# Differential Geometry

M. Kazarian

Classical differential geometry is often considered as an "art of manipulating with indices". In these lectures we develop a more geometric approach by explaining the true mathematical meaning of all introduced notions. Where possible, we try to avoid coordinates totally. But the correspondence to the traditional coordinate presentation is also explained. The usage of invariant language not only simplifies many arguments but also reduces the amount of computations in particular problems. The best example is a simple formula for the Gaussian curvature of a surface based on the concept of connection 1-form.

## Program of Differential Geometry course read in the Fall term 2005 of MIM

- **Plane and space curves.** *Length of a curve. Curvature. Curvature circle. Torsion. Frenet formulas.*
- **Surfaces in three-space.** *Riemannian structure. The* II*nd quadratic form. Principal curvatures. Gaussian curvature. Gaussian map.*
- **Curvature of a metric on the plane.** *Theorema egregium. Derivation lemma. Geometry of the sphere and the pseudosphere.*
- **Gauss-Bonnet formula.** *Connection and curvature forms. Parallel translate of vectors on a surface. Local Gauss-Bonnet formula. Global formula.*
- **Topological connection.** *Fibrations. Trivializations. Connection as an infinitesimal parallel translate.*
- **Vector bundles.** *Tangent and cotangent bundles. Tensor bundles. Sections. Index formalism. Raising and lowing indices. Convolution.*
- **Connection as a covariant derivative.** *Connection in a vector bundle. Parallel translate. Connection matrix. Curvature tensor. Cartan structure equation.*
- **Riemannian manifolds.** *Levi-Cevita connection. Riemann curvature tensor. Ricci tensor and the scalar curvature. Symmetries of the Riemann tensor.*
- **Geodesics.** *Variational interpretation. Exponential map. Normal coordinates. Conjugate points. Global geometry of Riemannian manifolds.*

# 1 Plane and space curves

Differential geometry of space curves $r : \mathbb{R} \to \mathbb{R}^n$ is the study of their invariants with respect to movements of the space $\mathbb{R}^n$ equipped with the standard Euclidean structure, $|x|^2 = \sum x_i^2$. The *length* of an arc $r(t) = (x_1(t), \dots, x_n(t))$, $a < t < b$, is given by the formula

$$l(a,b) = \int_a^b |r'| dt = \int_a^b \Big(\sum (x_i')^2\Big)^{1/2} dt.$$

The *arc length element* $dl = |r'|\, dt$ is invariant under reparametrizations of the curve provided the orientation of the curve is preserved. The value $\tau = l(a,t)$ can be taken as a parameter on the curve at the point $r(t)$. This parametrization is said to be *natural.* For the natural parameter $\tau$, one has $dl = d\tau$, that is

$$|\dot{r}| = 1$$

(the dot denotes the derivative with respect to the natural parameter). Differentiating the equality $(\dot{r}, \dot{r}) = 1$, one gets $(\dot{r}, \ddot{r}) = 0$, i.e., $\dot{r} \perp \ddot{r}$. The *curvature* is the rotation velocity of the tangent, $k = |\ddot{r}|$; the inverse value $R = k^{-1}$ is the *radius of curvature.*

**Problem 1.1.** Find the radius of curvature of a circle of radius $R$ on the plane $\mathbb{R}^2$.

**Problem 1.2.** Find expression for the curvature in arbitrary parametrization.

Generically, a circle tangent to a plane curve has the first order of tangency with it. If its radius equals the curvature radius, then the order of tangency is higher (two). Such a circle is called the *curvature circle.* The locus of centers of curvature circles is called the *evolute* of the curve. In geometric optic, it is also called the *caustic*, or the *focal set.*

Generically, the focal set admits singularities of semicubic type. Singularity points of the caustic correspond to the extremes of curvature (local maxima and minima). The curvature circle at an extremum point of the curvature has even higher order (three) of tangency with the curve.

Another description of the focal set is provided by the *generating family of functions.* Consider a plane curve $r$ and the distance square function $S(t) = |r(t) - q|^2$ on it, where $q$ is a given point of the plane $\mathbb{R}^2$.

**Problem 1.3.** (a) Show that $q$ belongs to the normal of the curve at the point $r(t) \Leftrightarrow S'(t) = 0$;

(b) $q$ is the curvature center $\Leftrightarrow S'(t) = S''(t) = 0$;

(c) $r(t)$ is an extremum of curvature $\Leftrightarrow S'(t) = S''(t) = S'''(t) = 0$.

**Problem 1.4.** Compute and draw the focal set of (a) parabola (b) ellipse.

A point of a space curve $r : \mathbb{R} \to \mathbb{R}^n$ is said to be *non-flattening* if the vectors $r', r'', \dots, r^{(n)}$ are linearly independent at this point (this condition is independent of the parametrization). A generic curve has only finitely many flattening points. At a non-flattening point, an *accompanying flag* formed by the *osculating planes* is well defined. The $k$th osculating plane is spanned by the vectors $r', r'', \dots, r^{(k)}$. Geometrically, the osculating plane has the highest order of tangency with the curve among all tangent planes of the same dimension.

The family of the orthonormal *Frenet frames* $(e_1, \dots, e_n)$ is obtained from the vectors $r', \dots, r^{(n)}$ by orthogonalization. The first $k$ vectors of the Frenet frame span the $k$th osculating plane, and this condition determines the vectors of the frame uniquelly up to a sign. For a curve in $\mathbb{R}^3$, these vectors are $e_1 = v = \dot r$, the *unit tangent vector*, $e_2 = n = \dot v/|v|$, the *normal*, and $e_3 = b = v \times n$, the *binormal*, respectively.

Consider the expansion of the derivatives $e_i'$ in terms of the same basis, $e_i' = \sum a_{ij} e_j$. The matrix $a_{ij}$ is *skew-symmetric*, since it is a derivative of an orthogonal matrix (more explicitly, $0 = \frac{d}{dt}(e_i, e_j) = (e_i', e_j) + (e_i, e_j') = a_{ij} + a_{ji}$). On the other hand, all possible non-zero off-diagonal elements of this matrix are on the two diagonals next to the principal one only, since the derivative $e_k'$ lies in the $(k+1)$st osculating plane. This implies the following *Frenet equations*:

$$e_i' = -k_{i-1} e_{i-1} + k_i e_{i+1}, \qquad i = 1, \dots, n, \quad (k_0 = k_{n+1} = 0, \quad e_0 = e_{n+1} = 0).$$

The functions $k_i$ computed in the natural parametrization are called the *higher curvatures*. In dimension $n = 3$, we have

$$\dot v = k\, n, \quad \dot n = -k\, v + \varkappa\, b, \quad \dot b = -\varkappa\, n.$$

These formulas show that in the first approximation the osculating 2-plane of the curve (having the normal $b$) rotates around the tangent line. The angular velocity $k_2 = \varkappa$ of this rotation is called the *torsion*.

It follows from the uniqueness theorem for solutions of ODE that an arbitrary collection of non-zero functions $k_1(\tau), \dots, k_n(\tau)$ gives rise to a curve determined up to a movement (given by the initial position of the Frenet frame). In other words, the higher curvatures form a *complete system of invariants* of a spatial curve.

**Problem 1.5.** Find the curvature and the torsion of the following curves:
(a) $(a\cos t, a\sin t, b\,t)$;
(b) $e^t\,(\cos t, \sin t, 1)$;
(c) $(t^3 + t, t^3 - t, \sqrt{3}\,t^2)$;
(d) $3x^2 + 15y^2 = 1$, $z = xy$.

**Problem 1.6.** Describe all curves with the constant curvatures $k$ and $\varkappa$.

**Problem 1.7.** Describe all curves with (a) $k \equiv 0$, (b) $\varkappa \equiv 0$.

**Problem 1.8.** A curve lies on a sphere and has a constant curvature. Prove that this curve is a circle.

## 2 Surfaces

One of the main objects of differential geometry is the Riemannian structure. A *Riemannian structure*, or a *metric* on a smooth manifold $M$ is a family of positively definite quadratic forms in the tangent spaces depending smoothly on the point of the manifold. In local coordinates $x^1, \dots, x^n$, the metric is given by a symmetric square matrix $g = \|g_{ij}\|$ so that its value on the tangent vector $\xi = \xi^1 \partial_{x^1} + \dots \xi^n \partial_{x^n}$ is given by $g(\xi) = g_{ij}\xi^i\xi^j$ (following tradition, we omit the summation sign for repeating indices).

It is convenient to represent the metric in the form $g_{ij}dx^i dx^j$ where the differentials $dx^i$ are considered as auxiliary independents variables. In this form there is no need to remember the transformation rule for a change of coordinates: it is sufficient to replace $dx^k$ by the differentials of the corresponding functions in the new coordinates:

$$g_{kl}dx^k dx^l = g_{kl}\left(\frac{\partial x^k}{\partial y^i}dy^i\right)\left(\frac{\partial x^l}{\partial y^j}dy^j\right) = g'_{ij}dy_i dy_j \ , \quad \text{i.e.,} \ g'_{ij} = g_{kl}\frac{\partial x_k}{\partial y_i}\frac{\partial x_l}{\partial y_j} \ .$$

Submanifolds of Riemannian manifolds inherit the Riemannian structure. The simplest example is provided by a surface in the standard Euclidean space $\mathbb{R}^3$. The induced structure is called also the *first quadratic form.* If the surface is given parametrically as $r(u,v) \in \mathbb{R}^3$, then the coefficients $A = g_{11}$, $B = g_{12} = g_{21}$, $C = g_{22}$ of the metric $g = A\,du^2 + 2B\,du\,dv + C\,dv^2$ can be computed as the scalar products of the basic tangent vectors

$$g_{ij} = (r_i, r_j), \quad r_1 = \frac{\partial r}{\partial u}, \quad r_2 = \frac{\partial r}{\partial v}.$$

Alternatively, the expression for the metric can be obtained from the Euclidean metric $dx^2 + dy^2 + dz^2$ by substituting the differentials of the coordinate functions $x(u,v), y(u,v), z(u,v)$ of the curve $r = (x,y,z)$ in terms of the parameters $u, v$ on the surface, $dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$, and so on.

**Problem 2.1.** Find the expression for the Euclidean metric $dx^2 + dy^2$ on the plane in polar coordinates.

**Problem 2.2.** Find the expression for the metric on the unit sphere $x^2 + y^2 + z^2 = 1$

(a) in spherical coordinates;

(b) in polar coordinates on the plane $z = 0$ after the stereographic projection from the point $(0,0,1)$;

(c) in Euclidean coordinates on the same plane.

The form $dx^2 + dy^2 - dz^2$ in $\mathbb{R}^3$ is not a metric since it is not positively definite. However, its restriction to the upper sheet $z > 0$ of the hyperboloid $z^2 - x^2 - y^2 = 1$ of two sheets is positively definite. This surface with the induced metric is called the *Lobachevski plane* or the *pseudosphere.* The stereographic projection from the point $(0,0,-1)$ maps the pseudosphere isomorphically to the disc $x^2 + y^2 \leqslant 1$ of the plane $z = 0$ (the Poincaré model for the Lobachevski plane).

**Problem 2.3.** Find the expression for the metric on the pseudosphere

(a) in pseudospherical coordinates

$$\begin{cases} x = \cos\varphi \sinh\psi \\ y = \sin\varphi \sinh\psi \qquad 0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant \psi < \infty \\ z = \cosh\psi \end{cases}$$

(b) in the Euclidean coordinates $(x,y)$ on the plane $z = 0$ after the stereographic projection.

If the first quadratic form is already determined, the embedding of the surface to $\mathbb{R}^3$ can be neglected. Properties of the surface determined by the Riemannian structure are called *intrinsic.* It is intuitively clear that bending a sheet of a paper we do not change its metric properties. The measurements of lengths, angles, areas are instances of intrinsic geometry of a surface. The *area element* $\sigma$ is determined by the condition that the area of a square with orthogonal unit edges is equal to 1. In coordinates, $\sigma = \pm\sqrt{|g|}\,du \wedge dv$, where $|g| = \det \|g_{ij}\|$. The sign $\pm$ is fixed by a choice of the *orientation* of the surface. Thus, the length of a curve $\gamma$ and the area of the domain $D$ are computed by the formulas

$$L = \int_\gamma |\dot r|\,dt = \int_\gamma \sqrt{A\dot u^2 + 2B\dot u\dot v + C\dot v^2}\,dt\,, \qquad S = \int_D \sigma = \int_D \sqrt{|g|}\,du \wedge dv\,.$$

As an example of an extrinsic invariant, we may consider the *second quadratic form* $h$, which measures the quadratic deviation of the surface from its tangent plane. Fix the unit normal vector $n$ at a point $r(v_0) \in M$ of the surface. Consider the orthogonal projection to $n$ as a smooth function $f(v) = (r(v), n)$ on $M$. The first differential of this function vanishes at $v_0$. Set $h(v_0) = d^2 f(v_0)$, the second differential of $f$ at the considered point. The value of the bilinear symmetric form $h$ on the pair $\xi, \eta$ of tangent vectors is given by the mixed partial derivative $h(\xi,\eta) = \partial_\xi\partial_\eta f(v_0) = (\partial_\xi\partial_\eta, n)$. If we choose the Euclidean coordinates $Oxyz$ in such a way that the plane $Oxy$ is tangent to the surface at the point under consideration, then the surface is represented as the graph of a function $z = f(x,y)$ and the second quadratic form is given by

$$h = f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}\,dy^2.$$

For an arbitrary parametrization of the surface, the coefficients $h_{ij}$ of the second quadratic form form in local coordinates $v_1, v_2$ are given by

$$h_{ij} = \Big(\frac{\partial^2 r}{\partial v_1 \partial v_2}, n\Big) = (\partial_{v_j} r_i, n),$$

where $r_i = \partial_{u_i} r$ is the corresponding basic tangent vector to $M$. Differentiating the identity $(r_j, n) = 0$, we observe that the coefficients of $h$ can be obtained alternatively by

$$h_{ij} = -\Big(r_i, \frac{\partial n}{\partial v_j}\Big).$$

Thus the tangent space $T_{v_0}M$ is equipped with the Euclidean structure $g$ and a supplementary quadratic form $h$. It is known from linear algebra that the eigenvalues of the quadratic form are real and the eigenvectors corresponding to distinct eignvalues are orthogonal (by the eigenvalues we mean the roots of the equation $\det\|h_{ij} - \lambda\, g_{ij}\| = 0$).

**Definition.** By the *principal curvatures* $k_1, k_2$ of the surface at a given point we mean the eigenvalues of $h$ with respect to $g$, and by the *principal directions* we mean its eigenspaces. The *Gaussian curvature* is the product $K = k_1 k_2 = \det\|h_{ij}\|/\det\|g_{ij}\|$ of principal curvatures and the *mean curvature* is $\frac{1}{2}(k_1 + k_2)$.

The surface is said to be *elliptic* (locally convex), if $K > 0$ (the principal curvatures have the same sign), and *hyperbolic*, if $K < 0$ (the signs of $k_1$ and $k_2$ are opposite). The elliptic and hyperbolic domains are separated, on a generic surface, by the *parabolic line* where one of the two principal curvatures vanishes. A famous '*theorema egregium*' of Gauss claims that in contrast to the mean curvature, the Gaussian curvature is an intrinsic invariant of the surface and is completely determined by the first quadratic form. It will be proved in the next section.

**Problem 2.4.** Compute the first, the second quadratic forms, and the Gaussian curvature of a surface given as the graph $z = f(x,y)$ of a smooth function.

**Problem 2.5.** (Euler's formula.) Let $l$ be a tangent line to a surface at a point $x$ and assume that $l$ forms the angle $\alpha$ with one of the principal directions. Compute the curvature of the curve cut out by the plane $\lambda$ passing through $l$, if:

(a) $\lambda$ contains the normal to the surface;

(b) $\lambda$ forms the angle $\beta$ with the surface.

Yet another interpretation of the second quadratic form is provided by the Gauss map. The *Gauss map* $G : M \to S^2$ takes a point $x$ on the surface to the normal unit vector $n_x$ translated to the origin. The tangent planes $T_xM$ and $T_{n_x}S^2$ are parallel (both are orthogonal to $n_x$) so they can be identified in a natural way. Therefore we may consider the derivative $G_*$ of the Gauss map as a linear transformation of the tangent plane $T_xM$.

**Problem 2.6.** Prove that $G_*$ is a self-conjugate operator corresponding to the quadratic form $-h$. In other words, the following equality holds: $h(\xi, \eta) = -(G_*(\xi), \eta)$. In particular, the eigenvalues of $G_*$ are $-k_1, -k_2$ and the Gaussian curvature $K$ is equal to the Jacobian $\det G_*$ of the Gauss map.

## 3 Gaussian curvature

Below, we provide an explicit formulas for the Gaussian curvature of a surface in terms of the Riemannian metric. From the first glance, the procedure looks very formal. The mathematical meaning of the presented formulas will be clarified later.

Consider the quadratic form $g$ and represent it as a sum of squares

$$g = u_1^2 + u_2^2$$

for some differential 1-forms $u_1$ and $u_2$. Locally, such reduction is always possible. It is equivalent to a choice of an orthonormal frame in the (co)tangent plane depending smoothly on the point of the surface. In particular, $\sigma = u_1 \wedge u_2$ is the *area form* on $M$. There is no reason for $u_1$ and $u_2$ to be differentials of some functions (even if the metric is Euclidean) so that the differentials $du_1$ and $du_2$ could be non-vanishing. Define functions $\alpha_1$ and $\alpha_2$ by the relations

$$du_1 = \alpha_1\,\sigma, \qquad du_2 = \alpha_2\,\sigma$$

and set

$$\theta = \alpha_1\,u_1 + \alpha_2\,u_2.$$

Differentiating again we find

$$d\theta = -K\,\sigma$$

for some function $K$. This function is called the *curvature* of the metric $g$. The form $-d\theta = K\,\sigma$ is called the *curvature form.*

**Theorem.** 1. *The curvature $K$ is uniquely determined by the metric and independent of the choice of the forms $u_1$ and $u_2$.*

2. *The metric is Euclidean (reduces to $dx^2 + dy^2$ in some local coordinates) if and only if $K \equiv 0$.*

3. *If the metric is the first quadratic form of a surface in $\mathbb{R}^3$, then $K$ coincides with the Gaussian curvature.*

*Proof.* 1. Any other orthonormal frame $u_1'$, $u_2'$ can be obtained from $u_1$, $u_2$ by rotating by some angle $\psi$ depending on a point of $M$,

$$u_1' = \cos\psi\, u_1 + \sin\psi\, u_2, \qquad u_2' = \sin\psi\, u_1 - \cos\psi\, u_2$$

(one immediately verifies the equalities $u_1'^2 + u_2'^2 = u_1^2 + u_2^2$ and $u_1' \wedge u_2' = u_1 \wedge u_2$). Set $d\psi = \beta_1\, u_1 + \beta_2\, u_2$. Then following the instruction above, we find

$$du_1' = (d\cos\psi) \wedge u_1 + \cos\psi\, du_1 + (d\sin\psi) \wedge u_2 + \sin\psi\, du_2,$$

that is,

$$\alpha_1' = \beta_2\, \sin\psi + \alpha_1\, \cos\psi + \beta_1\, \cos\psi + \alpha_2\, \sin\psi = (\alpha_1 + \beta_1)\, \cos\psi + (\alpha_2 + \beta_2)\, \sin\psi,$$

and similarly,

$$\alpha_2' = (\alpha_1 + \beta_1)\, \sin\psi - (\alpha_2 + \beta_2)\, \cos\psi.$$

Therefore,

$$\theta' = \alpha_1'\, u_1' + \alpha_2'\, u_2' = (\alpha_1 + \beta_1)\, u_1 + (\alpha_2 + \beta_2)\, u_2 = \theta + d\psi.$$

Differentiating we get $d\theta = d\theta'$, that is, $K' = K$.

2. If $K = 0$, then $d\theta = 0$. Therefore, locally $\theta$ is a differential of some function. Denote this function by $-\psi$, that is, $\theta + d\psi = 0$. If we rotate the frame $u_1, u_2$ by the angle $\psi$ then by the formula above we find that for the new frame $u_1', u_2'$ the form $\theta'$ vanishes that is $du_1' = du_2' = 0$. It follows that $u_1'$ and $u_2'$ are differentials of some functions, say, $x$ and $y$, respectively. In the coordinates $x, y$, the metric takes the required form $dx^2 + dy^2$.

3. For a tangent vector field $\xi$ of $M$ and any (not necessarily tangent) field $v$ on $M$ we denote by $\partial_\xi v$ the component-wise directional derivative, where we consider the vector $v \in \mathbb{R}^3$ as a column of three components. We are interested in the expansion of this derivative in terms of the orthonormal basis $e_1, e_2, e_3$, where $e_1$ and $e_2$ are the tangent vectors dual to the basis $u_1$, $u_2$, and $e_3 = n = e_1 \times e_2$ is the unit normal vector to the surface.

**Derivation Lemma.** *The derivatives $\partial_\xi e_i$ satisfy the equations*

$$\begin{cases} \partial_\xi e_1 = & & \theta(\xi)\, e_2 & +h_1(\xi)\, n \\ \partial_\xi e_2 = & -\theta(\xi)\, e_1 & & +h_2(\xi)\, n \\ \partial_\xi n = & -h_1(\xi)\, e_1 & -h_2(\xi)\, e_2 & \end{cases}$$

*where the differential 1-forms $h_i$ are determined by the second quadratic form, $h_i(e_j) = h_{ij} = h(e_i, e_j)$, and $\theta$ is the form defined at the beginning of this section.*

The fact that the matrix of the expansion of the fields $\partial_\xi e_i$ over this basis is skew-symmetric follows from the fact that it is a derivative of an orthogonal matrix: $0 = \partial_\xi\,(e_i, e_j) = (\partial_\xi e_i, e_j) + (e_i, \partial_\xi e_j)$. The entries of this matrix depend linearly on $\xi$, hence so it is clear that it should have the form of the lemma with certain 1-forms $h_1$, $h_2$, $\theta$. It remains to identify these forms.

Remark that the assertion concerning the forms $h_1$ and $h_2$ responsible for the normal component of the derivatives $\partial_\xi e_1$ and $\partial_\xi e_2$ is a reformulation of the definition of the second quadratic form, see the previous section. Concerning the form $\theta$, we have $(\partial_{e_2} e_1, e_1) = 0$ so that

$$\theta(e_1) = -(\partial_{e_1} e_2, e_1) = -(\partial_{e_1} e_2 - \partial_{e_2} e_1, e_1) = -([e_1, e_2], e_1),$$

where $[\cdot, \cdot]$ is the commutator of vector fields. The key observation is that this commutator is completely determined by the restriction of these fields to $M$ and does not depend on the embedding of $M$ into $\mathbb{R}^3$. It remains to notice that $-([e_1, e_2], e_1)$ coincides with the coefficient $\alpha_1$ defined at the beginning of the section. Indeed, by definition of the exterior differential (see Sect. 9), we have

$$\alpha_1 = du_1(e_1, e_2) = \partial_{e_1} u_1(e_2) - \partial_{e_2} u_1(e_1) - u_1([e_1, e_2]) = -u_1([e_1, e_2]) = -([e_1, e_2], e_1).$$

Similarly, we get $\theta(e_2) = \alpha_2$. The lemma is proved.

The lemma is applied as follows. Consider the identity $\partial_{[e_1,e_2]} = \partial_{e_1}\partial_{e_2} - \partial_{e_2}\partial_{e_1}$. Let us apply both sides to $e_1$ and compute the coefficient of $e_2$ in the expansion. For the left-hand side we have

$$(\partial_{[e_1,e_2]} e_1, e_2) = \theta([e_1, e_2]).$$

Now, we compute the right-hand side. Differentiating the equality $\partial_{e_2} e_1 = \theta(e_2)\, e_2 + h_1(e_2)\, n$ we get

$$(\partial_{e_1}\partial_{e_2}\, e_1, e_2) = \partial_{e_1}\theta(e_2) + h_1(e_2)\, (\partial_{e_1} n, e_2) = \partial_{e_1}\theta(e_2) - h_{12}h_{21}.$$

Similarly, we find

$$(\partial_{e_2}\partial_{e_1}\, e_1, e_2) = \partial_{e_2}\theta(e_1) - h_{22}h_{11}.$$

Thus, we proved the equality

$$\theta([e_1, e_2]) = \partial_{e_1}\theta(e_2) - \partial_{e_2}\theta(e_1) + h_{22}h_{11} - h_{12}h_{21}.$$

By the definition of the exterior differential, this is equivalent to

$$-d\theta(e_1, e_2) = \det \|h_{ij}\|.$$

The left-hand side of this equality is the definition of the curvature $K$ in the beginning of the section, while the right-hand side is the definition of the Gaussian curvature. The theorem is proved.

**Example.** Let us compute the Gaussian curvature of the metric $g = dx^2 + 2\cos\omega\, dx\, dy + dy^2$, where $\omega = \omega(x, y)$ is some function. Represent this metric in the form

$$g = (dx + \cos\omega\, dy)^2 + (\sin\omega\, dy)^2.$$

We may set $u_1 = dx + \cos\omega\, dy$, $u_2 = \sin\omega\, dy$ and get

$$\sigma = u_1 \wedge u_2 = \sin\omega\, dx \wedge dy.$$

Differentiating the basic forms we get

$$du_1 = -\sin\omega\, \omega_x\, dx \wedge dy, \quad du_2 = \cos\omega\, \omega_x\, dx \wedge dy.$$

Therefore,

$$\alpha_1 = -\omega_x, \quad \alpha_2 = \cot\omega, \quad \theta = \alpha_1\, u_1 + \alpha_2\, u_2 = -\omega_x\, dx.$$

Differentiating, we get

$$d\theta = \omega_{xy}\, dx \wedge dy, \qquad K = -\frac{\omega_{xy}}{\sin\omega}.$$

The metric is flat ($K = 0$) if $\omega_{xy} = 0$. Let us determine the Euclidean coordinates for the case $\omega = x + y$. In this case the form $\theta$ is exact, $\theta = -d\psi$ with $\psi(x, y) = x$. Rotating the frame $u_1, u_2$ by the angle $\psi$ we get

$$\begin{aligned} u_1' &= \cos\psi\, u_1 + \sin\psi\, u_2 = \cos x\, dx + (\cos x \cos(x+y) + \sin x \sin(x+y))\, dy \\ &= \cos x\, dx + \cos y\, dy = d(\sin x + \sin y), \\ u_2' &= \sin\psi\, u_1 - \cos\psi\, u_2 = \sin x\, dx + (\sin x \cos(x+y) - \cos x \sin(x+y))\, dy \\ &= \sin x\, dx - \sin y\, dy = d(-\cos x + \cos y) \end{aligned}$$

The desired Euclidean coordinates are $X = \sin x + \sin y$ and $Y = -\cos x + \cos y$.

**Problem 3.1.** Determine the Gaussian curvature for the metrics below. If $K = 0$, determine also the Euclidean coordinates.

(a) $dx^2 + \sin^2 x\, dy^2$;
(b) $dx^2 + \sinh^2 x\, dy^2$;
(c) $dx^2 + x^2\, dy^2$;
(d) $\dfrac{4\,(dx^2 + dy^2)}{(1 + k(x^2 + y^2))^2}$;
(e) conformal metric of the form $g(x, y)\,(dx^2 + dy^2)$, where $g(x, y) > 0$ is a function;
(f) $A^2\, dx^2 + B^2\, dy^2$, where $A = A(x, y) > 0$ and $B = B(x, y) > 0$ are some functions.

## 4 Connection as a parallel translate

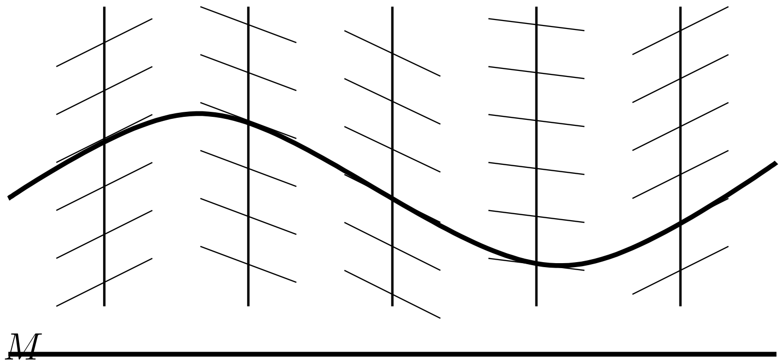
A map $\pi : W \to M$ is called a *locally trivial fibration*, or a *fiber bundle*, if every point $x \in M$ has a neighborhood $U$ admitting a homeomorphism $\varphi : \pi^{-1}U \simeq U \times F$ such that the map $\pi$ corresponds to the projection to the second factor under this isomorphism. Such an isomorphism is called a *trivialization*. The space $M$ is called the *base* of the bundle, $E$ is the *total space*, and $F \simeq \pi^{-1}(x)$ (which is independent of $x$ up to a homeomorphism) is the *fiber*. We shall always assume that all spaces $M$, $W$, and $F$ under consideration are smooth manifolds and the maps are infinitely smooth.

**Example.** Consider the 3-dimensional manifold $W$ formed by all vectors of length 1 tangent to a given surface $M$. This manifold is the total space of a fibration over $M$ with the fiber $S^1$. If $M$ is a connected compact oriented surface different from the torus, then this fibration is not trivial (that is, not isomorphic to the fibration of the form $M \times S^1 \to M$). Indeed, the triviality of the fibration would imply the existence of a non-vanishing vector field on $M$, but any vector field must have singular points whose number counted with appropriate signs is equal to $\chi(M) \neq 0$.

All fibers of a locally trivial fibration are isomorphic but the isomorphism is *not* canonical. Even for close points $x, y$ belonging to the same chart $U \subset M$, the isomorphisms $\pi^{-1}x \simeq \pi^{-1}(y) \simeq F$ depend on the choice of a trivialization over $U$. This ambiguity can be partially fixed by specifying a connection.

**Definition.** A (topological) *connection* on a fiber bundle $\pi : W \to M$ is a field of tangent $m$-planes $C_x$ in the space of the bundle $W$, that are transversal to the fibers of $\pi$ at any point $x \in W$, where $m = \dim M$.

The transversality condition means that for any $x \in W$ the tangent space $T_xW$ can be decomposed into the direct sum of the *horizontal* plane $C_x$ and the *vertical* one $V_x = T_xW_{\pi(x)}$, where $W_y$ is the fiber $\pi^{-1}(y)$ over $y$.




The projection $\pi_*$ takes $V_x$ to zero and maps $C_x$ isomorphically to $T_{\pi(x)}M$. It follows that any vector field on $M$ can be lifted in a unique way to a vector field on $W$ tangent to the planes of the connection. The lifted field provides an isomorphism of infinitesimally close fibers. Thus the connection 'connects' neighboring fibers, it shows in which direction a point of the fiber should go when the point of the base is moving.

A *section* is a map $s : M \to W$ such that $\pi \circ \sigma = \mathrm{Id}$. A section is called *covariantly constant* if it (or, more precisely, its image) is tangent to the planes of the connection field. A section defined over a point $x \in M$ extends in a unique way to a covariantly constant section over any path $\gamma$ starting at this point. This extension is given by the phase flow of the lifting of the field $\dot\gamma$, see the picture. The covariantly constant extension provides a diffeomorphism $h_\gamma : W_{\gamma(0)} \to W_{\gamma(1)}$. This diffeomorphism is called the *parallel translate* of the fiber $W_{\gamma(0)}$ to the fiber $W_{\gamma(1)}$. In general, the parallel translate depends on the path $\gamma$ connecting two given points.

**Problem 4.1.** Does the parallel translate depend on the parametrization of the path?

**Example.** Let $W$ be the bundle of unit vectors tangent to a surface $M \subset \mathbb{R}^3$. The *canonical connection* on the bundle $\pi : W \to M$ is determined by the tangent planes orthogonal to the fibers of $\pi$ in the sense of the standard Euclidean structure in the ambient space $T\mathbb{R}^3 = \mathbb{R}^6$.

**Problem 4.2.** Show that a field of tangent vectors $v(t)$ defined along a given curve $\gamma$ on $M$ is covariantly constant if and only if the coordinate-wise derivative $\frac{dv}{dt}$ is orthogonal to $M$ at any point.

The definition above uses extrinsic terms but we are going to show below that this connection is determined uniquely by the Riemannian structure (the first quadratic form). For example, the parallel translate on a sheet of paper (the Euclidean plane) is the usual parallel translate of vectors. Hence, the similar description holds if one bends the paper without stretching it.

**Problem 4.3.** Find the parallel translate along a directrix of a cone whose generatrix forms the angle $\alpha$ with the axis.

The fact that the parallel translate belongs to the intrinsic geometry of a surface implies its following geometric interpretation. Consider a curve on $M$, and let us attach a sheet of paper along this curve to the surface. The trace of the curve on the paper is called the *unfolding* of that curve. Since the surface and the paper are tangent along the curve, the restriction of the metric to the curve and to its unfolding coincide. Therefore, the parallel translate along the curve coincides with the parallel translate along its unfolding.

**Problem 4.4.** Find the parallel translate along the parallel with the latitude $\alpha$ on a sphere.

In order to compute the parallel translate in coordinates, choose an orthonormal frame $e_1, e_2$ of tangent vectors defined in a neighborhood of a curve. Then any field of unit vectors along the curve can be represented in the form

$$v(t) = \cos\psi\, e_1 + \sin\psi\, e_2$$

with the angle $\psi = \psi(t)$ depending on $t$.

**Problem 4.5.** Prove that the field $v$ is covariantly constant along $\gamma$ iff the function $\psi$ satisfies $\theta + d\,\psi \equiv 0$, i.e.,

$$\frac{d\psi}{dt} = -\theta(\dot\gamma),$$

where the $\theta$ is the 1-form used in the definition of the Gaussian curvature.

**Problem 4.6.** Using the formula of the previous problem, recompute the parallel translate along the parallel with the latitude $\alpha$ on a sphere.

Another interpretation of the parallel translate of tangent vectors on a surface uses the Gauss map. This interpretation reduces the problem of finding parallel translate on any surface to the case of a sphere.

**Problem 4.7.** Show that the family of tangent vectors $v(t)$ to a surface $M$ is parallel along a curve $\gamma$ if and only if this family is parallel along the image $G(\gamma)$ of this curve under the Gauss map $G : M \to S^2$.

Let $U$ be a domain on a surface homeomorphic to a disc.

**Theorem (local Gauss-Bonnet formula).** *The parallel translate along the boundary of the domain $U$ rotates tangent vectors by the angle*

$$\Delta\psi = \int_U K\,\sigma,$$

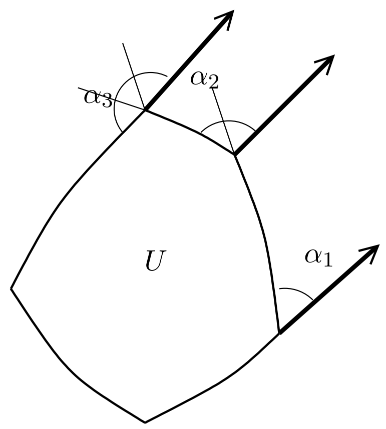
*where $K$ is the curvature and $\sigma$ is the area form.*

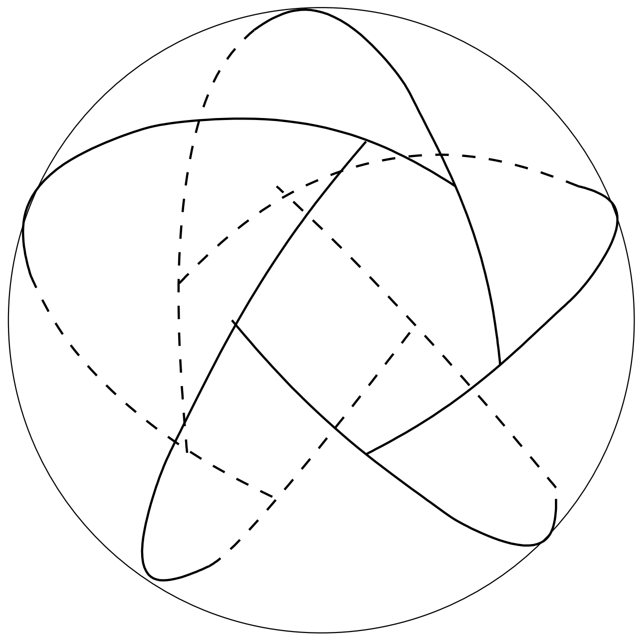
Recall that the curvature form $K\,\sigma$ is represented locally as $K\,\sigma = -d\theta$. Note that the curvature form is defined on the whole $M$ while the 1-form $\theta$ depends on the choice of an orthonormal frame $e_1, e_2$ on the surface and is defined only in the domain of definition of this frame. Without loss of generality, we may assume that the frame and the form $\theta$ are defined on $U$. Then the Stokes formula yields:

$$\Delta\psi = \oint_{\partial U} d\psi = -\oint_{\partial U} \theta = -\int_U d\theta = \int_U K\,\sigma.$$

This proves the theorem.

In the case when $M$ is the unit sphere, the formula can be proved also geometrically as follows. Let $U$ be a polygon whose $n$ edges are arcs of great circles. It is clear from



the picture that the rotation angle of the parallel translate along $\partial U$ is equal to $\Delta\psi = 2\pi - \sum \alpha_i$. Extend the edges of $U$ as in the picture. Then the sphere is represented as the union of the polygon $U$, its antipodal polygon $U'$, and $n$ two-gons with the angles $\alpha_1, \dots, \alpha_n$. Therefore,

$$4\pi = (\text{area of the sphere}) = 2\,(\text{area of } U) + 2\sum \alpha_i.$$

Therefore,

$$\Delta\psi = (\text{area of } U) = \int_U K\,\sigma$$

since on the unit sphere we have $K = 1$. The Gauss map $G$ reduces the proof of the local Gauss-Bonnet formula to the case of the sphere. Indeed, by the assertion of Problem 4.7, the Gauss map preserves the parallel translate, and by the assertion of Problem 2.6 the curvature form is represented as

$$K\,\sigma = G^*\Sigma$$

where $\Sigma$ is the area form on the sphere. In other words, the Gauss map preserves the integral $\int K\,\sigma$ as well.

The global version of the Gauss-Bonnet formula is formulated as follows. Let $M$ be a closed surface homeomorphic to the sphere with $g$ handles.

**Theorem (global Gauss-Bonnet formula).**

$$\frac{1}{2\pi}\int_M K\,s = \chi(M) = 2 - 2g.$$

The value $\chi(M) = 2 - 2g$ is called the *Euler characteristic* of the surface. It is known from topology that any surface can be glued from a polygon by identifying some of its edges. It follows that there exists a vector field on $M$ which is non-zero everywhere except for a single point and following a small circle around this point the field makes $\chi(M)$ full turns. This field can be used to construct an orthonormal frame with similar properties defined on the complement of the point. Applying the local Gauss-Bonnet formula to the polygon, we obtain the global one.

Alternatively, the theorem can be proved by applying the Gauss map. Since $K\,\sigma = G^*\Sigma$, where $\Sigma$ is the area form on the sphere, we have

$$\int_M K\,\sigma = 4\pi\, d,$$

where $4\pi$ is the area of the sphere and $d$ is the degree of the Gauss map. I leave it to the reader to verify that the degree of the Gauss map is equal to $\chi(M)/2 = 1 - g$.

**Problem 4.8.** Compute the degree of the Gauss map.

# 5 Connection as a covariant derivative

A *vector bundle* $E \to M$ is a locally trivial fiber bundle whose fibers $E_x$ are equipped with the structure of a vector space. A trivialization of a vector bundle is given by a collection of sections $e_1, \dots, e_n$ forming a base in each fiber. Each vector bundle admits local trivializations, but global ones do not necessarily exist. The *rank* $n = \operatorname{rk} E$ of the bundle is the dimension of its fibers.

A typical example is the *tangent bundle* $TM$ or the *cotangent bundle* $T^*M$. The sections of a vector bundle form a module $\Gamma(E)$ over the ring of smooth functions on the base (they can be added and multiplied by functions). For example, a section of the tangent bundle is a vector field, a section of the cotangent bundle is a differential 1-form. All operations on vector spaces like direct sum, tensor product, taking the quotient space etc. have obvious analogs for vector bundles.

In particular, one may consider the bundle of the form $T^{*\otimes p}M \otimes T^{\otimes q}M$, the tensor product of $p$ copies of the tangent and $q$ copies of the cotangent bundles. Sections of this bundle are called $(p,q)$*-type tensor fields.* For example, a Riemannian structure can be treated as a $(0,2)$-tensor. Locally, a $(p,q)$-tensor $T$ is given by a collection of its $n^{p+q}$ coordinates $T^{i_1 \dots i_p}_{j_1 \dots j_q}$,

$$T = T^{i_1 \dots i_p}_{j_1 \dots j_q} \partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q},$$

where $x^1, \dots, x^n$ is a local system of coordinates on $M$ (the summation sign over repeating indices is omitted). It is convenient to follow the tradition in placing top and bottom indices in order to compute correctly the transformation of components under a change of coordinates, see the Appendix. The Riemannian structure on $M$ (if present) provides an isomorphism

$$TM \to T^*M, \quad \xi \mapsto (\xi, \cdot).$$

In a coordinate form, this isomorphism takes a field with the coordinates $\xi^i$ to the field with the coordinates $\xi_i = g_{ij}\xi^j$. The inverse isomorphism $\xi_i \mapsto \xi^i = g^{ij}\xi_j$ is determined by the matrix $g^{ij}$ inverse to the matrix $g_{ij}$ of the metric. This isomorphism extends to an isomorphism of any two $(p,q)$-type tensor bundles with equal $p+q$. In coordinates, this leads to the formalism of raising and lowing indices of $(p,q)$-tensors with coinciding $p+q$:

$$T^I_{iJ} \longleftrightarrow \widetilde{T}^{iI}_J = g^{ij} T^I_{jJ}, \qquad T^{iI}_J \longleftrightarrow \widetilde{T}^I_{iJ} = g_{ij} T^{jI}_J,$$

where $I$, $J$ denote sets of indices that do not change.

There are, however, many important examples of vector bundles that are not related to the tangent bundle. In what follows, we consider a vector bundle $E \to M$ of rank $\operatorname{rk} E = n$ over a smooth base $M$ of dimension $\dim M = m$. It is important to distinguish between the coordinates along the fiber and the base of the bundle.

**Definition.** A *connection* in a vector bundle $E \to M$ is a correspondence assigning to any pair $(s, \xi)$ consisting of a section $s \in \Gamma(E)$ and a vector field $\xi \in \Gamma(TM)$ a new section $\nabla_\xi s \in \Gamma(E)$ called the *covariant derivative* of $s$ along $\xi$ such that the following properties hold: this correspondence is linear in $\xi$ (with respect to multiplication by a function), it is $\mathbb{R}$-linear in $s$ and satisfies the Leibnitz rule with respect to multiplication of the section by a function

$$\nabla_\xi f s = \partial_\xi f \; s + f \, \nabla_\xi s.$$

**Example.** $M = \mathbb{R}^n$, $E = T\mathbb{R}^n$, and $\nabla$ is the coordinate-wise derivative.

**Problem 5.1.** Find the derivative $\nabla_v u$ of the field $u = u_1 \partial_\rho + u_2 \partial_\theta$ along the field $v = v_1 \partial_\rho + v_2 \partial_\theta$ in polar coordinates on the plane $\mathbb{R}^2$. (Answer: $\nabla_v u = (\partial_v u_1)\, \partial_\rho + (\partial_v u_2)\, \partial_\theta - u_2 v_2 \, \rho \, \partial_\rho + (u_1 v_2 + u_2 v_1) \, \frac{1}{\rho} \, \partial_\theta$, where $\partial_u$ is the usual directional derivative of a function along a vector field).

**Example.** Let $M$ be a submanifold in the Euclidean space $\mathbb{R}^n$, $E = T\mathbb{R}^n$. Then we set $\nabla_u v = \mathrm{pr_M} \partial_u v$, the composition of the component-wise derivative in the ambient space $\mathbb{R}^n$ and the orthogonal projection to $TM$.

**Problem 5.2.** For the case of the unit sphere $M = S^2 \subset \mathbb{R}^3$, find the derivative $\nabla_v u$ of the field $u = u_1 \partial_\varphi + u_2 \partial_\theta$ along the vector field $v = v_1 \partial_\varphi + v_2 \partial_\theta$ in the spherical coordinates on $S^2$. (Answer: $\nabla_v u = (\partial_v u_1)\, \partial_\varphi + (\partial_v u_2)\, \partial_\theta) + u_2 v_2 \, \sin\varphi \cos\varphi \, \partial_\varphi - (u_1 v_2 + u_2 v_1) \, \tan\varphi \, \partial_\theta)$.

The exercises above suggest the general form of a local presentation of a connection. Fix a trivialization, that is, a local basis of sections $e_1, \ldots, e_n$. Expand the covariant derivatives of these sections in terms of the same basis:

$$\partial_\xi e_j = \theta^i_j(\xi) e_i.$$

The entries $\theta^i_j$ of the coefficient matrix depend linearly on $\xi$. Therefore, it can be thought as differential 1-forms.

**Definition.** The *connection matrix* $\Theta$ is an $(n \times n)$-matrix whose $(i,j)$th entry is the 1-form $\theta^i_j$.

To define the connection matrix, we do not have to introduce local coordinates on $M$ (but we do need a choice of a trivialization). If the coordinates are chosen, however, the components of the matrix can be written as $\theta^i_j = \Gamma^i_{jk} dx^k$. The coefficients $\Gamma^i_{jk}$ are called the *Christoffel symbols.* Note the the indices $i$ and $j$ in the Christoffel symbols run from 1 to $n = \operatorname{rk} E$, while $k$ runs from 1 to $m = \dim M$. Thus, the connection is determined locally by arbitrary $n^2 m$ functions $\Gamma^i_{jk}$.

By the Leibnitz rule, the covariant derivative is completely determined by the derivatives of basic sections: if $s = s^i e_i$, then

$$\nabla_\xi s^j e_j = (\partial_\xi s^j) e_j + \theta^i_j(\xi) e_i = (\partial_\xi s^i + \theta^i_j(\xi)) e_i,$$

or, in the matrix form,

$$\nabla_\xi s = \partial_\xi s + \Theta(\xi)s,$$

where the section $s$ is considered as a column of $n$ its components. Respectively, in the coordinate form, we have

$$\nabla_k s^i = (\nabla_{\partial_{x^k}} s)^i = \frac{\partial s^i}{\partial x^k} + \Gamma^i_{jk} s^j.$$

**Problem 5.3.** For the connections of Problems 5.1 and 5.2, determine the connection matrices and Christoffel symbols.

**Definition.** A section $s \in \Gamma(E)$ defined over a curve $\gamma$ on $M$ is said to be *covariantly constant* along $\gamma$ if $\nabla_{\dot\gamma} s = 0$. In coordinates, this results into the following linear ODE in the column of components of $s$:

$$\frac{ds}{dt} + \Theta(\dot\gamma)s(t) = 0.$$

Solutions to this equation provide a linear transformation of the fiber over $\gamma(0)$ to the fiber over $\gamma(1)$. This transformation is called the *parallel translate*. Thus the geometrically covariant derivative can be thought of as a topological connection whose parallel translates preserve the vector space structure on the fibers.

Consider the space of all connections. How many elements does it have?

**Problem 5.4.** Let $\nabla'$ and $\nabla''$ be connections, which of the following are connections: (a) $\nabla + \nabla'$; (b) $\frac{1}{2}(\nabla + \nabla')$; (c) $\nabla - \nabla'$?

One can notice that the expression $\nabla_\xi s - \nabla'_\xi s$ is linear both in $\xi$ and $s$. Hence it is a tensor. The inverse also is true: if $A : TM \otimes E \to E$ is an arbitrary tensor, then $\nabla + A$ is a connection. Summarizing, we can state that *connections on a given vector bundle $E$ form an affine space associated with the vector space of sections of the bundle* $\mathrm{Hom}(TM \otimes E, E) = T^*M \otimes E^* \otimes E$.

This means that the space of connections is isomorphic to the space of sections of the bundle $T^*M \otimes E^* \otimes E$, but this isomorphism is not canonical; it depends on the choice of the 'original' connection. Connections can be constructed over local charts on $M$ by choosing arbitrary Chritoffel symbols. Then, using the partition of unity, this local data can be glued to a globally defined connection: if $\nabla_1, \dots, \nabla_k$ are connections and $g_1, \dots, g_k$ are smooth functions satisfying $\sum g_i \equiv 1$, then $\nabla = g_1\nabla_1 + \dots + g_k\nabla_k$ is a connection.

The Leibnitz rule allows one to extend the notion of connection to the dual bundle $E^*$. The conjugate connection $\nabla^*$ on $E^*$ is determined by the condition

$$d(u,s) = (\nabla^* u, s) + (u, \nabla s), \quad \text{or} \quad (\nabla^*_\xi u)(s) = \partial_\xi u(s) - u(\nabla_\xi s)$$

for arbitrary sections $s \in \Gamma(E)$, $u \in \Gamma(E^*)$ of bundles $E$, $E^*$ and a field $\xi$. In the basis $\{f^i\}$ dual to the basis $\{e_i\}$, one has $0 = \delta(f^i, e_j) = \theta^i_j + \theta^{*i}_j$, i.e., $\Theta^* = -\Theta^\top$.

By similar argument, connections $\nabla^E$ and $\nabla^F$ on vector bundles $E$ and $F$ determine a connection on their tensor product:

$$\nabla^{E\otimes F}_\xi u \otimes v = (\nabla^E_\xi u) \otimes v + u \otimes (\nabla^E_\xi v).$$

We shall use the same symbol $\nabla$ to denote a connection on a vector bundle $E$ and on all its tensor powers.

## 6 The curvature tensor

A connection is said to be *flat* if the bundle admits (locally) a trivialization for which the connection matrix vanishes identically. Such a basis consists of covariantly constant sections, and the covariant derivative for it coincides with the coordinate-wise derivative. The existence of such a basis is equivalent to the condition that the parallel translate remains unchanged under continuous deformations of a path connecting any two given points.

The curvature $R$ is a tensor intrinsically determined by a connection that serves as an obstruction to flatness. As a tensor, it is a section of the bundle $\Lambda^2 T^*M \otimes \mathrm{Hom}(E, E)$, hence it can be thought of as a family

$$R(\xi, \eta) : E_x \to E_x$$

of linear transformations of the fiber $E_x$ depending in a bilinear and skew-symmetric way on two tangent vectors $\xi, \eta$ of the base. Below we give several interpretations of this tensor. Either of these descriptions could be taken for a definition, then the others would turn into properties. We leave the verification of equivalence of all these definitions to the reader as an (extremely useful, and almost obligatory) exercise.

1. The differential operator

$$R(\xi, \eta) = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]}$$

acts on sections of the bundle $E$.

**Problem 6.1.** Show that the operator $R(\xi, \eta)$ is linear with respect to multiplication of a section by a function. In other words, it corresponds to a globally defined section of the bundle $\mathrm{Hom}(E, E)$. Moreover, it depends linearly on the fields $\xi, \eta$.

2. For a chosen trivialization of $E$, the linear transformation $R(\xi, \eta)$ is given by an $(n \times n)$-matrix. The entries of this matrix depend skew-symmetrically on $\xi, \eta$, hence they can be thought of as differential 2-forms.

**Problem 6.2.** Show that the matrix of the curvature operator consisting of 2-forms is determined by the equation

$$R = d\Theta + \Theta \wedge \Theta.$$

This equation is known as the *Cartan structure equation.*

3. In coordinates, the curvature matrix $R$ has the entries $R^i_j = R^i_{jkl} dx^k \wedge dx^l$.

**Problem 6.3.** Show that the coefficients $R^i_{jkl}$ of the curvature tensor are given by the formula

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{pk}\Gamma^p_{jl} - \Gamma^i_{pl}\Gamma^p_{jk} \,.$$

What is the range of the indices $i, j, k, l$ in this equality?

**Problem 6.4.** Using a coordinate system, let us identify a neighborhood of some point on $M$ with a neighborhood of the origin in the tangent plane at this point. Consider the parallelogram on $M$ spanned by the tangent vectors $\varepsilon\xi, \varepsilon\eta$. Consider the parallel translate

$h$ of the fiber under the path going along the boundary of this parallelogram. Prove that this transformation of the fiber has the form

$$h = \mathrm{Id} - \varepsilon^2 R(\xi, \eta) + \dots,$$

where the dots denote terms of higher order in $\varepsilon$.

**Theorem.** *The connection is flat if and only if the curvature tensor $R$ vanishes identically.*

For simplicity, we shall consider only the case where the dimension of the base is $\dim M = 2$ (and the bundle has an arbitrary rank). The general case differs from the one under consideration only by more complicated notations.

Choose a trivialization of the bundle and a coordinate system $x, y$ on $M$. Then covariantly constant sections are solutions to the system of linear differential equations

$$\begin{cases} \dfrac{\partial s}{\partial x} = A\, s \\ \dfrac{\partial s}{\partial y} = B\, s \end{cases} \qquad (*)$$

with respect to the column $s$, where $A$ and $B$ are matrices depending on $x$ and $y$ and given by $A = -\Theta(\partial_x)$, $B = -\Theta(\partial_y)$. It is not difficult to extend solutions of this system along the lines $x = \mathrm{const}$ and $y = \mathrm{const}$. The problem is to verify that such an extension is unique for any initial value of $s$ at the origin. Differentiating the first equation with respect to $y$ and the second one with respect to $x$, from the equality of mixed partial derivatives we get

$$\frac{\partial A}{\partial y} s + ABs = \frac{\partial B}{\partial x} s + BAs, \qquad \text{or} \qquad \frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} + AB - BA = 0.$$

This equation is called the *compatibility condition* for the system $(*)$. Note that it is exactly the equality $R = 0$, but in a different notation. We would like to show that the compatibility condition guarantees existence of solutions to this system.

For any value of $s$ at the point $x = y = 0$, we extend the solution along the line $x = 0$ using the second equation and then along the lines $y = \mathrm{const}$ using the first one. By construction, the first equation $\frac{\partial s}{\partial x} = As$ is satisfied. We need to verify that the second equation is satisfied as well. Set $z = \frac{\partial s}{\partial y} - Bs$. Then the compatibility condition yields

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial y}\frac{\partial s}{\partial x} - \frac{\partial B}{\partial x} s - B\frac{\partial s}{\partial x} \\ &= \left(\frac{\partial A}{\partial y} s + A\frac{\partial s}{\partial y}\right) - \frac{\partial B}{\partial x} s - BAs = A\frac{\partial s}{\partial y} - ABy = Az. \end{aligned}$$

Thus, $z$ satisfies the linear equation $\frac{\partial z}{\partial x} = Az$. By construction, $z$ vanishes for $x = 0$. Therefore, we conclude from the uniqueness of solutions of ODE that $z$ vanishes identically (since $z \equiv 0$ is an obvious solution of $\frac{\partial z}{\partial x} = Az$). This completes the proof of the theorem.

## 7 The Levi-Cevita connection

It is important to remember that the connections can be introduced for any vector bundle, not necessarily related to the tangent one or to the Riemannian structure. In general, there is no special reason to choose one or another connection. However, if $M$ is a *Riemannian* manifold, then there exists a canonical connection on the *tangent* bundle determined uniquely by two conditions: *symmetry* and the *compatibility with the metric.* The curvature tensor of this connection provides important information about the metric.

Let $E = TM$ be the tangent bundle. A connection $\nabla$ is said to be *symmetric* if one of the following two equivalent conditions is satisfied:

1) the Christoffel symbols are symmetric with respect to the bottom indices, $\Gamma^i_{j,k} = \Gamma^i_{k,j}$;

2) the identity $\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$ holds for any two vector fields $\xi$ and $\eta$.

**Problem 7.1.** Prove the equivalence of these conditions.

**Problem 7.2.** Prove that for a symmetric connection, any covariantly constant 1-form (if any) is closed.

**Problem 7.3.** Prove that there is a symmetric connection on any manifold. How many independent functions determine a symmetric connection locally?

For any connection on the tangent bundle, the operation $T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$ is linear with respect to each of the arguments. Therefore, it provides a $(1, 2)$-tensor. This tensor is called the *torsion tensor.* The coordinates of the torsion tensor are given by $T^i_{j,k} = \Gamma^i_{k,j} - \Gamma^i_{j,k}$. Thus, the connection is symmetric iff its torsion tensor is trivial.

**Theorem.** *A symmetric connection is locally Euclidean, i.e., its Christoffel symbols are zero in some chart, if and only if its curvature tensor vanishes identically.*

*Proof.* If the curvature tensor is trivial, then the connection is flat, i.e., there exists a basis $e_i(x)$ consisting of covariantly constant vector fields. The dual basis $e_i^*$ of 1-forms is also covariantly constant. If the connection is symmetric, then the forms $e_i^*$ are closed, i.e., locally, $e_i^* = dx_i$ for some functions $x_i$. The functions $x_i$ thus constructed form a coordinate system for which $e_i = \partial_{x_i}$. Therefore, the Christoffel symbols vanish, as desired.

In contrast to symmetric connections that are defined for the tangent bundle only, the condition of compatibility with the Riemannian structure makes sense for any vector bundle $E \to M$.

A Riemannian structure on a bundle $E$ is a family of positively definite symmetric bilinear forms $g_x$ on its fibers $E_x$. For example, a Riemannian structure on a manifold is, by definition, a Riemannian structure on its tangent bundle. A Riemannian structure provides an isomorphism $E \to E^* : v \mapsto g_x(v, \cdot)$. Having this in mind, we shall always denote by $(\cdot, \cdot)$ both the scalar product and the pairing between vectors and covectors.

A connection $\nabla$ is said to be *compatible with the Riemannian structure* if the parallel translate preserves the Riemannian structure on the fibers.

**Problem 7.4.** Prove that a connection is compatible with a Riemannian structure if and only if one of the following equivalent conditions hold:

(1) $\nabla$ coincides with the conjugate connection $\nabla^*$ under the isomorphism $E \simeq E^*$ above;

(2) the identity $\partial_\xi(u, v) = (\nabla_\xi u, v) + (\xi, \nabla_\xi v)$ holds for arbitrary vector field $\xi$ and sections $u$ and $v$;

(3) the metric $g$ considered as a section of the bundle $E^* \otimes E^*$ is covariantly constant, $\nabla_\xi g = 0$;

(4) the matrix $\Theta$ of the connection is skew-symmetric in the basis of orthonormal sections.

**Problem 7.5.** Prove that a connection compatible with a Riemannian structure exists on any manifold. How many independent functions determine it locally?

**Theorem.** *On the tangent bundle to a Riemannian manifold, there exists a unique symmetric connection compatible with the Riemannian structure.*

This connection is called the *Levi-Cevita connection.*

**Example.** Let $M$ be a surface. Choose an orthonormal basis $e_i, e_2$ of vector fields. The compatibility of the connection with the metric is equivalent to the skew-symmetry of the connection matrix $\Theta$. Therefore, it should have the form

$$\Theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}, \qquad \nabla_\xi e_1 = \theta(\xi)\, e_2, \qquad \nabla_\xi e_2 = -\theta(\xi)\, e_1,$$

for some 1-form $\theta$. By the symmetry condition, one has

$$[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1 = -\theta(e_1)e_1 - \theta(e_2)e_2.$$

This condition determines the form $\theta$ uniquely. Moreover, it is exactly the form used in the definition of the Gaussian curvature, see Sect. 3.

*Proof of the theorem.* In order to determine the derivative $\nabla_{\xi_1}\xi_2$ of a vector field along another vector field it is sufficient to determine the scalar product $(\nabla_{\xi_1}\xi_2, \xi_3)$ of this derivative with arbitrary third one. Let us fix a triple of vector fields $\xi_1, \xi_2, \xi_3$ and consider six quantities of the form $(\nabla_{\xi_i}\xi_j, \xi_k)$, where $(i, j, k)$ is a permutation of the sequence $(1, 2, 3)$. Compatibility with the metric provides three linear relations between these quantities:

$$(\nabla_{\xi_k}\xi_j, \xi_i) + (\nabla_{\xi_k}\xi_i, \xi_j) = \xi_k(\xi_i, \xi_j)\ .$$

The symmetry condition $\nabla_{\xi_k}\xi_j - \nabla_{\xi_j}\xi_k = [\xi_k, \xi_j]$ gives another three relations:

$$(\nabla_{\xi_k}\xi_j, \xi_i) - (\nabla_{\xi_j}\xi_k, \xi_i) = ([\xi_k, \xi_j], \xi_i)\ .$$

Solving the system of six linear equations thus obtained in three indeterminates, we find the solution

$$(\nabla_{\xi_k}\xi_j, \xi_i) = \frac{1}{2}\Big(\xi_k(\xi_i, \xi_j) - \xi_i(\xi_j, \xi_k) + \xi_j(\xi_i, \xi_k) - ([\xi_j, \xi_k], \xi_i) + ([\xi_i, \xi_k], \xi_j) + ([\xi_i, \xi_j], \xi_k)\Big)\ .$$

One have to verify that the expression on the right hand side is linear with respect to $\xi_k$ and $\xi_i$ and satisfies the Leibnitz identity with respect to $\xi_j$. This can be done by direct computations.

The formula thus obtained applied to the fields of some base gives explicit expressions for the coefficients of the Levi-Cevita connection.

**Problem 7.6.** Show that in the base of commuting vector fields the Christoffel symbols of the Levi-Cevita connection are given by

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{il}(\partial_{x^k} g_{lj} - \partial_{x^l} g_{jk} + \partial_{x^j} g_{lk})\ ,$$

or, in the matrix form, $\Theta = \frac{1}{2}g^{-1}\|dg_{ij} - \partial_i u_j + \partial_j u_i\|_{ij}$, where $u_i = g_{ij}dx^i$ are the dual 1-forms and $\partial_{x_i}$ is the component-wise partial derivative.

**Problem 7.7.** Show that in a base of orthonormal vector fields the Christoffel symbols of the Levi-Cevita connection are given by

$$\Gamma^i_{jk} = \frac{1}{2}\,(c^i_{kj} + c^j_{ik} + c^k_{ij})\,, \qquad \text{where } [\xi_i, \xi_j] = c^k_{ij}\xi_k\,.$$

Here is another important corollary of the explicit formula above for $(\nabla_{\xi_k}\xi_j, \xi_i)$.

**Theorem.** *Let $N \subset M$ be a submanifold of a Riemannian manifold $M$. Consider the connection on $N$ given by differentiation of the fields in the Levi-Cevita connection on $M$ followed by the orthogonal projection to $TN$. This connection coincides with the Levi-Cevita connection on $N$.*

Indeed, if the fields $\xi_1$, $\xi_2$ and $\xi_3$ are tangent to the submanifold $N$, then the restriction to $N$ of their commutators, Lie derivatives of the functions, scalar products etc. are determined by the restrictions of these fields to $N$ only.

This shows, in particular, that the operation of parallel translate of tangent vectors along a curve on a surface in $\mathbb{R}^3$ (see Problem 4.2) belongs to the intrinsic geometry of the surface.

# 8 Riemann curvature tensor

The *Riemann curvature tensor* on a Riemannian manifold is the curvature tensor of the Levi-Cevita connection. By definition, it is a linear transformation $R(X,Y)$ of the tangent space invariantly determined by the Riemannian structure and depending in a linear and skew-symmertic way on two additional tangent vectors $X$ and $Y$. It is an important invariant of the metric serving as an obstruction to the exitance of Euclidean coordinates.

**Theorem.** *A Riemannian metric is locally Euclidean, i.e., can be reduced to the form $\sum(dx^i)^2$ by an appropriate change of coordinates, if and only if its curvature tensor vanishes identically.*

*Proof.* Assume that the curvature tensor is zero. Then there exists a covariantly constant basis of tangent vector fields. The symmetry condition of the Levi-Cevita connection implies that this basis consists of coordinate vector fields of some coordinate system (i.e., the dual basis of one-forms consists of closed forms). In these coordinates, all Christoffel symbols vanish identically and the connection is given by coordinate wise derivative. Then, compatibility with the metric implies that the coefficients $g_{ij}$ of the metric are constant functions in these coordinates. Therefore, it can be reduced to the sum of squares by a linear change of coordinates.

The Riemann curvature tensor can be viewed as a 4-linear form on the tangent space given by

$$(X, Y, Z, U) \mapsto (R(X,Y)Z, U).$$

This form has the coefficients

$$R_{ijkl} = (R(\partial_{x^k}, \partial_{x^l})\,\partial_{x^j}, \partial_{x^i}) = g_{ip}R^p_{jkl}.$$

The fact that this form originates from a Riemannian metric results in its additional symmetry properties.

**Theorem.** *The Riemannian curvature tensor satisfies the following identities:*
(1) $R(X,Y) = -R(Y,X)$;
(2) $R(X,Y)\,Z + R(Y,Z)\,X + R(Z,X)\,Y = 0$;
(3) $(R(X,Y)\,Z,U) + (Z,R(X,Y)\,U) = 0$;
(4) $(R(X,Y)\,Z,U) = (R(Z,U)\,X,Y)$.

In coordinates, this theorem asserts that the tensor is skew-symmetric with respect to the first and the last pairs of indices; it is invariant with respect to exchange of these pairs of indices; and the sum of its components under a cyclic permutation of arbitrary three indices is equal to zero,

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}\ , \quad R_{ijkl} + R_{iklj} + R_{iljk} = 0\ .$$

*Proof.* To simplify the arguments, we observe that by polylinearity of tensors it is sufficient to assume that the vector field $X$, $Y$, $Z$, $U$ are pairwise commuting.

Equation (1) is assumed by definition. It holds for the curvature tensor of any connection on any vector bundle.

Equation (2) follows from the definition, $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X$ and the equality $\nabla_X Y = \nabla_Y X$ valid for commuting vector fields (compatibility with the metric is not used in the proof of this property).

Equation (3) means that the parallel translate along a small loop preserves the Riemannian structure. More formally, compatibility with the metric yields

$$\begin{aligned}\partial_X\partial_Y(Z,U) &= \partial_X(\nabla_Y Z,U) + \partial_X(Z,\nabla_Y U)\\ &= (\nabla_X\nabla_Y Z,U) + (\nabla_Y Z,\nabla_X U) + (\nabla_X Z,\nabla_Y U) + (Z,\nabla_X\nabla_Y U).\end{aligned}$$

Computing $\partial_Y\partial_X(Z,U)$ in the same way and subtracting the expressions thus obtained we get the desired equality (the fact that the connection is symmetric is not used in the proof of this property).

Finally, Equation (4) is a formal consequence of the Eqns. (2) and (3). Its proof is left as an exercise.

The *space of curvature tensors* on a vector space of dimension $n$ is the space of 4-linear forms satisfying properties (1)–(4).

**Problem 8.1.** Find the dimension of the space of curvature tensors for $n = 2; 3; 4$; arbitrary $n$.

The *Ricci tensor* is the result of convolution of the Riemann tensor, $\mathrm{Ric}_{ql} = R^i_{qil}$. The *scalar curvature* is obtained by convolving of the Ricci tensor,

$$R = g^{ql}\mathrm{Ric}_{ql} = g^{ql}R^i_{qil}.$$

**Theorem.** *Let $m$ be a surface. Then the scalar curvature is related to the Gaussian curvature by $R = 2\,K$.*

*Proof.* For $n = 2$, the connection matrix is given in an orthonormal basis by $\Theta = \left(\begin{smallmatrix} 0 & -\theta \\ \theta & 0 \end{smallmatrix}\right)$ where the 1-form $\theta$ is defined in Sect. 3. Then it follows that the curvature matrix is given by

$$R = d\theta + \theta\wedge\theta = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & K\sigma \\ -K\sigma & 0 \end{pmatrix}.$$

Hence the curvature tensor is determined completely by the component $R^1_{212} = K$. Convolving, we obtain $R = R^1_{212} + R^2_{121} = 2K$. □

**Problem 8.2.** Prove that in the case $n = 2$, for arbitrary basis, one has $K = R/2 = \frac{R_{1212}}{\det |g|}$.

**Example.** Let us compute the Gaussian curvature of the metric $g = dx^2 + 2\cos w(x,y)\,dx\,dy + dy^2$.

(1) Determine the matrix $g = \|g_{ij}\|$ which is inverse to the matrix $g^{-1}$, and the forms $u_i = g_{ij}dx^j$ (the coefficients of these forms are given by the rows of $g$):

$$g = \begin{pmatrix} 1 & \cos w \\ \cos w & 1 \end{pmatrix}, \quad g^{-1} = \frac{1}{\sin^2 w}\begin{pmatrix} 1 & -\cos w \\ -\cos w & 1+\varphi^2 \end{pmatrix}, \quad \begin{matrix} u_1 = dx + \cos w\,dy \\ u_2 = \cos w\,dx + dy \end{matrix}.$$

(2) Compute the matrix

$$\begin{aligned} g\theta &= \frac{1}{2}\|dg_{ij} - \partial_i u_j + \partial_j u_i\| = \frac{1}{2}\begin{pmatrix} dg_{11} & dg_{12} - \partial_1 u_2 + \partial_2 u_1 \\ dg_{21} + \partial_1 u_2 - \partial_2 u_1 & dg_{22} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\sin w\, w_y\,dy \\ -\sin w\, w_x\,dx & 0 \end{pmatrix} \end{aligned}$$

(here, $\partial_i$ is the componentwise partial derivative). Multiplying by $g^{-1}$ on the left, we obtain the connection matrix

$$\theta = \begin{pmatrix} \Gamma^1_{1k}dx^k & \Gamma^1_{2k}dx^k \\ \Gamma^2_{1k}dx^k & \Gamma^2_{2k}dx^k \end{pmatrix} = \frac{1}{\sin w}\begin{pmatrix} \cos w\, w_x\,dx & -w_y\,dy \\ -w_x\,dx & \cos w\, w_y\,dy \end{pmatrix}.$$

Therefore, $\Gamma^1_{11} = \operatorname{ctg} w\, w_x$, $\Gamma^2_{22} = \operatorname{ctg} w\, w_y$, $\Gamma^2_{11} = -w_x/\sin w$, $\Gamma^1_{22} = -w_y/\sin w$, and the remaining four Christoffel symbols vanish.

(3) Compute the curvature matrix

$$R = d\theta + \theta\wedge\theta = \begin{pmatrix} -\cos w & -1 \\ 1 & \cos w \end{pmatrix}\frac{w_{xy}}{\sin w}\,dx\wedge dy.$$

If the computations are made without mistakes, then the matrix $gR = R_{ijkl}\,dx_k \wedge dx_l$ must be skew-symmetric. Indeed, multiplying the matrices we obtain

$$gR = \begin{pmatrix} 0 & -\sin w\, w_{xy}\,dx\wedge dy \\ \sin w\, w_{xy}\,dx\wedge dy & 0 \end{pmatrix},$$

that is, $R_{1212} = -\sin w\, w_{xy}$, $K = \frac{R_{1212}}{\det g} = -\frac{w_{xy}}{\sin w}$.

(4) Assume that $w = x + y$. Then the curvature vanishes and the metric is Euclidean. Let us find the Euclidean coordinates. Covariantly constant 1-forms can be written as $u\,dx + v\,dy$, where the coefficients $u$ and $v$ satisfy the system of equations

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \theta^\top\begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{or} \quad \begin{cases} \frac{\partial u}{\partial x} = \operatorname{ctg} w\, w_x\, u - \frac{w_x}{\sin w}\, v \\ \frac{\partial v}{\partial x} = 0 \end{cases}, \quad \begin{cases} \frac{\partial u}{\partial y} = 0 \\ \frac{\partial v}{\partial y} = -\frac{w_y}{\sin w}\, u + \operatorname{ctg} w\, w_y\, v \end{cases}.$$

In the case $w = x + y$, this system admits a solution

$$u = c_1\cos x - c_2\sin x, \quad v = c_1\cos y + c_2\sin y,$$

where $c_1$ and $c_2$ are certain constants. Thus, the covariantly constant 1-forms are given by

$$\begin{aligned} u\,dx + v\,dy &= c_1\,(\cos x\,dx + \cos y\,dy) + c_2\,(-\sin x\,dx + \sin y\,dy) \\ &= c_1\,d(\sin x + \sin y) + c_2\,d(\cos x - \cos y), \end{aligned}$$

and the desired Euclidean coordinates are $X = \sin x + \sin y$ and $Y = \cos x - \cos y$.

**Problem 8.3.** For the metrics given below, determine the connection matrix $\Theta$, the curvature matrix $R$, and the Gaussian curvature $K$. In the case $K \equiv 0$, find the Euclidean coordinates.

(a) $g = (1+\varphi^2)\,dx^2 + 2\varphi\,dx\,dy + dy^2$, $\varphi = \varphi(x)$.

(b) $g = dx^2 + \cos^2 x\,dy^2$;

(c) $g = dx^2 + x^2\,dy^2$.

**Problem 8.4.** Prove that in the case $n = 3$, the curvature tensor is determined by the Ricci tensor.

# 9 Geodesics

A curve $g(t)$ on a Riemannian manifold $M$ is called a *geodesic* if its tangent vectors are parallel along the curve, $\nabla_{\dot\gamma}\dot\gamma = 0$. In coordinates, the equation of geodesics can be written as

$$\ddot\gamma = -\theta(\dot\gamma)\dot\gamma, \quad \text{or} \quad \ddot x^i = -\Gamma^i_{jk}\dot x^i \dot x^j.$$

These relations provide an ODE (i.e., the vector field) on the $2n$-dimensional total space $TM$ of the cotangent bundle. The initial vector (together with its attachment point) determines the whole geodesic completely.

The equation of geodesics admits a number of interpretations in physics and control theory. Consider the metric $g$ as the function $T(\xi) = \frac{1}{2}g(\xi,\xi)$ on the total space of the tangent bundle. This function, which is quadratic on the fibers, is often referred to as the *kinetic energy.*

**Problem 9.1.** Prove that geodesics are trajectories of free particles with kinetic energy $T$, namely, the following assertions are equivalent (the equivalence of these assertions is usually proved on one of the first lectures of the standard course of mechanics).

(a) Geodesics connecting given points $a$ and $b$ on $M$ are extremals of the length and action functionals

$$l(\gamma) = \int_\gamma dl = \int_\gamma \sqrt{g_{ij}(x)\dot x^i \dot x^j}\,dt\,, \qquad S(\gamma) = \int_\gamma T(\dot\gamma)\,dt = \frac{1}{2}\int_\gamma g_{ij}(x)\dot x^i \dot x^j\,dt$$

defined on the space of all smooth curves connecting points $a$ and $b$.

(b) The equation of geodesics can be written in the form of the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot x^i}\right) - \frac{\partial T}{\partial x^i} = 0\,.$$

(c) Under the isomorphism $TM = T^*M$ provided by the Riemannian structure, the equation of geodesics can be written in the form of Hamiltonian equations

$$\dot p_i = \frac{\partial T}{\partial x^i}; \quad \dot x^i = -\frac{\partial T}{\partial p_i}, \qquad p_i = g_{ij}\dot x^j = \frac{\partial T}{\partial \dot x^j}.$$

In some cases, the correspondence described above can be used in the opposite direction: a mechanical problem can be reduced to the study of the geodesic flow of some Riemannian manifold. Yet another interpretation of the geodesic flow is as follows: it describes the propagation of some perturbation in a nonhomogeneous medium whose properties are described by the metric: according to the Huygens principle, the perturbation propagates by minimizing the length, i.e., along geodesics.

Isometries of Riemannian manifolds transform geodesics to geodesics. Therefore, they can be used to define invariants, both local and global. For example, they almost uniquely determine coordinates in a neighborhood of any point $x \in M$. The *exponential map* takes a tangent vector $\xi \in T_xM$ to the point $\gamma(1)$ on the geodesic determined by the initial condition $\dot\gamma(0) = \xi$. It maps a neighborhood of the origin in the tangent space $T_xM$ to a neighborhood of the point $x$ in $M$. Therefore, it can be thought as a chart. This chart is called the *normal coordinate system*. Normal coordinates are defined up to an action of the group $\mathrm{O}(n)$.

**Problem 9.2.** Compute the derivative (i.e., the Jacobi matrix) of the exponential map. Prove that this map is invertible in a small neighborhood of the point $x$. Prove that any two sufficiently closed points can be connected by a unique geodesic that does not quit the neighborhood.

**Problem 9.3.** Prove that in the normal coordinates the Christoffel symbols $\Gamma^i_{jk}(0)$ vanish. Moreover, one has

$$g_{ij} = \delta_{ij} + \frac{1}{3}R_{iklj}x^k x^l + o(|x|^2), \qquad \Gamma^i_{jk}(x) = \frac{1}{3}(R^i_{jlk} + R^i_{klj})x^l + o(|x|).$$

Geodesics minimize locally the distance between two points until these two points are not conjugate to each other. Point $A$ of a geodesic is said to be *conjugate* to point $B$ of the same geodesic if another infinitely closed geodesic exiting the point $A$ also passes through $B$. For example, the South pole on a sphere is conjugate to the North pole along any geodesic. The conjugate point depends on a geodesic: after a small perturbation of the spherical metric, the set of points conjugate to the North pole forms a small closed curve situated near the South pole. After crossing the conjugate point the geodesic is still an extremal point of the action functional but not a local minimum. The index of the quadratic variation of the action is changed by one (attains a negative square).

According to the more formal definition conjugate points are critical points of the exponential map. They can be found by varying the equation of geodesics. Consider a geodesic $\gamma = \gamma(t)$ parametrized by the natural parameter and let $V(t) = \dot\gamma(t)$ be the field of unit tangent vectors. The field $J(t)$ along $\gamma$ is said to be *Jacobian* if it is the field of initial velocities of some family of geodesics $\gamma_s(t)$, $J = \frac{\partial\gamma_s}{\partial s}|_{s=0}$. Each Jacobian field satisfies the equation

$$\ddot J - R(V, J)V = 0\ ,$$

where the dot is the covariant derivative with respect to the natural parameter. Indeed, the fields $J = \partial_s$ and $V = \partial_t$ commute on the plane of the variables $s, t$, therefore,

$$R(V, J)V = \nabla_V\nabla_J V - \nabla_J\nabla_V V = \nabla_V\nabla_V J \qquad \square$$

(we used here the equality $\nabla_V V \equiv 0$ and the property $\nabla_J V = \nabla_V J$ that hold for symmetric connections). Now, decompose $J$ as the sum of the tangent and normal components. Both are Jacobian fields. The tangent component is responsible for reparametrizations of the geodesic. Therefore, we may assume that it is equal to zero, that is, $J(t) = y(t)N$, where $N = N(t)$ is the unit normal vector. Note that $N(t)$ is covariantly constant along $\gamma$ (since $V$ is covariantly constant). Besides, $R(V, N)\,V = -k\,N$ where $k$ is the Gaussian curvature. Hence $\ddot{J} - R(V, J)V = (\ddot{y} + k(t)y)N$, i.e., the function $y(t)$ satisfies the equation

$$\ddot{y} + k(t)y = 0.$$

**Problem 9.4.** Prove that in the case $k(t) \leqslant k_0 < 0$ any solution of the equation $\ddot{y}+k(t)y = 0$ has at most one zero.

**Problem 9.5.** Prove that in the case $k(t) \geqslant k_0 > 0$ the distance between any two consecutive zeros of any solution of the equation $\ddot{y} + k(t)y = 0$ is at most $\pi/\sqrt{k_0}$.

Let me list some global assertions obtained by considering geodesics. Assume that a given Riemannian manifold is *complete*, i.e., any geodesic can be extended infinitely in any direction. Then any two points can be connected by a geodesic of the smallest length.

(a) If $k \leqslant 0$, then $M$ may have no pairs of conjugate points. Then it follows that the universal cover of $M$ is diffeomorphic to $\mathbb{R}^2$, and $M$ is the quotient space of $\mathbb{R}^2$ by a discrete group of diffeomorphisms.

(b) If $k \geqslant k_0 > 0$, then the length of the shortest geodesic connecting any two points is uniformly bounded. Then it follows that $M$ is compact (therefore, it is diffeomorphic to the sphere $S^2$).

**Problem 9.6.** Prove that any isometry of a connected Riemannian manifold acting identically on some tangent space is the identical diffeomorphism.

**Problem 9.7.** Compute the isometry group of (a) sphere; (b) the conical surface given by $x^2 + y^2 - z^2 = 0$, $z > 0$.

Recall (see page 4) that the *Lobachevsky plane*, $L$ is the connected component $z > 0$ of the hyperboloid $z^2 - x^2 - y^2 = 1$ equipped with the metric induced from the pseudo-Riemannian metric $dx^2 + dy^2 - dz^2$.

**Problem 9.8.** (a) Find the Gaussian curvature of the Lobachevsky plane.

(b) Describe the geodesics in all models (a hyperboloid of one sheet, the unit circle, the upper-half plane.

(c) Determine the isometry group.

# Appendix A.
# Calculus on manifolds
# (short review of main statements)

**1.** ***Vector fields.*** A vector field $v = \sum v_i(x)\frac{\partial}{\partial x_i}$ is given in coordinates by $n$ functions $v_i$, where $n$ is the dimension of the manifold. The *derivative* of a function along the field $v$, $f \mapsto \partial_v f = \sum v_i\, \partial f/\partial x_i$ is a *derivation* of the ring of functions, that is, it is $\mathbb{R}$-linear and satisfies the Leibnitz rule, $\partial_v(fg) = f\,\partial_v g + g\,\partial_v f$. Moreover, there is a one-to-one correspondence between the spaces of vector fields and derivations.

**2.** ***The Lie derivative*** $L_v$ along the field $v$ acts on various objects of tensor nature as follows: the phase flow $g^t$ of the field $v$ translates the given tensor field $w$ and we get a family of tensor fields $w_t$. By definition, $L_v w = \frac{d}{dt} w_t \Big|_{t=0}$.

**3.** ***The commutator*** $[u,v]$ of vector fields $u = \sum u_i \frac{\partial}{\partial x_i}$ and $v = \sum v_j \frac{\partial}{\partial x_j}$ is defined as the derivation $f \mapsto \partial_u\partial_v f - \partial_v\partial_u f$. Equivalently, it can be defined as the Lie derivative, $[u,v] = L_u v = \frac{d}{dt} v_t \Big|_{t=0}$, where $v_t(x) = \left(g^t\right)_*^{-1} v(g^t x)$, and the family of diffeomorphisms $g^t$ is the flow of $u$. In coordinates, the commutator is written as

$$[u,v] = \sum (uv_j)\frac{\partial}{\partial x_j} - \sum (vu_i)\frac{\partial}{\partial x_i} = \sum \left( u_j \frac{\partial v_i}{\partial x_j} - v_j \frac{\partial u_i}{\partial x_j}\right) \frac{\partial}{\partial x_i}.$$

The commutator is not linear with respect to the multiplication of the fields by functions but satisfies the identities

$$[u, gv] = g[u,v] + (\partial_u g)\, v, \qquad [fu, v] = f[u,v] - (\partial_v f)\, u.$$

**4.** ***Differential forms*** of degree $k$ are multilinear skew symmetric functions of a collection of $k$ vector fields. The simplest example is the 1-form $df$, where $f$ is a function. By definition, $df(\xi) = \partial_\xi f$.

**5.** ***The exterior product*** of two forms $\alpha$ and $\beta$ of degrees $k$ and $l$, respectively, is the $(k+l)$-form $\alpha \wedge \beta$ given by

$$\alpha^k \wedge \beta^l(\xi_1, \dots, \xi_{k+l}) = \sum_{\substack{\sigma_1 < \cdots < \sigma_k \\ \sigma_{k+1} < \cdots < \sigma_{k+l}}} (-1)^\sigma \alpha^k(\xi_{\sigma_1}, \dots, \xi_{\sigma_k})\, \beta^l(\xi_{\sigma_{k+1}}, \dots, \xi_{\sigma_{k+l}}) \,,$$

where $(-1)^\sigma$ is the sign of the permutation $(1, \dots, k+l) \mapsto (\sigma_1, \dots, \sigma_{k+l})$. This operation is *associative and (super)commutative*, $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$. Using the exterior product, any $k$-form can be uniquely written locally in the form

$$\alpha = \sum_{i_1 < \cdots < i_k} a_{i_1, \dots, i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $x_i$ are the coordinate functions of the chosen coordinate system.

**6.** ***The Lie derivative*** of differential forms $L_v\alpha$ satisfies the identity

$$\partial_v \alpha(\xi_1, \dots, \xi_k) = (L_v\alpha)(\xi_1, \dots, \xi_k) + \alpha([v,\xi_1], \xi_2, \dots, \xi_k) + \cdots + \alpha(\xi_1, \dots, \xi_{k-1}, [v, \xi_k]).$$

**7.** The Lie derivative is defined also by the following properties
(1) it is $\mathbb{R}$-linear;
(2) for a 0-form $f$, that is, for a function, $L_v f = \partial_v f$;
(3) for 1-forms of the form $df$, we have $L_v df = d(\partial_v f)$;
(4) $L_v(\alpha \wedge \beta) = (L_v\alpha) \wedge \beta + \alpha \wedge (L_v\beta)$.

**8.** ***The differential*** of a $k$-form $\omega$ is the $(k+1)$-form defined by the relation

$$d\omega(v_0, ..., v_k) = \sum_{i=0}^{k} (-1)^i \partial_{v_i}\omega(v_0, ..., \hat{v}_i, ..., v_k) + \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j}\omega([v_i, v_j], v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_k\ .)$$

In particular,

$$\begin{aligned} k = 1: \quad & d\alpha(\xi, \eta) = && \partial_\xi \alpha(\eta) - \partial_\eta \alpha(\xi) - \alpha([\xi, \eta]) \\ k = 2: \quad & d\alpha(\xi_1, \xi_2, \xi_3) = && \partial_{\xi_1}\alpha(\xi_2, \xi_3) - \alpha([\xi_1, \xi_2], \xi_3) + \circlearrowleft, \end{aligned}$$

where the terms $\circlearrowleft$ are given by cyclic permutations of the indices $1, 2, 3$.

**9.** In an equivalent way, the differential can be defined by the following axioms
(1) it is $\mathbb{R}$-linear;
(2) for 0-forms, that is, for a function, $df(\xi) = \partial_\xi f$;
(3) $d(\alpha^k \wedge \beta^l) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$;
(4) $d^2 = 0$.

**10.** The second definition implies the *coordinate representation* of the differential,

$$d\Big(\sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\Big) = \sum_{i_1 < \cdots < i_k, j} \frac{\partial a_{i_1, \ldots, i_k}}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

**11.** ***The Cartan identity*** $L = id + di$**.** Denote by $i_v$ the operation of the substitution to the first place of the field $v$ to the form (this operation decreases the degree of the form by one). Then the following identity holds

$$L_v\alpha = i_v d\alpha + d i_v \alpha.$$

It can be proven either directly from the definition or by induction decomposing the form into forms of smaller degrees.

**12.** ***The integral*** of a $k$-form over a $k$-dimensional domain $U \subset \mathbb{R}^n$ is defined as the multiple integral

$$\int_U a(x)\, dx_1 \wedge \cdots \wedge dx_k = \iint_U a(x)\, dx_1 \ldots dx_k.$$

This definition does not depend on the choice of the coordinates whenever the coordinate change *preserves the orientation.* The integral of a $k$-form over a compact *oriented* $k$-dimensional manifold $M$ is determined by cutting the manifold into small pieces or by means of a partition of unity.

**13.** ***The Stokes formula***. Let $M$ be an oriented manifoldwith the boundary $\partial M$; then for any $(k-1)$-form $\omega$ one has

$$\int_M d\omega = \int_{\partial M} \omega.$$

The boundary $\partial M$ is oriented by the rule: '**the exterior normal to the first place**', namely, a frame $\xi_2, \ldots, \xi_k$ orients the boundary $\partial M$ positively iff the frame $\xi_1, \xi_2, \ldots, \xi_k$ orients $M$ positively, where the vector $\xi_1$ is directed outwards.

# Appendix B. The correspondence between invariant and coordinate notations

| | |
|---|---|
| Tensor $T = T^{i_1\dots i_p}_{j_1\dots j_q}\partial_{x^{i_1}}\otimes\dots\otimes dx^{j_q}$ | Collection of $n^{p+q}$ functions $T^{i_1\dots i_p}_{j_1\dots j_q}(x)$ |
| Its presentation in a new basis<br>$T=\overline{T}^{i_1\dots i_p}_{j_1\dots j_q}\partial_{y^{i_1}}\otimes\dots\otimes dy^{j_q}$ | Coordinate change rule<br>$\overline{T}^{k_1\dots k_p}_{l_1\dots l_q}=T^{i_1\dots i_p}_{j_1\dots j_q}\frac{\partial y^{k_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{k_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial y^{l_1}}\cdots\frac{\partial x^{j_q}}{\partial y^{l_q}}$ |
| Transposition isomorphism<br>$T^{\otimes 2}M\to T^{\otimes 2}M\colon\ u\otimes v\mapsto v\otimes u$ | Transposition of indices<br>$T^{ij}\mapsto T^{ji}$ |
| Riemannian structure $g_{ij}\,dx^i\,dx^j$ | Symmetric positively definite matrix $g_{ij}$<br>The matrix $g^{ij}$ is its inverse: $g_{ij}g^{jk}=\delta_i^k$ |
| Isomorphism $T_xM\to T_x^*M\colon\ v\mapsto g_x(v,\cdot)$ and its extension to the higher rank tensors | Operations of raising and lowing indices<br>$T^{iI}_J\mapsto\overline{T}^I_{jJ}=g_{ij}T^{iI}_J$ и $T^I_{jJ}\mapsto\overline{T}^{iI}_J=g^{ij}T^I_{jJ}$ |
| Trace of an operator $f: u\otimes a\mapsto u(a)$,<br>$f\in\mathrm{Hom}(TM,TM)\cong T^*M\otimes TM$ | Convolution of indices $T^i_j\mapsto T^i_i$ |
| its generalization $T^*M\otimes TM\otimes E\to E\colon$<br>$u\otimes a\otimes w\mapsto u(a)\,w$ | $T^{iI}_{jJ}\mapsto\overline{T}^I_J=T^{iI}_{iJ}$ |
| Connection in a vector bundle,<br>its matrix of 1-forms $\theta=\theta^i_j=\Gamma^i_{jk}\,dx^k$ | Christoffel symbols $\Gamma^i_{jk}$ |
| Covariant derivative<br>$\nabla s^ie_i=(ds^i+\theta^i_js^j)\otimes e_i$, or<br>$\nabla_\xi s=\partial_\xi s+\Theta(\xi)s$<br>$d(u,v)=(\nabla u,v)+(u,\nabla v)$,<br>$\nabla u\otimes v=(\nabla u)\otimes v+u\otimes(\nabla v)$ | $\nabla_kT^i=\frac{\partial T^i}{\partial x^k}+\Gamma^i_{jk}T^j$,<br>$\nabla_kT_j=\frac{\partial T_j}{\partial x^k}-\Gamma^i_{jk}T_i$,<br>$\nabla_kT^I_J=\frac{\partial T^I_J}{\partial x^k}-T^I_{jj_2\dots j_q}\Gamma^j_{j_1k}-\dots-T^I_{j_1\dots j_{q-1}j}\Gamma^j_{j_qk}+T^{ii_2\dots i_p}_J\Gamma^{i_1}_{ik}+\dots+T^{i_1\dots i_{p-1}i}_J\Gamma^{i_p}_{ik}$ |
| Change of the trivialization $e'_i=a^j_ie_j$,<br>$\Theta'=a^{-1}da+a^{-1}\Theta a$ | Coordinate change $x^i\rightsquigarrow y^{i'}$,<br>$\Gamma^{i'}_{j'k'}=\frac{\partial y^{i'}}{\partial x^i}(\Gamma^i_{jk}\frac{\partial x^j}{\partial y^{j'}}\frac{\partial x^k}{\partial y^{k'}}+\frac{\partial^2x^i}{\partial y^{j'}\partial y^{k'}})$ |
| Levi-Cevita connection in $TM$<br>$g\Theta=\frac{1}{2}\Vert dg_{ij}-\partial_iu_j+\partial_ju_i\Vert$ $(u_i=g_{ik}dx^k)$ | $\Gamma^i_{jk}=\frac{1}{2}g^{il}(\frac{\partial g_{lj}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}+\frac{\partial g_{lk}}{\partial x^j})$ |
| Torsion tensor<br>$T(\xi,\eta)=\nabla_\xi\eta-\nabla_\eta\xi-[\xi,\eta]$ | $T^i_{jk}=\Gamma^i_{kj}-\Gamma^i_{jk}$ |
| Curvature tensor<br>$R=d\Theta+\Theta\wedge\Theta=R^i_{jkl}dx^k\otimes dx^l$<br>$R(\xi,\eta)=\nabla_\xi\nabla_\eta-\nabla_\eta\nabla_\xi-\nabla_{[\xi,\eta]}$ | $R^i_{jkl}=\frac{\partial\Gamma^i_{jl}}{\partial x^k}-\frac{\partial\Gamma^i_{jk}}{\partial x^l}+\Gamma^i_{pk}\Gamma^p_{jl}-\Gamma^i_{pl}\Gamma^p_{jk}$<br>$R(\xi,\eta)=R^i_{jkl}\xi^k\eta^l$ |

## Differential Geometry
## Examination, 9.12.2005

1. [2] Prove the equality $R_{ijkl} = R_{klij}$ for the Riemann curvature tensor (assuming the other symmetries established).

2. [4] A closed space curve $C \subset \mathbb{R}^3$ has the curvature $k$ and the torsion $\varkappa$ (both depending on the point of the curve). Let $U_\varepsilon$ be the tubular $\varepsilon$-neighborhood of the curve. Prove that the volume of $U_\varepsilon$ is a polynomial in $\varepsilon$. Find the coefficients of this polynomial in terms of $k$ and $\varkappa$.

3. [4] Under the assumptions of the previous problem, let $S$ be the surface swept by the tangent lines of the curve. Find the Gaussian curvature of $S$ (at its smooth points).

4. [5] Sections of the trivial vector bundle with the fiber $\mathbb{R}^2 = \mathbb{C}$ over the punctured plane $M = \mathbb{R}^2 \setminus \{0\}$ can be considered as complex-valued functions. Consider the connection in this bundle given by the formula

$$\nabla_\xi u = \xi u + a \frac{\xi_1 + i\xi_2}{x_1 + ix_2} u,$$

   where $\xi = \xi_1 \partial_{x_1} + \xi_2 \partial_{x_2}$ and $a \in \mathbb{C}$ is a constant. Is this connection flat? Find the parallel translate along the circle of radius 2 centered at $(0, 1)$.

5. [6] Draw the geodesics of the metric $g - a\,dx^2 + dy^2$, $a = 1 - 3\,y + 9\,y^2$, issuing from the origin $x = y = 0$ in the three following directions: $dy/dx = 1, 2, 3$. Are there conjugated points to $(0, 0)$ along these geodesics? (The three curves must be infinitely prolongated and the difference between the three asymptotic behaviors must be described).