Quotients of birational automorphism groups of rationally connected threefolds

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Everything over $\mathbb{C}$.

Let X be an algebraic variety. We denote by $\mathrm{Bir}(X)$ its group of birational automorphisms.

During the last few decades, there have been numerous papers concerning various properties of these groups. A case of particular interest is $X = \mathbb{P}^n$, when the group $\mathrm{Bir}(\mathbb{P}^n)$ is called the Cremona group of rank n.

Most of the results about $\mathrm{Bir}(X)$ deal with the case $X = \mathbb{P}^2$.

Non-simplicity of Cremona groups

Our goal is to get some insights into the structure of Bir(X) for X a rationally connected threefold. One of possible motivations is the following

Question (S. Cantat)

Is there an algebraic variety X such that the group Bir(X) is "very large" and has no proper normal subgroups?

- The non-simplicity of Bir($\mathbb{P}^2_{\mathbb{C}}$) was proved by S. Cantat and S. Lamy using an action of Bir($\mathbb{P}^2$) on some infinite-dimensional hyperbolic space.

- The non-simplicity of Bir($\mathbb{P}^n_{\mathbb{C}}$), $n \geqslant 3$, was proved in 2018 by J. Blanc, S. Lamy and S. Zimmermann.
They used the Sarkisov program and recent achievements in the BAB conjecture.

Non-simplicity of Cremona groups

In fact, they proved the following

Theorem (Blanc-Lamy-Zimmermann, 2018)

Let $B \subseteq \mathbb{P}^m$ be a smooth projective complex variety, $P \longrightarrow \mathbb{P}^m$ a decomposable $\mathbb{P}^2$-bundle (projectivisation of a decomposable rank 3 vector bundle) and $X \subset P$ a smooth closed subvariety such that the projection to $\mathbb{P}^m$ gives a conic bundle $\eta \colon X \longrightarrow B$. Then there exists a group homomorphism

$$\mathrm{Bir}(X) \twoheadrightarrow \bigoplus_{\mathbb{Z}} \mathbb{Z}/2.$$

In particular, applying this to $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$, we get that Cremona groups $\mathrm{Bir}(\mathbb{P}^n)$ are not simple.

Threefolds

Let X be a rationally connected three-dimensional algebraic variety. Run the MMP on X. Recall that we stay in the category of projective normal varieties with at worst terminal $\mathbb{Q}$-factorial singularities. Since X is rationally connected, on the final step we get a Fano-Mori fibration

$$f : \widetilde{X} \longrightarrow Z,$$

which means that $\dim Z < \dim X$, Z is normal, f has connected fibers, $-K_{\widetilde{X}}$ is ample over Z, and $\rho(\widetilde{X}/Z) = 1$. Then we have the following possibilities:

(1) Z is a rational surface and a general fiber of f is a conic;

(2) $Z \cong \mathbb{P}^1$ and a general fiber of f is a smooth del Pezzo surface;

(3) Z is a point and $\widetilde{X}$ is a $\mathbb{Q}$-Fano threefold.

Del Pezzo fibrations

We see that Blanc-Lamy-Zimmermann covers the first case of this trichotomy, so it is natural to proceed with the cases of del Pezzo fibrations and Fano threefolds. Moreover, as the following result shows, we can focus only del Pezzo fibrations of degrees 3, 2 or 1.

Proposition

Let B be a projective curve. A del Pezzo fibration X/B of degree $\geqslant 4$ is birational to a conic bundle.

Proof.

Let F be the generic fiber of X/B. This is a del Pezzo surface of degree $\geqslant 4$ over $\mathbb{C}(B)$. Since B is a curve, $\mathbb{C}(B)$ is a c_1-field. Thus F has a rational point. By Iskovskikh's theorem, F is then rational over $\mathbb{C}(B)$ for $d \geqslant 5$. In particular, it is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, hence to a conic bundle. For $d = 4$ we are done by Alexeev's theorem. □

The strategy

(1) Any birational map between two Mori fiber spaces is a composition of birational maps, called Sarkisov links.

(2) The relations between Sarkisov links are generated by so-called elementary relations.

(3) Replace the group Bir by groupoid BirMori(X) of all birational maps between Mori fiber spaces birational to X. Then (1) and (2) give the presentation of this groupoid.

(4) One constructs a morphism from BirMori(X) and restricts it to Bir(X).

(5) By construction, most links are mapped to 0. Those which are not — links of a very particular **type**. If we want our map to be a well-defined morphism, we should check that it sends elementary relations to 0. But then the **type** should be chosen so that elementary relations involving links of this **type** are "easy".

From now on: X is a del Pezzo fibration of degree 3

We will call our "type" a Bertini type. For a link χ of Bertini type we introduce an integer number $g(\chi)$ which measures the "complexity" of χ. Our main result is then the following one:

Theorem (Blanc-Y.)

There is an integer $d \geq 0$ such that for each del Pezzo fibration X of degree 3, there exists a non-trivial group homomorphism

$$\mathrm{Bir}(X) \longrightarrow *_{[\chi]\in\mathcal{B}} \ \mathbb{Z}/2$$

which sends every Sarkisov link χ of Bertini type with $g(\chi) \geq d$ to the generator indexed by equivalence class of $[\chi]$, and all other Sarkisov links and all automorphisms of Mori fiber spaces birational to X onto zero. The set $\mathcal{B}$ of equivalence classes of Bertini type links is infinite.

Rank r fibrations

The key notion in all this story is a notion of rank r fibrations. It puts together the notions of Mori fiber space, Sarkisov link and elementary relation for $r = 1$, 2 and 3 respectively.

Definition

A morphism $\eta \colon X \longrightarrow B$ is a rank r fibration if the following conditions are satisfied:

(1) X/B is a Mori dream space;

(2) $\dim X > \dim B \geq 0$ and $\rho(X/B) = r$;

(3) X is $\mathbb{Q}$-factorial and terminal, and for any divisor D on X, the output of any D-MMP from X over B is still $\mathbb{Q}$-factorial and terminal;

(4) B has klt singularities.

(5) $-K_X$ is η-big.

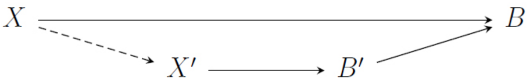
Rank r fibrations

A useful fact for the future:

Lemma

If $\eta : X \longrightarrow B$ is a rank r fibration, then for a general point $b \in B$ the fiber $X_b = \eta^{-1}(b)$ is pseudo-isomorphic to a weak Fano variety.

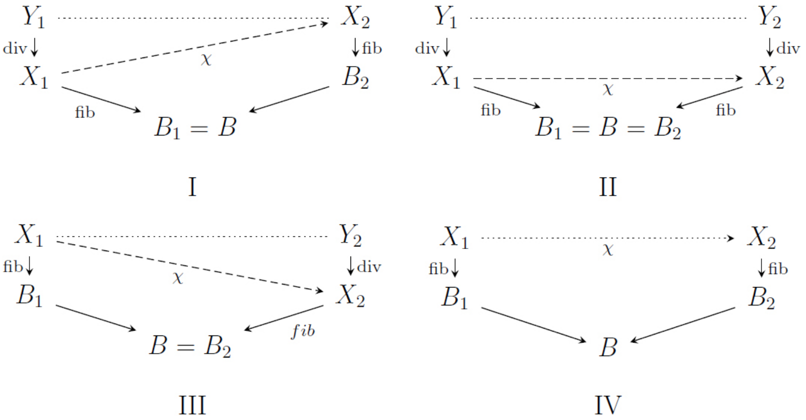
We say that a rank r fibration X/B factorises through a rank r' fibration X'/B', or that X'/B' is dominated by X/B, if the fibrations X/B and X'/B' fit in a commutative diagram



where $X \dashrightarrow X'$ is a birational contraction, and $B' \longrightarrow B$ is a morphism with connected fibres.

Rank 2 fibrations: Sarkisov links

Rank 1 fibrations are just terminal Mori fiber spaces. The notion of a rank 2 fibration recovers the notion of Sarkisov link. Indeed, it is easy to show that a rank 2 fibration Y/B factorises through exactly two rank 1 fibrations X_1/B_1 and X_2/B_2. We have 4 types of the corresponding birational map $\chi : X_1 \dashrightarrow X_2$.



$$\begin{array}{ccc} Y_1 & \overset{\beta}{\cdots\cdots\cdots\cdots} & Y_2 \\ \eta_1\downarrow & & \downarrow\eta_2 \\ X_1 & \overset{\chi}{\dashrightarrow} & X_2 \\ & \searrow \qquad \swarrow & \\ & B = \mathbb{P}^1 & \end{array}$$

Definition

We say that the Sarkisov link $\chi : X_1 \dashrightarrow X_2$ is a Bertini link if

(1) χ has type II with $B = \mathbb{P}^1$;

(2) η_1 is a blow-up of an irreducible curve $\Gamma_1 \subset X_1$ of the fibration X_1/B, so that the generic fiber of Y_1/B is a del Pezzo surface of degree 1;

We call $g(\chi) := p_a(\Gamma_1)$ a genus of χ.

Bertini links

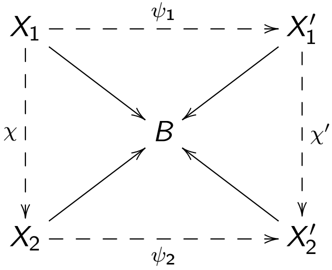
$$
\begin{array}{ccccc}
Y_1 & & \overset{\beta}{\cdots\cdots\cdots\cdots} & & Y_2 \\
\eta_1 \downarrow & & & & \downarrow \eta_2 \\
X_1 & & \overset{\chi}{\dashrightarrow} & & X_2 \\
 & \underset{dP_n}{\searrow} & & \underset{dP_n}{\swarrow} & \\
 & & B = \mathbb{P}^1 & &
\end{array}
$$

Remark

The definition implies that the birational map between the generic fibres of X_1/B and X_2/B induced by χ is equal, up to an isomorphism of the target, to a birational involution that lifts to the Bertini involution on the generic fibre of Y_1/B, and implies that X_1/B and X_2/B are del Pezzo fibrations of the same degree $d \geqslant 2$. This motivates the terminology.

Bertini links

We say that two Bertini links $\chi : X_1 \dashrightarrow X_2$ and $\chi' : X_1' \dashrightarrow X_2'$ of are equivalent, if there exists a commutative diagram



with ψ_1 and ψ_2 inducing isomorphisms of generic fibers of the corresponding del Pezzo fibrations. The equivalence class of χ will be denoted $[\chi]$.

Rank 3 fibrations: elementary relations

The notion of rank 3 fibration recovers the notion of an elementary relation between Sarkisov links.

Proposition

Let T/B be a rank 3 fibration. Then there are only finitely many Sarkisov links χ_i dominated by T/B, and they fit in a relation

$$\chi_t \circ \cdots \circ \chi_1 = \mathrm{id}\,.$$

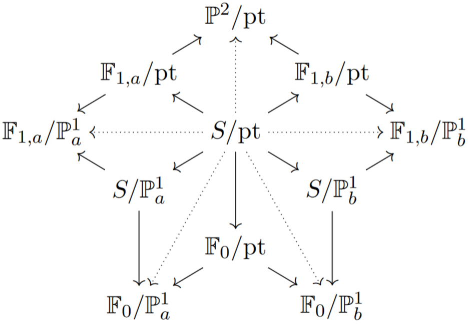
We say that

$$\chi_t \circ \cdots \circ \chi_1 = \mathrm{id}$$

is an *elementary relation* between Sarkisov links, coming from the rank 3 fibration T/B.

Example

Let S be the surface obtained by blowing-up $\mathbb{P}^2$ in two distinct points a, b. Denote by $\mathbb{F}_{1,a}/\mathbb{P}^1_a$, $\mathbb{F}_{1,b}/\mathbb{P}^1_b$ the two intermediate Hirzebruch surfaces with their fibrations to $\mathbb{P}^1$. Finally, denote by $\mathbb{F}_0$ the surface obtained by contracting the strict transform on S of the line through a and b.



The big loop here is an elementary relation coming from S/pt.

Crucial theorem

Now we are ready to state a keynote result, which allows us to study the structure of birational automorphism groups. Its first part is basically the Sarkisov program, following C. Hacon and J. McKernan. The second part is inspired by results of A.-S. Kaloghiros.

Let X/B a Mori fibre space. We denote by $\mathrm{BirMori}(X)$ the Mori groupoid of birational maps between Mori fibre spaces birational to X. Note that $\mathrm{Bir}(X)$ is a subgroupoid of $\mathrm{BirMori}(X)$.

Theorem (Blanc-Lamy-Zimmermann)

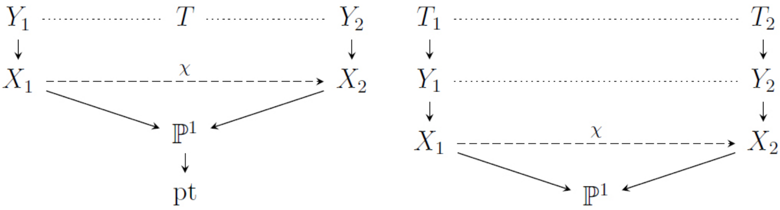
Let X/B be a terminal Mori fibre space.

(1) *The groupoid* $\mathrm{BirMori}(X)$ *is generated by Sarkisov links and automorphisms.*

(2) *Any relation between Sarkisov links in* $\mathrm{BirMori}(X)$ *is generated by elementary relations.*

Elementary relations involving Bertini links

Let $\chi_t \circ \cdots \circ \chi_1 = \text{id}$ be an elementary relation between Sarkisov links, coming from a rank 3 fibration T/B. Assume that one of χ_i has type II. Depending on whether $B = pt$ or $B = \mathbb{P}^1$, there are only two possible ways in which T/B can factorize through χ_i:



Over a point

Over a curve

Relations over a point

Lemma (whose proof relies on BAB)

Let X be a terminal threefold, and $\eta : Y \longrightarrow X$ be the blow-up of a reduced not necessarily irreducible curve $\Gamma \subset X$. Assume that both X and Y are pseudo-isomorphic to weak Fano terminal threefolds having a small anticanonical morphism. Then there exists a positive integer N (independent of X and Γ) such that $p_a(\Gamma) < N$.

Proposition

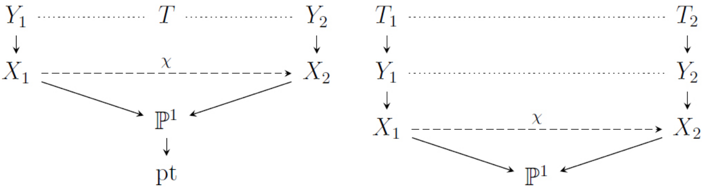
There exists some positive integer r such that no Bertini link χ with $g(\chi) > r$ occurs in a non-trivial elementary relation over a point.

Proof.

Note that Y_1/pt and X_1/pt are rank 3 and rank 2 fibrations respectively. Moreover, they both pseudo-isomorphic to weak Fano terminal varieties. It remains to apply Lemma and take $r = N$. □

Relations over a curve

Now consider elementary relations over a curve.



Over a point   Over a curve

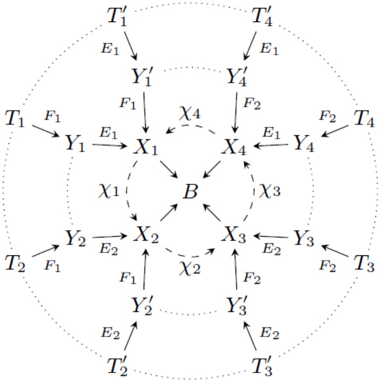
Note that the general fiber of $T_1 \longrightarrow B$ is isomorphic to a weak del Pezzo surface. But the general fiber of $T_1 \longrightarrow B$ is already a del Pezzo surface of degree 1! It means that $T_1 \longrightarrow Y_1$ is the blow-up of a curve in a special fiber.

Proposition

Let $\chi_1 : X_1/B \dashrightarrow X_2/B$ be a Bertini link appearing in some non-trivial elementary relation. Then this relation has the form

$$\chi_4 \circ \chi_3 \circ \chi_2 \circ \chi_1 = \mathrm{id},$$

where χ_1 and χ_3 are Bertini links, equivalent via χ_2 and χ_4.



Proof of the main result.

Let X be a del Pezzo fibration of degree 3. Fix $r \in \mathbb{Z}_{>0}$ from the Proposition about links/pt. Consider the map

$$\Phi : \mathrm{BirMori}(X) \longrightarrow \mathop{\ast}_{[\chi] \in \mathcal{B}} \mathbb{Z}/2$$

which sends each Sarkisov link χ of Bertini type with $g(\chi) > r$ to the generator indexed by $[\chi]$, and all other Sarkisov links and automorphisms of Mori fibre spaces are sent to zero. To check that Φ is a well-defined groupoid morphism, we need to show that every elementary relation is sent to the neutral element.

Let $\chi_n \circ \cdots \circ \chi_1 = \mathrm{id}$ be a non-trivial elementary relation between the Sarkisov links. We may assume that it involves a Bertini link χ_i with $g(\chi_i) > r$. Then our relation is a relation over a curve with $n = 4$. Moreover, χ_1 and χ_3 are equivalent links of Bertini type, while χ_2 and χ_4 are not of Bertini type. Thus, our elementary relation is sent to the neutral element. □