Using Cut Elimination

Logic II, University of Pennsylvania, Spring 2017

Gödel – Gentzen Translation Syntactically

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- $\varphi^N = \neg\neg\varphi$ for atomic φ;
- $(\varphi_1 \wedge \varphi_2)^N = \varphi_1^N \wedge \varphi_2^N$;
- $(\varphi \to \psi)^N = \varphi^N \to \psi^N$;
- $(\forall x\, \varphi(x))^N = \forall x\, \varphi^N(x)$;
- $(\varphi_1 \vee \varphi_2)^N = \neg(\neg\varphi_1^N \wedge \neg\varphi_2^N)$;
- $(\exists x\, \varphi(x))^N = \neg\forall x\, \neg\varphi^N(x)$.

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- $(\exists x\, \varphi(x))^N = \neg\forall x\, \neg\varphi^N(x)$.

Theorem

FO-CL $\vdash \varphi$ *iff FO-Int* $\vdash \varphi^N$.

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Theorem
FO-CL $\vdash \varphi$ *iff FO-Int* $\vdash \varphi^N$.

Theorem
If FO-CL$_G \vdash \Gamma \Rightarrow \Delta$, *then FO-Int*$_G \vdash \Gamma^N, \Delta^{N\neg} \Rightarrow \bot$.

Classical	Intuitionistic

Classical	Intuitionistic
$$\frac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge L)$$	

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge L)$	$\dfrac{\Gamma^N, \varphi_i^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \varphi_1^N \wedge \varphi_2^N, \Delta^{N\neg} \Rightarrow \bot}\ (\wedge L)$

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge L)$	$\dfrac{\Gamma^N, \varphi_i^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, (\varphi_1 \wedge \varphi_2)^N, \Delta^{N\neg} \Rightarrow \bot}\ (\wedge L)$

Classical	Intuitionistic
$$\frac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge L)$$	$$\frac{\Gamma^N, \varphi_i^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, (\varphi_1 \wedge \varphi_2)^N, \Delta^{N\neg} \Rightarrow \bot}\ (\wedge L)$$
$$\frac{\Gamma \Rightarrow \Delta, \varphi_1 \quad \Gamma \Rightarrow \Delta, \varphi_2}{\Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2}\ (\wedge R)$$	

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge L)$	$\dfrac{\Gamma^N, \varphi_i^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, (\varphi_1 \wedge \varphi_2)^N, \Delta^{N\neg} \Rightarrow \bot}\ (\wedge L)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi_1 \quad \Gamma \Rightarrow \Delta, \varphi_2}{\Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2}\ (\wedge R)$	$\dfrac{}{\Gamma^N, \Delta^{N\neg}, \neg(\varphi_1^N \wedge \varphi_2^N) \Rightarrow \bot}$

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge L)$	$\dfrac{\Gamma^N, \varphi_i^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, (\varphi_1 \wedge \varphi_2)^N, \Delta^{N\neg} \Rightarrow \bot}\ (\wedge L)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi_1 \quad \Gamma \Rightarrow \Delta, \varphi_2}{\Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2}\ (\wedge R)$	$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \Delta^{N\neg}, \neg\varphi_1^N \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg\varphi_1^N} \quad \dfrac{\Gamma, \Delta^{N\neg}, \neg\varphi_2^N \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg\varphi_2^N}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg\varphi_1^N \wedge \neg\neg\varphi_2^N}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg(\varphi_1^N \wedge \varphi_2^N)}\ (cut)}{\Gamma^N, \Delta^{N\neg}, \neg(\varphi_1^N \wedge \varphi_2^N) \Rightarrow \bot}$

Classical	Intuitionistic

Classical

$$\frac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge L)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi_1 \quad \Gamma \Rightarrow \Delta, \varphi_2}{\Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2}\ (\wedge R)$$

Intuitionistic

$$\frac{\Gamma^N, \varphi_i^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, (\varphi_1 \wedge \varphi_2)^N, \Delta^{N\neg} \Rightarrow \bot}\ (\wedge L)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \Delta^{N\neg}, \neg\varphi_1^N \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg\varphi_1^N} \quad \dfrac{\Gamma, \Delta^{N\neg}, \neg\varphi_2^N \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg\varphi_2^N}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg\varphi_1^N \wedge \neg\neg\varphi_2^N}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg(\varphi_1^N \wedge \varphi_2^N)}\ (cut)}{\Gamma^N, \Delta^{N\neg}, \neg(\varphi_1^N \wedge \varphi_2^N) \Rightarrow \bot}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha, \beta \Rightarrow \alpha \wedge \beta \quad \bot \to \bot}{\alpha, \beta, \neg(\alpha \wedge \beta) \Rightarrow \bot}}{\alpha, \neg(\alpha \wedge \beta) \Rightarrow \neg\beta} \quad \bot \to \bot}{\alpha, \neg\neg\beta, \neg(\alpha \wedge \beta) \Rightarrow \bot}}{\neg\neg\beta, \neg(\alpha \wedge \beta) \Rightarrow \neg\alpha} \quad \bot \to \bot}{\neg\neg\alpha, \neg\neg\beta, \neg(\alpha \wedge \beta) \Rightarrow \bot}}{\neg\neg\alpha \wedge \neg\neg\beta \Rightarrow \neg\neg(\alpha \wedge \beta)}$$

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge L)$	$\dfrac{\Gamma^N, \varphi_i^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, (\varphi_1 \wedge \varphi_2)^N, \Delta^{N\neg} \Rightarrow \bot}\ (\wedge L)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi_1 \quad \Gamma \Rightarrow \Delta, \varphi_2}{\Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2}\ (\wedge R)$	$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \Delta^{N\neg}, \neg\varphi_1^N \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg\varphi_1^N} \quad \dfrac{\Gamma, \Delta^{N\neg}, \neg\varphi_2^N \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg\varphi_2^N}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg\varphi_1^N \wedge \neg\neg\varphi_2^N}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\neg(\varphi_1^N \wedge \varphi_2^N)}\ (cut)}{\Gamma^N, \Delta^{N\neg}, \neg(\varphi_1 \wedge \varphi_2)^N \Rightarrow \bot}$
	$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha, \beta \Rightarrow \alpha \wedge \beta \quad \bot \to \bot}{\alpha, \beta, \neg(\alpha \wedge \beta) \Rightarrow \bot}}{\alpha, \neg(\alpha \wedge \beta) \Rightarrow \neg\beta} \quad \bot \to \bot}{\alpha, \neg\neg\beta, \neg(\alpha \wedge \beta) \Rightarrow \bot}}{\neg\neg\beta, \neg(\alpha \wedge \beta) \Rightarrow \neg\alpha} \quad \bot \to \bot}{\neg\neg\alpha, \neg\neg\beta, \neg(\alpha \wedge \beta) \Rightarrow \bot}}{\neg\neg\alpha \wedge \neg\neg\beta \Rightarrow \neg\neg(\alpha \wedge \beta)}$

Classical	Intuitionistic
$$\frac{\Gamma, \varphi_1 \Rightarrow \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta} \; (\vee L)$$	

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$\dfrac{\Gamma, \varphi_1 \Rightarrow \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta}\ (\vee L)$	$\dfrac{}{\Gamma^N, \neg(\neg\varphi_1^N \wedge \neg\varphi_2^N), \Delta^{N\neg} \Rightarrow \bot}$

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$$\frac{\Gamma, \varphi_1 \Rightarrow \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta}\ (\vee L)$$	$$\dfrac{\dfrac{\dfrac{\Gamma^N, \varphi_1^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\varphi_1^N} \quad \dots}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\varphi_1^N \wedge \neg\varphi_2^N}}{\Gamma^N, \neg(\neg\varphi_1^N \wedge \neg\varphi_2^N), \Delta^{N\neg} \Rightarrow \bot}$$

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$\dfrac{\Gamma, \varphi_1 \Rightarrow \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta}\ (\vee L)$	$\dfrac{\dfrac{\dfrac{\Gamma^N, \varphi_1^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\varphi_1^N} \quad \ldots}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\varphi_1^N \wedge \neg\varphi_2^N}}{\Gamma^N, (\varphi_1 \vee \varphi_2)^N, \Delta^{N\neg} \Rightarrow \bot}$

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$$\frac{\Gamma, \varphi_1 \Rightarrow \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta}\ (\vee L)$$	$$\frac{\dfrac{\dfrac{\Gamma^N, \varphi_1^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\varphi_1^N} \quad \ldots}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\varphi_1^N \wedge \neg\varphi_2^N}}{\Gamma^N, (\varphi_1 \vee \varphi_2)^N, \Delta^{N\neg} \Rightarrow \bot}$$
$$\frac{\Gamma \Rightarrow \Delta, \varphi_i}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}\ (\vee R)$$	

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi_1 \Rightarrow \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta}\ (\vee L)$	$\dfrac{\dfrac{\dfrac{\Gamma^N, \varphi_1^N, \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\varphi_1^N} \quad \dots}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\varphi_1^N \wedge \neg\varphi_2^N}}{\Gamma^N, (\varphi_1 \vee \varphi_2)^N, \Delta^{N\neg} \Rightarrow \bot}$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi_i}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}\ (\vee R)$	$\dfrac{\dfrac{\dfrac{\Gamma^N, \Delta^{N\neg}, \neg\varphi_i^N \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg}, \neg\neg\neg\varphi_i^N \Rightarrow \bot}}{\Gamma^N, \Delta^{N\neg}, \neg\neg\neg\varphi_1^N, \neg\neg\neg\varphi_2^N \Rightarrow \bot}}{\Gamma^N, \Delta^{N\neg}, \neg\neg(\neg\varphi_1^N \wedge \varphi_2^N) \Rightarrow \bot}\ (cut)$

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$$\frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2} \; (\to L)$$	

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$$\frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2}\ (\to L)$$	$$\frac{}{\Gamma_1^N, \Gamma_2^N, \varphi^N \to \psi^N, \Delta_1^{N\neg}, \Delta_2^{N\neg} \Rightarrow \bot}$$

Classical	Intuitionistic
$$\dfrac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2} \ (\to L)$$	$$\dfrac{\dfrac{}{\Gamma_1^N, \Delta_1^{N\neg} \Rightarrow \varphi^N} \quad \Gamma_2^N, \psi^N, \Delta_2^{N\neg} \Rightarrow \bot}{\Gamma_1^N, \Gamma_2^N, \varphi^N \to \psi^N, \Delta_1^{N\neg}, \Delta_2^{N\neg} \Rightarrow \bot}$$

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$$\dfrac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2}\ (\to L)$$	$$\dfrac{\dfrac{\Gamma_1^N, \Delta_1^{N\neg}, \neg\varphi^N \Rightarrow \bot}{\Gamma_1^N, \Delta_1^{N\neg} \Rightarrow \varphi^N} \quad \Gamma_2^N, \psi^N, \Delta_2^{N\neg} \Rightarrow \bot}{\Gamma_1^N, \Gamma_2^N, \varphi^N \to \psi^N, \Delta_1^{N\neg}, \Delta_2^{N\neg} \Rightarrow \bot}$$

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$\dfrac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2} (\to L)$	$\dfrac{\dfrac{\Gamma_1^N, \Delta_1^{N\neg}, \neg\varphi^N \Rightarrow \bot}{\Gamma_1^N, \Delta_1^{N\neg} \Rightarrow \neg\neg\varphi^N} \quad \Gamma_2^N, \psi^N, \Delta_2^{N\neg} \Rightarrow \bot}{\Gamma_1^N, \Gamma_2^N, \varphi^N \to \psi^N, \Delta_1^{N\neg}, \Delta_2^{N\neg} \Rightarrow \bot}$

Lemma

$\neg\neg\varphi^N \leftrightarrow \varphi^N$.

Classical	Intuitionistic
$$\dfrac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2} (\to L)$$	$$\dfrac{\dfrac{\Gamma_1^N, \Delta_1^{N\neg}, \neg\varphi^N \Rightarrow \bot}{\Gamma_1^N, \Delta_1^{N\neg} \Rightarrow \neg\neg\varphi^N} \quad \Gamma_2^N, \psi^N, \Delta_2^{N\neg} \Rightarrow \bot}{\Gamma_1^N, \Gamma_2^N, \varphi^N \to \psi^N, \Delta_1^{N\neg}, \Delta_2^{N\neg} \Rightarrow \bot}$$

Lemma

$\neg\neg\varphi^N \leftrightarrow \varphi^N$.

$$\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi} (\to R)$$	

Classical	Intuitionistic
$\dfrac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2}\ (\to L)$	$\dfrac{\dfrac{\Gamma_1^N, \Delta_1^{N\neg}, \neg\varphi^N \Rightarrow \bot}{\Gamma_1^N, \Delta_1^{N\neg} \Rightarrow \neg\neg\varphi^N} \quad \Gamma_2^N, \psi^N, \Delta_2^{N\neg} \Rightarrow \bot}{\Gamma_1^N, \Gamma_2^N, \varphi^N \to \psi^N, \Delta_1^{N\neg}, \Delta_2^{N\neg} \Rightarrow \bot}$

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$\neg\neg\varphi^N \leftrightarrow \varphi^N$.

$\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\to R)$	$\dfrac{\dfrac{\overline{\Gamma^N, \varphi^N, \Delta^{N\neg} \Rightarrow \psi^N}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \varphi^N \to \psi^N} \quad \bot \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg}, \neg(\varphi^N \to \psi^N) \Rightarrow \bot}$

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$$\dfrac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2}\ (\to L)$$	$$\dfrac{\dfrac{\Gamma_1^N, \Delta_1^{N\neg}, \neg\varphi^N \Rightarrow \bot}{\Gamma_1^N, \Delta_1^{N\neg} \Rightarrow \neg\neg\varphi^N} \quad \Gamma_2^N, \psi^N, \Delta_2^{N\neg} \Rightarrow \bot}{\Gamma_1^N, \Gamma_2^N, \varphi^N \to \psi^N, \Delta_1^{N\neg}, \Delta_2^{N\neg} \Rightarrow \bot}$$

Lemma

$\neg\neg\varphi^N \leftrightarrow \varphi^N$.

$$\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\to R)$$	$$\dfrac{\dfrac{\overline{\Gamma^N, \varphi^N, \Delta^{N\neg} \Rightarrow \neg\neg\psi^N}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \varphi^N \to \psi^N} \quad \bot \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg}, \neg(\varphi^N \to \psi^N) \Rightarrow \bot}$$

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$\dfrac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2}\ (\to L)$	$\dfrac{\dfrac{\Gamma_1^N, \Delta_1^{N\neg}, \neg\varphi^N \Rightarrow \bot}{\Gamma_1^N, \Delta_1^{N\neg} \Rightarrow \neg\neg\varphi^N} \quad \Gamma_2^N, \psi^N, \Delta_2^{N\neg} \Rightarrow \bot}{\Gamma_1^N, \Gamma_2^N, \varphi^N \to \psi^N, \Delta_1^{N\neg}, \Delta_2^{N\neg} \Rightarrow \bot}$

Lemma

$\neg\neg\varphi^N \leftrightarrow \varphi^N$.

$\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\to R)$	$\dfrac{\dfrac{\dfrac{\Gamma^N, \varphi^N, \Delta^{N\neg}, \neg\psi^N \Rightarrow \bot}{\Gamma^N, \varphi^N, \Delta^{N\neg} \Rightarrow \neg\neg\psi^N}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \varphi^N \to \psi^N} \quad \bot \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg}, \neg(\varphi^N \to \psi^N) \Rightarrow \bot}$

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi(y) \Rightarrow \Delta}{\Gamma, \exists x\, \varphi(x) \Rightarrow \Delta}\ (\exists L)$	$\dfrac{\dfrac{\dfrac{\Gamma^N, \varphi^N(y), \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\varphi^N(y)}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \forall x\, \neg\varphi^N(x)} \quad \bot \Rightarrow \bot}{\Gamma^N, \neg\forall x\, \neg\varphi^N(x), \Delta^{N\neg} \Rightarrow \bot}$
$\dfrac{\Gamma \Rightarrow \varphi(t), \Delta}{\Gamma \Rightarrow \exists x\, \varphi(x), \Delta}\ (\exists R)$	$\dfrac{\dfrac{\dfrac{\Gamma^N, \Delta^{N\neg}, \neg\varphi^N(t) \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg}, \forall x\, \neg\varphi^N(x) \Rightarrow \bot} \quad \bot \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg} \Rightarrow \neg\forall x\, \neg\varphi^N(x)}}{\Gamma^N, \Delta^{N\neg}, \neg\neg\forall x\, \neg\varphi^N(x) \Rightarrow \bot}$

Classical	Intuitionistic
$$\frac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x\, \varphi(x) \Rightarrow \Delta}\ (\forall L)$$	

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x\, \varphi(x) \Rightarrow \Delta}\ (\forall L)$	$\dfrac{\Gamma^N, \varphi^N(t), \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \forall x\, \varphi^N(x), \Delta^{N\neg} \Rightarrow \bot}$

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x\, \varphi(x) \Rightarrow \Delta}\ (\forall L)$	$\dfrac{\Gamma^N, \varphi^N(t), \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \forall x\, \varphi^N(x), \Delta^{N\neg} \Rightarrow \bot}$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi(y)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\ (\forall R)$	

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x\, \varphi(x) \Rightarrow \Delta}\ (\forall L)$	$\dfrac{\Gamma^N, \varphi^N(t), \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \forall x\, \varphi^N(x), \Delta^{N\neg} \Rightarrow \bot}$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi(y)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\ (\forall R)$	$\dfrac{\dfrac{\Gamma^N, \Delta^{N\neg} \Rightarrow \varphi^N(x)}{\Gamma^N, \Delta^{N\neg} \Rightarrow \forall x\, \varphi^N(x)} \quad \bot \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg}, \neg(\forall x\, \varphi^N(x)) \Rightarrow \bot}$

Classical	Intuitionistic
$\dfrac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x\, \varphi(x) \Rightarrow \Delta}\ (\forall L)$	$\dfrac{\Gamma^N, \varphi^N(t), \Delta^{N\neg} \Rightarrow \bot}{\Gamma^N, \forall x\, \varphi^N(x), \Delta^{N\neg} \Rightarrow \bot}$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi(y)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\ (\forall R)$	$\dfrac{\dfrac{\dfrac{\Gamma^N, \Delta^{N\neg}, \neg\varphi^N \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg}, \Rightarrow \neg\neg\varphi^N(x)}}{\Gamma^N, \Delta^{N\neg} \Rightarrow \forall x\, \varphi^N(x)} \quad \bot \Rightarrow \bot}{\Gamma^N, \Delta^{N\neg}, \neg(\forall x\, \varphi^N(x)) \Rightarrow \bot}$

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$\varphi \Rightarrow \varphi$	$\varphi^N, \neg\varphi^N \Rightarrow \bot$

Classical	Intuitionistic
$\varphi \Rightarrow \varphi$	$\varphi^N, \neg\varphi^N \Rightarrow \bot$
$\bot \Rightarrow$	$\bot^N \Rightarrow \bot$

Classical	Intuitionistic
$\varphi \Rightarrow \varphi$	$\varphi^N, \neg\varphi^N \Rightarrow \bot$
$\bot \Rightarrow$	$\bot^N \Rightarrow \bot$

Structural rules all move to the left.

Summary

- Disjunctive property
- Existential property
- Harrop's theorems
- Herbrand's theorem
- Gödel – Gentzen translation
- Interpolation and Robinson's theorem