$
4. $(\neg q \to \neg p) \to (p \to q)$
5. $p \to \neg\neg p$
6. $\neg\neg p \to p$
7. $\neg\neg\neg p \to \neg p$
8. $(p \to q) \to (\neg p \vee q)$
9. $(\neg p \vee q) \to (p \to q)$
10. $(p \to q) \vee (q \to r) \vee (r \to p)$
11. $(\neg p \vee \neg q) \to \neg(p \wedge q)$
12. $\neg(p \wedge q) \to (\neg p \vee \neg q)$
13. $(\neg p \wedge \neg q) \to \neg(p \vee q)$
14. $\neg(p \vee q) \to (\neg p \wedge \neg q)$
15. $((p \to q) \to q) \to p$
16. $((((p \to q) \to p) \to p) \to q) \to q$

Exercise 2.

1. Is the formula $(p \to q) \vee (q \to p)$ derivable in Int?
2. Does adding the axiom scheme $(A \to B) \vee (B \to A)$ to Int yield CL?

Exercise 3. *Glivenko's theorem.* Prove that for any formula A the following holds: $\vdash_{\mathrm{CL}} A$ if and only if $\vdash_{\mathrm{Int}} \neg\neg A$. (Compare with the double negation translation.)

Exercise 4*. Construct a formula that can be falsified by a Kripke model of depth n, but is true in all Kripke models of depth less than n. (The *depth* of a model is the length of the longest sequence $x_1, \ldots, x_n$ of possible worlds, where $x_i R x_{i+1}$, but not $x_{i+1} R x_i$ (in particular, all the elements of such a sequence are distinct).)
Hint: first try small n's: $n = 1, 2, \ldots$

Exercise 5*. Show that adding the axiom scheme $\neg A \vee \neg\neg A$ ("weak excluded middle") to Int yields a logic that is strictly stronger than Int (i.e., $\not\vdash_{\mathrm{Int}} \neg p \vee \neg\neg p$) and strictly weaker than CL (i.e., it doesn't yet derive the original law of excluded middle).

Exercise 6.

1. Prove that for any formula A the following holds: $\vdash_{\mathrm{CL}} \neg A$ if and only if $\vdash_{\mathrm{Int}} \neg A$.
2. Prove that if A is built from variables using only $\neg$ and $\wedge$, then $\vdash_{\mathrm{CL}} A$ if and only if $\vdash_{\mathrm{Int}} A$.