Lax operator algebras: unexpected outcome, and a new tool of the theory of integrable systems

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Outline

1. Lax operator algebras
2. Lax integrable systems
3. Quantization of Lax integrable systems and 2D CFT
SECTION 1. LAX OPERATOR ALGEBRAS
Geometrical data

- Riemann surface $\Sigma$
- Classical Lie alg. $g$ over $\mathbb{C}$
- Marked points of two kinds: $P_1, \ldots, P_N \in \Sigma$ and $\gamma_1, \ldots, \gamma_K \in \Sigma$
- Vectors $\alpha_1, \ldots, \alpha_K \in \mathbb{C}^n$ associated with $\gamma$’s.

Pairs $\gamma, \alpha$ are referred to as Tyurin parameters, due to the

**Theorem (A.N. Tyurin):** Let $g = \text{genus } \Sigma$, $n \in \mathbb{Z}_+$. Then there is a 1–1 correspondence between the pairs of sets $\gamma_1, \ldots, \gamma_{ng} \in \Sigma$; $\alpha_1, \ldots, \alpha_{ng} \in \mathbb{C}P^{n-1}$, and the equivalence classes of the semi-stable holomorphic rank $n$ vector bundles on $\Sigma$. 
Lax operator with the spectral parameter on a Riemann surface

— it is a meromorphic function \( L \) on \( \Sigma \) with arbitrary poles at \( P_i \)'s, simple or double poles at the \( \gamma \)'s, holomorphic at the other points, and having the expansion at a \( \gamma \) of the form

\[
L(z) = \frac{L_{-2}}{(z - z_\gamma)^2} + \frac{L_{-1}}{(z - z_\gamma)} + L_0 + L_1(z - z_\gamma) + O((z - z_\gamma)^2)
\]

where \( z \) is a local coordinate at \( \gamma \), \( z_\gamma = z(\gamma) \), and

\[
L_{-1} = \begin{cases}
\alpha \beta^t, & \text{if } \mathfrak{g} = \mathfrak{gl}(n) \\
\alpha \beta^t - \beta \alpha^t, & \text{if } \mathfrak{g} = \mathfrak{so}(n) \\
(\alpha \beta^t + \beta \alpha^t)\sigma, & \text{if } \mathfrak{g} = \mathfrak{sp}(2n)
\end{cases}
\]

\[
\beta^t \sigma \alpha = 0 \\
L_0 \alpha = k \alpha \\
L_{-2} = \nu \alpha \alpha^t \sigma, \quad \mathfrak{g} = \mathfrak{sp}(2n)
\]

\( \alpha, \beta \in \mathbb{C}^n \), \( \alpha \) is associated with \( \gamma \), \( \beta \) arbitrarily, \( \sigma \) — is the matrix of the quadratic form
**THEOREM:** For given \( g, (\alpha, \gamma, P_i) \), the Lax operators form a Lie algebra with respect to the point-wise matrix commutator 
\[
[L, L'](P) = [L(P), L'(P)] \quad (P \in \Sigma)
\] (Krichever-Sh., 2007).

This algebra is referred to as **Lax operator algebra**, and is denoted by \( \mathcal{L} \), or by \( \bar{g} \).

**Examples:**

1) If \( g = 0, P_1 = 0, P_2 = \infty, \{\gamma\} = \emptyset \) then \( \bar{g} = g \otimes \mathbb{C}[z, z^{-1}] \) — **a loop algebra**.

2) If \( g > 0, \{\gamma\} = \emptyset \) then \( \bar{g} = g \otimes \mathcal{A} \) where \( \mathcal{A} \) consists of meromorphic functions holomorphic outside \( \{P\} \) — **a Krichever–Novikov current algebra**.
Almost graded structure

Lax operator algebras are by no means graded. Define $\mathfrak{g}_m \subset \mathfrak{g}$ by

\[
(\mathfrak{g}_m)_l = -m \sum_{i=1}^{N} P_i \\
(\mathfrak{g}_m)_o = \sum_{j=1}^{M} (a_j m + b_{j,m}) Q_j,
\]

where $\sum_{j=1}^{M} a_j = N$,

$\sum_{j=1}^{M} b_{j,m} = N + g - 1$,

$|b_{j,m}| \leq B \ \forall \ j, m \ (N \geq M)$

**Theorem:**

1. $\dim \mathfrak{g}_m = N \dim \mathfrak{g}$;
2. $\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m$
3. $\exists$ a constant integer $R > 0$ such that

\[
[\mathfrak{g}_m, \mathfrak{g}_k] \subseteq \bigoplus_{h=m+k}^{m+k+R} \mathfrak{g}_h.
\]
A skew-symmetric bilinear form \( \gamma \) on \( \mathcal{L} \) is called a 2-cocycle if
\[
\gamma([X, Y], Z) + \gamma([Z, X], Y) + \gamma([Y, Z], X) = 0
\]
for all \( X, Y, Z \in \mathcal{L} \).

Given a Lie algebra \( \mathcal{L} \), and a cocycle \( \gamma \), we can construct a new Lie algebra \( \hat{\mathcal{L}} \) called central extension of \( \mathcal{L} \). As a linear space
\[
\hat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C} \cdot t
\]
and the bracket is given by
\[
[X + c_1 t, Y + c_2 t] = [X, Y]_{\mathcal{L}} + \gamma(X, Y)t, \quad [X, t] = 0 \ \forall X \in \mathcal{L}.
\]

Example. \( \mathcal{L} = \) loop algebra, \( \gamma(X, Y) = \text{res}_0 \text{tr}(XdY) \).
Equivalence of central extensions

Equivalently, a central extension of $\mathcal{L}$ is a short exact sequence of Lie algebras

$$0 \to \mathbb{C} \xrightarrow{i} \hat{\mathcal{L}} \xrightarrow{p} \mathcal{L} \to 0 \quad (2)$$

where $\text{Im}(i) = \ker(p)$ is the center of $\hat{\mathcal{L}}$. Then $\hat{\mathcal{L}}$ is a central extension of $\mathcal{L}$ in the above defined sense.

Two central extensions $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}'$ are called equivalent if there exists an isomorphism $e$ (equivalence) such that the following diagram is commutative

$$\begin{array}{ccc}
0 & \to & \mathbb{C} & \xrightarrow{i} & \hat{\mathcal{L}} & \xrightarrow{p} & \mathcal{L} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & e & & & & & & \\
& & \hat{\mathcal{L}}' & & \hat{\mathcal{L}} & & \mathcal{L} & & 0.
\end{array} \quad (3)$$
Classification of almost graded central extensions

**Geometric cocycle:**
\[ \gamma(L, L') = \sum_{i=1}^{N} \text{res}_{P_i} \text{tr}(L \nabla^{(\theta)} L') = \sum_{j=1}^{M} \text{res}_{Q_j} \text{tr}(L \nabla^{(\theta)} L') \]
where \( \theta \) is a \( g \)-valued 1-form, \( \nabla^{(\theta)} = d + \text{ad} \theta \).

**Theorem:**
1) If \( \theta \) possesses the same expansion as \( L \) at any \( \gamma \)-point, with \( \beta^t \sigma \alpha = 1 \), then \( \gamma \) gives an almost graded central extension.

2) If \( g \) is simple, the only central extension of \( \bar{g} \), up to equivalence and normalization of the central generator, is given by the above cocycle \( \gamma \).

3) If \( g = gl(n) \) then \( \bar{g} \) has only one more central extension given by \( \gamma' = \sum_{i=1}^{N} \text{res}_{P_i} \text{tr}(L)\text{tr}(\nabla^{(\theta)} L') \).

Example. For a loop algebra, or a KN current algebra \( \theta = 0 \), and \( \gamma(X, Y) = \text{tr}(XdY) \).
Canonical representation of $\mathcal{L}$

$\mathcal{F}$ — the space of meromorphic functions on $\Sigma$ with the same singularities as $L$. Expansion at a $\gamma$ is of the form

$$\varphi = \text{const} \cdot \frac{\alpha}{z - z_\gamma} + \varphi_0 + O(z - z_\gamma).$$

$\mathcal{L}$ has a natural representation in $\mathcal{F}$. This representation is almost graded.

Almost grading in $\mathcal{F}$ is given by means a certain base (the Krichever–Novikov base).

Example. For two points $P$, $Q$ the degree is essentially given by the order at $P$.

The $\mathcal{L}$-module $\mathcal{F}$ by no means is of vacuum type.
Vacuum representation

\( \mathcal{F}^{\infty}/2 \) — space of semi-infinite forms on \( \mathcal{F} \),\n\[ \mathcal{F}^{\infty}/2 = \bigoplus \mathcal{F}_k^{\infty}/2, \]

\[ \mathcal{F}_k^{\infty}/2 = \{ f_{i_1} \wedge \ldots \wedge f_{i_k} \wedge f_{i_{k+1}} \wedge f_{i_{k+2}} \ldots \} \]

where \( f_j \) is the Krichever–Novikov base in \( \mathcal{F} \).

\( \mathcal{F}_k^{\infty}/2 \) is a vacuum representation of \( \mathcal{L} \)

where the vacuum is given by the semi-infinite monomial \( f_{k+1} \wedge f_{k+2} \wedge \ldots \), and the action by Leibniz rule.
Given an $L$-operator, its spectral curve $\Sigma_L$ is defined by means the equation $\det(L(z) - \kappa) = 0$. It is an $n$-fold branch covering of $\Sigma$.

Spectrum $K$ of $L$ is a diagonal matrix defined by the relation $\Psi L = K \Psi$. The spectrum defines the function $\kappa$ on $\Sigma_L$.

**Lemma:** $\kappa$ is meromorphic and holomorphic outside the pre-images of $P$-$Q$-points (in particular, at pre-images of $\gamma$’s).

Vise versa, the direct image of such a function is a spectrum of the operator $L(\kappa) = \psi^{-1} K(\kappa) \psi$.

**Theorem:** The maximal subalgebra in $\mathcal{L}$ commuting with an $L \in \mathcal{L}$ is isomorphic to the algebra of meromorphic functions on $\Sigma_L$, holomorphic outside the pre-images of $P$-$Q$-points.
SECTION 2. LAX INTEGRABLE SYSTEMS
Lax equations

Lax equation: \[ \dot{L} = [L, M] \]

where

\[ L = L(z, \alpha, \beta, \gamma, \ldots), \quad M = M(z, \alpha, \mu, \gamma, \ldots). \]

It is regarded to as a collection of equations on \( \alpha \)'s, \( \beta \)'s, \( \gamma \)'s, \( \kappa \)'s and the main parts of the expansions of \( L \) at \( P_i \)'s.

\( M \) is defined by the same constrains as \( L \), excluding \( \beta^t \sigma \alpha = 0 \) and \( L_0 \alpha = k \alpha \), namely

\[ M = \frac{M_{-2}}{(z - z_\gamma)^2} + \frac{M_{-1}}{z - z_\gamma} + M_0 + M_1(z - z_\gamma) + O((z - z_\gamma)^2) \]

where

\[ M_{-2} = \lambda \alpha \alpha^t \sigma \quad M_{-1} = (\alpha \mu^t + \varepsilon \mu \alpha^t) \sigma \]
Hierarchy of commuting flows

\[ D := \sum m_i P_i \ (i = 1, \ldots, N), \ \text{s.t. supp } D \cap \{\gamma\} = \emptyset. \]

\[ \mathcal{L}^D := \{L \in \mathcal{L} \mid (L) + D \geq 0 \text{ outside } \gamma's\}. \]

Upon a certain (effective) condition a Lax equation defines a flow on \( \mathcal{L}^D \).

**Theorem:** Given a generic \( L \), there is a family of \( M \)-operators \( M_a = M_a(L) \) \((a = (P_i, n, m), n > 0, \ m > -m_i)\) uniquely defined up to normalization, such that outside the \( \gamma \)-points \( M_a \) has pole at the point \( P_i \) only, and in the neighborhood of \( P_i \)

\[ M_a(w_i) = w_i^{-m} L^n(w_i) + O(1), \]

The equations

\[ \partial_a L = [L, M_a], \ \partial_a = \partial/\partial t_a \]

(4)

define a family of commuting flows on \( \mathcal{L}^D \).
We define an external 2-form on $\mathcal{L}^D$. For $L \in \mathcal{L}^D$ let $\Psi$ be a matrix-valued function formed by the eigenvectors of $L$: $\Psi L = K \Psi$ ($K$ — diagonal).

$$\Omega := \text{tr}(\Psi^{-1} \delta L \wedge \delta \Psi - \Psi^{-1} \delta \Psi \wedge \delta K) = \delta \text{tr}(\Psi^{-1} L \delta \Psi)$$

where $\delta \Psi$ is the differential of $\Psi$ in $\alpha, \beta, \ldots$.

Let $\omega_0$ be a holomorphic 1-form on $\Sigma$ and

$$\omega := \sum \text{res}_{\gamma_s} \Omega \omega_0 + \sum \text{res}_{P_i} \Omega \omega_0$$

**Theorem**: $\omega$ is a symplectic form on a certain invariant manyfold $\mathcal{P}^D \subset \mathcal{L}^D$. 
Theorem: The equations of the above commutative family are Hamiltonian with respect to the Krichever-Phong symplectic structure on $\mathcal{L}^D$, with the Hamiltonians given by

$$H_a = -\frac{1}{n+1} \text{res}_{P_i} \text{tr}(w_i^{-m}L^{n+1})dw_i$$

Example. Let $D = (\omega_0)$. Then $H_a$ are Hitchin Hamiltonians.
Calogero-Moser systems, $\mathfrak{g} = \mathfrak{gl}(n)$:

Lax operator:

$$L_{i,j} := \frac{\sigma(z + q_i - q_j)\sigma(z - q_i)\sigma(q_j)}{\sigma(z)\sigma(z - q_j)\sigma(q_i - q_j)\sigma(q_i)} \quad (i \neq j), \quad L_{ii} = p_i$$

2d order Hamiltonian:

$$H = -\frac{1}{2} \text{res}_{z=0} \text{tr}(z^{-1}L^2) = -\frac{1}{2} \sum_{j=1}^{n} p_j^2 + \sum_{i<j} \wp(q_i - q_j).$$

Tyurin parameters: $(q_i, e_i), \ e_i = (\ldots, \delta_{ij}, \ldots)^t, \ \omega = \sum dp_i \wedge dq_i.$
Calogero-Moser systems, $\mathfrak{g} = \mathfrak{so}(2n)$:

Lax operator: $L = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{so}(2n), \quad B^t = -B, \quad C^t = -C.$

$A_{i,j}$ is the same as $L_{i,j}$ above.

$B_{ij} = \frac{\sigma(z + q_j + q_i)\sigma(z - q_j)}{\sigma(z)\sigma(z + q_i)\sigma(q_i + q_j)}, \quad C_{j,i} = -\frac{\sigma(z - q_j - q_i)\sigma(z + q_i)}{\sigma(z)\sigma(z - q_j)\sigma(q_i + q_j)}, \quad i < j.$

2d order Hamiltonian:

$H = -\frac{1}{2} \text{res} \, \text{tr}(z^{-1}L^2) = -\sum_{i=1}^{n} p_i^2 + 2 \sum_{i<j} \wp(q_i - q_j) + 2 \sum_{i<j} \wp(q_i + q_j)$. 
Other examples

1) \( g = 0, \alpha = 0 \) (i.e. \( \Sigma = \mathbb{CP}^1 \), the bundle is trivial), \( P_1 = 0, P_2 = \infty \). Then \( \bar{g} = g \otimes \mathbb{C}[z, z^{-1}] \) — loop algebra.

It yields a conventional Lax equation with a rational spectral parameter:

\[
L_t = [L, M], \quad L, M \in g \otimes \mathbb{C}[\lambda^{-1}, \lambda], \quad \lambda \in \mathcal{D}^1
\]

(I.Gelfand, L.Dikii, I.Dorfman, A.Reyman, M.Semenov-Tian-Shanskii, V.Drinfeld, V.Sokolov, V.Kac, P. van Moerbeke).

Majority of known integrable cases of motion and hydrodynamics of a solid body.

2) Arbitrary genus, \( D = (\omega_0) \) where \( \omega_0 \) is a holomorphic 1-form. Then we obtain Hitchin systems.
SECTION 3. QUANTIZATION OF LAX INTEGRABLE SYSTEMS AND 2D CFT
Sheaf of conformal blocks

\[ \dot{L} = [L, M] \]

\( \mathcal{P} = \{L\} \) — phase space.

\( \{\Sigma_L\} \) — family of spectral curves over \( \mathcal{P} \)

Over every \( \{\Sigma_L\} \) take the canonical vacuum \( \mathcal{L} \)-module \( \mathcal{F}_L^\infty/2 \)

\( \mathcal{A}_L \) acts in \( \mathcal{F}_L^\infty/2 \) by means

\[ \chi \rightarrow L(\chi) = \Psi^{-1} K(\chi) \Psi \]

Conformal blocks:

\[ C = \mathcal{F}_L^\infty/2 / \mathcal{A}_L^{\text{reg}} \mathcal{F}_L^\infty/2. \]
Kodaira–Spencer cocycle

\( X \in Vect(P), \, d_L \in \text{Diff}_{\text{loc}} \) — transition function on \( \Sigma_L \).

Kodaira–Spencer cocycle:

\[
\rho(X) = d_L^{-1} \cdot \partial_X d_L
\]

**Theorem (Sch-Sh, ’05):**

\( \rho(X) \) continues to a KN vector field on \( \Sigma_L \)

(i.e. a global meromorphic vector field on \( \Sigma_L \) holomorphic outside \( P-Q \)-points).

Notation: \( \mathcal{V}_L \) is the Lie algebra of KN vector fields over \( L \).
**Sugawara construction**

\( V - \) vacuum \( \mathcal{L} \)-module. \underline{Example:} \( V = \mathcal{F}^\infty_L / 2 \)

Let \( \{ A_j \}, \{ \omega^j \} \) be the Krichever–Novikov bases of functions and 1-forms, resp., on \( \Sigma \), \( \{ u_i \} \).

\textbf{Notation:} \( a(m) \) is the representation operator of \( A_m \) in \( V \).

\textbf{Energy–momentum tensor:} \( T = \sum_{m,n} :a(m)a(n): \omega^m \omega^n \)

For \( e \in V \) let \( T(e) = \sum_{i \in I} \text{res}_{P_i}(Te) \).

\textbf{Theorem:}

1) \( e \rightarrow T(e) \) defines a projective representation of \( V \):

\[
T([e_1, e_2]) = [T(e_1), T(e_2)] + \lambda(e_1, e_2)id
\]

2) \( [T(e), a(m)] = \text{const} \cdot a(eA_m) \)
Representation of the algebra \( C^\infty(\mathcal{P}) \) of observables

\[ f \in C^\infty(\mathcal{P}), \quad X_f \quad \text{— the corresponding Hamiltonian vector field,} \]
\[ \nabla X_f = \partial X_f + T(\rho(X_f)) \quad \text{— the corresponding KZ-type operator.} \]

**Theorem:**
1) \( f \rightarrow \nabla X_f \) is a projective representation of the Poisson algebra of observables;
2) if \( f, g \) depend only on the action variables then
\[ [\nabla X_f, \nabla X_g] = 0. \]
3) \( f \rightarrow \nabla X_f \) is a unitary representation.

**Proof.** 1) follows from
\[ [\nabla X, \nabla Y] = \nabla [X, Y] + \lambda(X, Y) \cdot \text{id}, \quad \forall X, Y. \]
2) by invariance of the complex structure on \( \Sigma_L \) along trajectories (the spectral curve is an integral of motion).
3) by invariance of the volume form \( \omega^p/p! \), \( p = \text{dim} \mathcal{P}/2 \) resp. Hamiltonian flows, and by unitarity of the Sugawara rep.
Partition function of a gauge theory: \( \tilde{F} = \int e^{i\hbar S(\psi)} D\psi \) where 

\( S \) is the Yang–Mills action, \( \psi \) runs over gauge fields.

Seiberg–Witten prepotential \( F \) is a low energy limit of \( \tilde{F} \).

There is a family of pairs \((\Sigma, \lambda)\) where \( \Sigma \) is a Riemann surface, \( \lambda \) is a 1-form on it. Let \( a_i, b_i \) be the periods of \( \lambda \). Then \( F \) is a function of \( \{a_i, b_i\} \) satisfying the following

Seiberg–Witten equations:

\[
b_i = \frac{\partial F}{\partial a_i}.
\]
Prepotential and integrable systems

Observation (Gorski–Krichever–Marshakov–Mironov –Morozov): in all known cases the family of Riemann surfaces is just the family of the spectral curves of an integrable system fibered over the phase space of the latter. As for the 1-forms,

$$\lambda = \nu \cdot \omega_0$$

$\nu$ is the spectrum of $\mathcal{L}$ (function on $\Sigma_L$), $\omega_0$ is fixed in Lecture 2. Then

$$a_i = \oint_{A_i} \nu \cdot \omega_0, \quad b_i = \oint_{B_i} \nu \cdot \omega_0$$

are integrals of motion (depend only on the spectrum of Lax operator). They are dependent, so that

$$\sum_i da_i \wedge db_i = 0.$$ 

Hence the 1-form $\sum_i b_i da_i$ has potential $\mathcal{F}$ obviously satisfying the Seiberg–Witten equations.
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*Current algebras on Riemann surfaces*

M. Schlichenmaier

*Higher genus conformal algebras*
DeGruyter, in process.