Lambek Categorial Grammars
Day 3

Stepan Kuznetsov

Steklov Mathematical Institute, RAS

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Gentzen-style Lambek Calculus

$$A \to A$$

$$\frac{A\,\Pi \to B}{\Pi \to A \setminus B}\text{, } \Pi \text{ is not empty} \qquad \frac{\Pi\,A \to B}{\Pi \to B\,/\,A}\text{, } \Pi \text{ is not empty}$$

$$\frac{\Pi \to A \quad \Gamma\,B\,\Delta \to C}{\Gamma\,\Pi\,(A \setminus B)\,\Delta \to C} \qquad \frac{\Pi \to A \quad \Gamma\,B\,\Delta \to C}{\Gamma\,(B\,/\,A)\,\Pi\,\Delta \to C}$$

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$$\frac{\Gamma\,A\,B\,\Delta \to C}{\Gamma\,(A \cdot B)\,\Delta \to C} \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma\,\Delta \to A \cdot B}$$

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$$\frac{\Pi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, \Pi\, (A \backslash B)\, \Delta \to C} \qquad \frac{\Pi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, (B / A)\, \Pi\, \Delta \to C}$$

$$\frac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C} \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}$$

$$\frac{\Pi \to A \quad \Gamma\, A\, \Delta \to C}{\Gamma\, \Pi\, \Delta \to C}\ (\mathrm{cut})$$

Cut Elimination

Theorem (Lambek '58)

Every sequent derivable in L *can be derived without using the* (cut) *rule.*

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Proof.

Routine induction on the complexity of the sequent. □

Lambek Grammars and Context-Free Grammars

Theorem (Gaifman '61)

Every context-free language without the empty word can be generated by an AB*-grammar.*

Theorem (Pentus '92)

Every language generated by a Lambek categorial grammar is context-free.

Greibach Normal Form

A context-free grammar is in Greibach normal from if every rule of this grammar has one of the following forms:

- $A \to a$
- $A \to aB$
- $A \to aBC$

Theorem (Greibach '65)

Every context-free language without the empty word can be generated by a context-free grammar in Greibach normal form.

From Context-Free Grammars to AB-grammars: Proof

$N \ni A \rightsquigarrow p_A \in \mathrm{Pr}.$

Categorial vocabulary:

$A \to a$	$\langle p_A, a \rangle$
$A \to aB$	$\langle p_A / p_B, a \rangle$
$A \to aBC$	$\langle (p_A / p_C) / p_B, a \rangle$

$H = p_S$

Pentus' Theorem Proof: Outline

- Interpolation lemma
- Free group interpretation, thin sequents
- BR-lemma
- Translating derivations into Lcut

Complexity Counters

$\|A\|_p$ is the number of occurrences of p in A.

$$\|A\| = \sum_{p \in \mathrm{Pr}} \|A\|_p$$

Interpolation Lemma

Theorem (Roorda '91)

Let $\mathrm{L} \vdash \Phi\, \Theta\, \Psi \to C$, *where* Θ *is not empty. Then there exists such a type* E *(the* interpolant*) that*

- $\mathrm{L} \vdash \Theta \to E$;
- $\mathrm{L} \vdash \Phi\, E\, \Psi \to C$;
- $\|E\|_p \leq \min\{\|\Theta\|_p, \|\Phi\, \Psi\, C\|_p\}$ *for every* $p \in \mathrm{Pr}$.

Thin Sequents

Definition

A sequent $\Gamma \to C$ is called *thin* if $\|\Gamma\, C\|_p \leq 2$ for every $p \in \mathrm{Pr}$.

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Primitive type substitution: $\phi \colon \mathrm{Pr} \to \mathrm{Pr}$.

$\phi(A \cdot B) = \phi(A) \cdot \phi(B)$;

$\phi(A \setminus B) = \phi(A) \setminus \phi(B)$;

$\phi(B \mathbin{/} A) = \phi(B) \mathbin{/} \phi(A)$.

$\phi(A_1 \ldots A_k \to B) = \phi(A_1) \ldots \phi(A_k) \to \phi(B)$.

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Lemma

If $\mathrm{L} \vdash \Pi \to C$, *then* $\mathrm{L} \vdash \phi(\Pi \to C)$ *for any primitive type substitution* ϕ.

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Lemma

If $\mathrm{L} \vdash \Pi \to C$*, then there exist such* $\Pi' \to C'$ *and* ϕ *that* $\Pi' \to C'$ *is derivable and thin and* $\Pi \to C = \phi(\Pi' \to C')$*.*

Free Group Interpretation

Let FG be the free group generated by Pr. Then define $[\![A]\!] \in \mathrm{FG}$ for every type A.

- $[\![p]\!] = p$ for $p \in \mathrm{Pr}$;
- $[\![A \cdot B]\!] = [\![A]\!][\![B]\!]$;
- $[\![A \setminus B]\!] = [\![A]\!]^{-1}[\![B]\!]$;
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Lemma

If $\mathrm{L} \vdash \Gamma \to C$, *then* $[\![\Gamma]\!] = [\![C]\!]$.

Interpolation for Thin Sequents

Lemma

Let $\mathrm{L} \vdash \Phi\, \Theta\, \Psi \to C$, *where* Θ *is not empty and the whole sequent is thin. Then there exists such a type* E *that*

- $\mathrm{L} \vdash \Theta \to E$ *and* $\mathrm{L} \vdash \Phi\, E\, \Psi \to C$;
- $\Theta \to E$ *and* $\Phi\, E\, \Psi \to C$ *are thin sequents;*
- $\|E\| = |[\![\Theta]\!]|$ *(where* $|u|$ *for* $u \in \mathrm{FG}$ *is the length of* u *as a word after all cancellations).*

The BR-lemma

Lemma

Let $u_1, \dots, u_n \in \mathrm{FG}$*,* $n \geq 2$*, and* $u_1 \dots u_n = \mathbf{1}_{\mathrm{FG}}$*. The there exists such* $k < n$ *that* $|u_k u_{k+1}| \leq \max\{|u_k|, |u_{k+1}|\}$*.*

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If $\Gamma \to C$ is derivable in L, $\Gamma \in \mathrm{Tp}_m^+$, and $C \in \mathrm{Tp}_m$, then this sequent is derivable in Lcut_m.

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Lcut_m-grammars are context-free!

One More Translation of Context-Free Grammars to Lambek Grammars

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Let $N \subset \mathrm{Pr}$.
$N \ni t \mapsto I(t) \subset \mathrm{Tp}$:

- $t \in I(t)$;
- if $p \Rightarrow qr$ is a rule of the grammar, then $(q \backslash p) \in I(t)$;
- if $t \in N$ and $p \Rightarrow qr$ is a rule of the grammar, then $(q \backslash p) / (t \backslash r) \in I(t)$.

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- if $t \in N$ and $p \Rightarrow qr$ is a rule of the grammar, then $(q \setminus p) / (t \setminus r) \in I(t)$.

$\mathcal{D} = \{\langle A, a\rangle \mid t \Rightarrow a \text{ is a rule of the grammar and } A \in I(t)\}$.
$H = s$.

Translating Montague Semantics

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Buszkowski's construction also can be enriched with semantics.