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Uniform approximation of partial sums of a Dirichlet series by shorter sums and Φ-widths

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Abstract. It is shown that each Dirichlet polynomial P of degree N which is bounded in a certain natural Euclidean norm, admits a nontrivial uniform approximation on the corresponding interval on the real axis by a Dirichlet polynomial with spectrum containing significantly fewer than N elements. Moreover, this spectrum is independent of P.

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§ 1. Introduction

In this paper we prove the following result.

Theorem 1. *Let $T \in \mathbb{R}$, $T \geqslant 1$, $N \in \mathbb{Z}$, $N \geqslant 1$, $T > (\log N)^{10}$ and let $\rho \in (0,1)$. Then there exists a set of integers*

$$\Lambda \subset \mathbb{Z} \cap \left[1, N\left(1 + \frac{(\log N)^7}{T}\right)\right]$$

of cardinality $|\Lambda| \leqslant \rho \min\{N, T\}$ such that for any polynomial

$$f(t) = \sum_{n=1}^{N} a_n n^{it} \tag{1}$$

satisfying

$$\sum_{n=1}^{N} |a_n|^2 \leqslant 1, \qquad \frac{1}{2T}\int_{-T}^{T} |f(t)|^2\,dt \leqslant 1 \tag{2}$$

there exists a polynomial

$$g(t) = \sum_{n \in \Lambda} b_n n^{it} \tag{3}$$

for which

$$\sup_{|t| \leqslant T/2} |g(t) - f(t)| \leqslant C \log^{15}(N+T) \cdot \rho^{-1/2}$$

(*here and throughout*, $C, c, C_1, \dots$ *are various absolute constants*).

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This result, which is a slight generalization of a theorem announced in [1], can be regarded as an estimate for some special kind of width, similar to the trigonometric width introduced in [2]. However, the main reason why we consider Theorem 1 is a certain analogy with some results playing an important role in analytic number theory, rather than this connection with estimates for widths. We mean here theorems claiming that a Dirichlet polynomial of the form (1) has a nontrivial uniform approximation on $[-T,T]$ by a much 'shorter' polynomial of the form (3), with $|\Lambda|$ significantly smaller than N. In applying the known results of this type to the theory of the Riemann zeta-function and other areas of analytic number theory the authors usually assume that the coefficients $\{a_n\}$ of (1) display some regular behaviour and prove that then the coefficients $\{b_n\}$ are also regular and, moreover, Λ is a fairly short segment of the natural number sequence (see [3], Ch. 3 and [4] for details). By contrast, Theorem 1 can be applied to arbitrary polynomials (1) satisfying the mild condition (2), and the set Λ is independent of the coefficients $\{a_n\}$, but on the other hand we have no control over the behaviour of $\{b_n\}$ or the structure of the set Λ.

Note that about 10 years ago Karatsuba was interested in finding general results on approximating Dirichlet polynomials by shorter ones.

For example, applying Theorem 1 to the zeta polynomial $\sum_{n=1}^N n^{it}$ yields the following result: for $\varepsilon > 0$ and $N = 1, 2, \dots$ there exists a polynomial $g(t)$ of the form (3) with $|\Lambda| \leqslant N^{1-\varepsilon}$ such that

$$\biggl|\sum_{n=1}^N n^{it} - g(t)\biggr| \leqslant CN^{1/2}|T|^{\varepsilon}, \qquad |t| \leqslant T, \quad 1 \leqslant N \leqslant \sqrt{T}.$$

In this connection we note that the conjectured estimate

$$\biggl|\sum_{n=1}^N n^{it}\biggr| \leqslant CN^{1/2}|t|^{\varepsilon}, \qquad 1 \leqslant N \leqslant \sqrt{|t|},$$

is equivalent to Lindelöf's classical conjecture (see [3] for details).

§ 2. Concepts and results relating to Theorem 1 and used in this paper

We use the standard notation: $\mathbb{R}$ and $\mathbb{Z}$ are the sets of real numbers and integers, respectively; $\mathbb{R}^N$ and $\mathbb{C}^N$, $N = 1, 2, \dots$, are the N-dimensional real and complex spaces, respectively. For $x = \{x_j\}_{j=1}^N \in \mathbb{C}^N$ and $1 \leqslant p < \infty$,

$$\|x\|_{L_p^N} = \biggl(\frac{1}{N}\sum_{j=1}^N |x_j|^p\biggr)^{1/p} \quad \text{and} \quad \|x\|_{L_\infty^N} = \|x\|_{l_\infty^N} = \max_{1\leqslant j\leqslant N} |x_j|.$$

Let B_2^N, $N = 1, 2, \dots$, be the unit ball in $(\mathbb{C}^N, \|\cdot\|_{L_2^N})$. We denote the distance of an element f of a normed space E from a set L by $\operatorname{dist}_E(f, L)$. Finally, $\operatorname{span}(\{\varphi_i\}_{i\in\Lambda})$ is the linear span of the system of elements $\{\varphi_i\}_{i\in\Lambda}$ in a linear space.

Recall the following definition.

Definition 1. The Kolmogorov n-width of a subset F of a normed space E is the quantity

$$d_n(F,E) = \inf_{L_n} \sup_{f\in F} \operatorname{dist}_E(f, L_n),$$

where inf is taken over all the n-dimensional subspaces of E.

In [2], in discussing the approximation of smooth functions by subspaces spanned by n elements of the trigonometric system, Ismagilov introduced the following idea.

Definition 2. The trigonometric n-width of a set $F \subset L^\infty(-\pi,\pi)$ is the quantity

$$d_n^T(F, L^\infty) = \inf_{G_n} \sup_{f\in F} \operatorname{dist}_{L^\infty}(f, G_n),$$

where inf is taken over all the subspaces

$$G_n = \operatorname{span}(\{e^{ikt}\}_{k\in\Lambda}), \qquad \Lambda \subset \mathbb{Z}, \quad |\Lambda| = n.$$

In a similar way we can define the Φ-width by taking a system of elements $\Phi = \{\varphi_k\}$ of a normed space E in place of the trigonometric system and the space $L^\infty(-\pi,\pi)$ and using only the subspaces spanned by n elements of Φ in the approximation. In Theorem 1 in this paper we look at the case when $\Phi = \{n^{it}\}_{n=1}^N \subset L^\infty(-T,T)$, and also at the case of an arbitrary system of discrete functions $\Phi = \{\varphi_k\}_{k=1}^n \subset L_\infty^N$ with $\|\varphi_k\|_{L_\infty^N} \leqslant K$, $k = 1,\dots,n$ (see Proposition 1 below).

Usually, in estimates for the widths of subclasses of functions in $L^\infty(-T,T)$ or $L^p(-T,T)$ the authors reduce the problem to estimates for the widths of finite-dimensional sets. In particular, in the most important case for applications, when the test for being a member of the subclass takes the form of an inequality for the norm in some Hilbert space (for example, see (2)), we arrive at the problem of finding estimates for $d_n(B_2^N, L_\infty^N)$ and $d_n^T(B_2^N, L_\infty^N)$ (in the second case we take the discrete trigonometric system).

Fairly precise estimates for the widths $d_n(B_2^N, L_\infty^N)$ were found in [5] (see also [6]). It turned out there that the ball B_2^N can be well-approximated in the metric of L_∞^N by a subspace of much smaller dimension than N. Much more elaborate techniques were required to estimate the widths $d_n^T(B_2^N, L_\infty^N)$. The first result, stating the existence of subspaces of $\mathbb{C}^N$ spanned by $n \leqslant (1-\delta)N$ elements of the discrete trigonometric system (where $\delta > 0$ is an absolute constant and $N = 2,3,\dots$) and approximating the ball B_2^N well in the metric of L_∞^N, was due to Bourgain and was subsequently refined by Talagrand [6]. The order of approximation by these subspaces is only worse by a logarithmic factor than the order of approximation by the subspaces giving the Kolmogorov width. More precisely, Bourgain's and Talagrand's theorems were stated in the dual form and claimed the existence of subspaces of $L^1(-\pi,\pi)$ of the form

$$\operatorname{span}\{e^{ikt}\}_{k\in\Lambda}, \qquad \Lambda \subset \{1,\dots,N\}, \quad |\Lambda| \geqslant \delta N,$$

such that the L^1 and L^2-norms of their elements differ by a logarithmic factor at worst.

From the standpoint of approximation theory it is natural to look at the case when the approximating subspace has a much smaller dimension than the approximated set.

For the Φ-widths connected with an arbitrary uniformly bounded orthonormal basis in L_2^N this case was considered by Guédon, Mendelson, Pajor and Tomczak-Jaegermann [8]. More precisely, in [8] they proved the following result, which is of importance to us.

Theorem A. *Let $\{\varphi_j\}_{j=1}^N$ be an orthonormal basis in L_2^N and $\|\varphi_j\|_{L_\infty^N} \leqslant K$, $j = 1,\dots,N$. Then for each integer m, $1 \leqslant m \leqslant N$, there exists a set $\Lambda \subset \{1,\dots,N\}$ such that $|\Lambda| = N - m$ and for any coefficients $\{a_j\} \in \mathbb{C}^N$*

$$\left(\sum_{j\in\Lambda} |a_j|^2\right)^{1/2} \leqslant C \cdot K(\log N)^2 \sqrt{\frac{N}{m}} \cdot \left\|\sum_{j\in\Lambda} a_j\varphi_j\right\|_{L_1^N}.$$

Theorem A and standard duality arguments readily yield an estimate for the Φ-width of the ball B_2^N in the norm of L_∞^N. Note also that from Theorem A we easily deduce the bound $d_n^T(W_2^r, L^\infty(-\pi,\pi)) \leqslant Cn^{-r}(\log n)^\gamma$, where W_2^r is the Sobolev class and γ is an absolute constant. The question of whether this inequality is valid has long been open.

In [7] and [8], in the proofs of the cited results the authors used modern versions of Kolmogorov's chaining method and fairly precise estimates for the ε-entropy of finite-dimensional compact sets. In this paper we use a similar scheme in the following result.

Proposition 1. *Let $\{\varphi_i\}_{i=1}^n \subset L_2^N$, $N \geqslant n$, be a set of elements such that*

$$\|\varphi_i\|_{L_\infty^N} \leqslant K, \qquad i = 1,2,\dots,n. \tag{4}$$

Then for each integer k, $1 \leqslant k \leqslant n$, there exists a subset Λ of $\{1,\dots,n\}$ such that

$$|\Lambda| = k \tag{5}$$

and

$$\sup_{\{a_i\}_{i=1}^n, \sum_1^n |a_i|^2 \leqslant 1} \operatorname{dist}_{L_\infty^N}\left(\sum_{i=1}^n a_i\varphi_i,\ \operatorname{span}(\{\varphi_i, i\in\Lambda\})\right) \leqslant CK(\log N)^{7/2}\sqrt{\frac{n}{k}}. \tag{6}$$

The difference between Proposition 1 and the results in [8] is that we eliminate the condition that the system $\{\varphi_i\}$ be orthogonal. To do this we require a general result on the duality of the entropy numbers of operators, which was proved in [9], Theorem 6.

We also point out that, because the system $\Phi = \{n^{it}\}_{n=1}^N$ has some special features, for the estimate of the Φ-widths which is in fact provided by Theorem 1, apart from geometric results we have to use several nontrivial facts from harmonic analysis.

§ 3. Estimates for Φ-widths

We derive Proposition 1 as a simple consequence of the following result of a dual nature.

Lemma 1. *Let $\{\varphi_i\}_{i=1}^n \subset L_2^N$ be a set of linearly independent functions satisfying (4). The for each integer k, $c(\log N)^{3/2} \leqslant k \leqslant n/2$ there exists a set $I \subset \{1,\dots,n\}$ such that*

$$|I| = n - k', \qquad |k-k'| \leqslant \frac{k}{10}, \tag{7}$$

and for each set of coefficients $a = \{a_i\}_{i=1}^n \in \mathbb{C}^n$ satisfying

$$\operatorname{supp} a = \{i : a_i \neq 0\} \subset I \tag{8}$$

the following inequality holds:

$$\|a\|_{l_2^n} = \biggl(\sum_I |a_i|^2\biggr)^{1/2} \leqslant C\cdot K(\log N)^{7/2}\sqrt{\frac{n}{k}}\,\|a\|_*, \tag{9}$$

where by definition

$$\|a\|_* = \max\biggl\{\biggl|\sum_{i=1}^n a_i\bar b_i\biggr| : \biggl\|\sum_{i=1}^n b_i\varphi_i\biggr\|_{L_\infty^N} \leqslant 1\biggr\}.$$

Before we prove Lemma 1, we make several observations.

a) It is clear from the definition that $(\mathbb{C}^n, \|\cdot\|_*)$ is the dual space of $(\mathbb{C}^n, |\!|\!|\cdot|\!|\!|)$, where for $b = \{b_i\}_{i=1}^n$ we set

$$|\!|\!|b|\!|\!| \equiv \Bigl\|\sum b_i\varphi_i\Bigr\|_{L_\infty^N}. \tag{10}$$

The space $(\mathbb{C}^n, |\!|\!|\cdot|\!|\!|)$ is isomorphic to the subspace of L_∞^N spanned by the discrete functions φ_i, $1 \leqslant i \leqslant n$.

It is well known that the type-2 constant of L_∞^N, and therefore also of the space $(\mathbb{C}^n, |\!|\!|\cdot|\!|\!|)$ has an upper bound in terms of $(\log N)^{1/2}$:

$$T_2(\mathbb{C}^n, |\!|\!|\cdot|\!|\!||) \leqslant T_2(L_\infty^N) \leqslant C(\log N)^{1/2} \tag{10$'$}$$

(the estimate (10′) is a simple consequence of inequality (27); as concerns the quantities $T_2(E)$, see, for instance, [10]).

b) If $\{\varphi_i\}_{i=1}^n$ is an orthonormal system in L_2^N, then

$$\|a\|_* \leqslant \Bigl\|\sum a_i\varphi_i\Bigr\|_{L_1^N}.$$

In fact, writing $\langle\cdot\,,\cdot\rangle$ for the inner product in L_2^N in this case we obtain

$$\biggl|\sum_{i=1}^n a_i\bar b_i\biggr| = \Bigl|\Bigl\langle\sum a_i\varphi_i, \sum b_i\varphi_i\Bigr\rangle\Bigr| \leqslant \Bigl\|\sum a_i\varphi_i\Bigr\|_{L_1^N}\cdot\Bigl\|\sum b_i\varphi_i\Bigr\|_{l_\infty^N}.$$

c) Below we use estimates for the ε-entropy of subsets of metric spaces. Recall the definition of the covering numbers $N(A,\varepsilon)$ and entropy numbers $e_s(A)$ of a compact subset A of a metric space E with metric γ:

$$N(A,\varepsilon)=\inf\biggl\{r: A\subset\bigcup_{j=1}^{r}B(y_j,\varepsilon)\text{ for some }y_j\text{ in }E,\ j=1,\dots,r\biggr\},$$

here $B(y,\varepsilon)=\{z\in E:\gamma(y,z)\leqslant\varepsilon\}$; and

$$e_s(A)=e_s(A,\gamma)=\inf\{\varepsilon: N(A,\varepsilon)\leqslant 2^s\},\qquad s=0,1,\dots .$$

If P is an operator from a Banach space X with unit ball B_X to a Banach space Y, then

$$e_s(P)\equiv e_s(P(B_X),\|\cdot\|_Y).$$

The numbers $N(P,\varepsilon)$ are defined in a similar way.

Proof of Lemma 1. For fixed $\rho>0$ let

$$F_\rho\equiv\{a\in\mathbb{C}^n:\|a\|_*\leqslant 1,\ \|a\|_2=\rho\},\tag{11}$$

where $\|a\|_2=\|a\|_{l_2^n}=(\sum_{i=1}^n|a_i|^2)^{1/2}$.

Similarly to [8] (see also [11]) we shall prove that there exists a set

$$\Lambda=\{i_1,\dots,i_{k'}\}\subset\{1,\dots,n\}$$

such that $|k-k'|\leqslant(1/10)k$, all the integers i_ν are distinct and

$$\sum_{i\in\Lambda}\biggl(\frac{\rho^2}{n}-|a_i|^2\biggr)\leqslant 0.4\frac{k}{n}\rho^2\tag{12}$$

for each $a\in F_\rho$, where

$$\rho=CK(\log N)^{7/2}\sqrt{\frac{n}{k}}\tag{13}$$

(the constant C is specified below). Once we have found such a set, for elements of F_ρ we obtain

$$\sum_{i\in\Lambda}|a_i|^2\geqslant|\Lambda|\frac{\rho^2}{n}-0.4\frac{k}{n}\rho^2\geqslant\frac{\rho^2}{n}(k_0'-0.4k)\geqslant\frac{\rho^2}{n}\,\frac{k}{2}>0,$$

so that F_ρ contains no vectors with support in $I=\{1,\dots,n\}\setminus\Lambda$. In other words,

$$\|a\|_*\geqslant\frac{1}{\rho}\|a\|_2\quad\text{if }\operatorname{supp}a\in I,$$

as required.

Note that we assume that if ρ is as given in (13), then F_ρ is nonempty; otherwise (9) holds for each a. Now fix some $a^0\in F_\rho$.

We seek a set Λ satisfying (12) using probability arguments. Let $\Omega = \{(0,1),\mu\}$ be a standard probability space (μ is Lebesgue measure) and let $\{\delta_i(\omega)\}_{i=1}^n$ be a set of independent, equidistributed random variables taking the values 0 and 1:

$$\mathsf{P}(\delta_i = 1) = \frac{k}{n} \qquad \mathsf{P}(\delta_i = 0) = 1 - \frac{k}{n}.$$

We look at a stochastic process indexed by points in F_ρ:

$$X_\omega(a) = \sum_{i=1}^{n} \left(\frac{\rho^2}{n} - |a_i|^2\right)\delta_i(\omega), \qquad a \in F_\rho. \tag{14}$$

We claim that the expectation satisfies the inequality

$$J \equiv \mathsf{E}_\omega \sup_{a\in F_\rho} X_\omega(a) \leqslant \frac{k}{3n}\rho^2 \tag{15}$$

Before we prove (15), we shall verify that it implies $\omega_0 \in \Omega$ exists such that

$$\begin{cases} \sup\limits_{a\in F_\rho} X_{\omega^0}(a) \leqslant 0.4\dfrac{k}{n}\rho^2 \\ |\{i : \delta_i(\omega_0) = 1\}| \equiv k_0' \in [0.9k, 1.1k] \end{cases} \tag{16}$$

Let $G = \{\omega \in \Omega : 0.9k \leqslant \sum_{i=1}^n \delta_i(\omega) \leqslant 1.1k\}$. Then by Bernstein's inequality (see, for instance, [12], Ch. 3)

$$\mu\{\Omega \setminus G\} < e^{-k/500}. \tag{17}$$

Furthermore, note that

$$X_\omega(a) \geqslant -\rho^2 \quad \text{for all} \;\; \omega \in \Omega,\; a \in F_\rho. \tag{18}$$

Hence it follows from (15) that

$$\int_G \sup_{a\in F_\rho} X_\omega(a)\, d\omega \leqslant \frac{k}{3n}\rho^2 + \rho^2\mu(\Omega\setminus G) \leqslant \rho^2\left(\frac{k}{3n} + e^{-k/500}\right) \leqslant 0.35\left(\frac{k}{n}\right)\rho^2. \tag{19}$$

(We have used the fact that $k \geqslant C(\ln N)^{3/2}$ is large if $N > 1$ and the constant C is sufficiently large.) It follows from (19) that

$$\min_{\omega\in G} \sup_{a\in F_\rho} X_\omega(a) \leqslant 0.35\frac{k}{n}\rho^2(\mu G)^{-1} \leqslant 0.4\frac{k}{n}\rho^2.$$

The last inequality and the definition of the set G yield (16). Now we return to the proof of (15). Note that if $a^0 \in F_\rho$ is the vector fixed above, then

$$\int_\Omega X_\omega(a^0)\, d\omega = 0.$$

Hence

$$J = \int_\Omega \Big\{ \sup_{a\in F_\rho} X_\omega(a) - X_\omega(a^0) \Big\}\, d\omega = \int_\Omega \sup_{a\in F_\rho} \{X_\omega(a) - X_\omega(a^0)\}\, d\omega$$
$$\leqslant \int_\Omega \sup_{a,a'\in F_\rho} \{X_\omega(a) - X_\omega(a')\}\, d\omega \leqslant \int_\Omega \sup_{a,a'\in F_\rho} |X_\omega(a) - X_\omega(a')|\, d\omega \equiv J^{(1)}. \quad (20)$$

Next,

$$J^{(1)} = \mathsf{E}_\omega \sup_{a,a'\in F_\rho} \Big| \sum_{i=1}^n \{|a_i'|^2 - |a_i|^2\}\delta_i(\omega) \Big|$$
$$= \mathsf{E}_\omega \sup_{a,a'\in F_\rho} \Big| \sum_{i=1}^n \{|a_i'|^2 - |a_i|^2\}(\delta_i(\omega) - \mathsf{E}\delta_i(\omega)) \Big|. \quad (21)$$

Let $\delta_i'(\omega')$ be an independent copy of the random variable δ_i, $1 \leqslant i \leqslant n$, which is defined on (Ω', μ'). Then from (21) we obtain

$$J^{(1)} = \mathsf{E}_\omega \sup_{a,b\in F_\rho} \Big| \sum_{i=1}^n \{|a_i|^2 - |b_i|^2\}\mathsf{E}_{\omega'}(\delta_i(\omega) - \delta_i'(\omega')) \Big|$$
$$\leqslant \mathsf{E}_\omega \sup_{a,b\in F_\rho} \mathsf{E}_{\omega'} \Big| \sum_{i=1}^n \{|a_i|^2 - |b_i|^2\}(\delta_i(\omega) - \delta_i'(\omega')) \Big|$$
$$\leqslant \mathsf{E}_\omega \mathsf{E}_{\omega'} \sup_{a,b\in F_\rho} \Big| \sum \{|a_i|^2 - |b_i|^2\}(\delta_i(\omega) - \delta_i'(\omega')) \Big|. \quad (22)$$

Using again the standard symmetrization trick (see [11], pp. 9 and 34) we transform the right-hand side of (22) into

$$\mathsf{E}_\omega \mathsf{E}_{\omega'} \mathsf{E}_{\omega''} \sup_{a,b\in F_\rho} \Big| \sum \{|a_i|^2 - |b_i|^2\}\varepsilon_i(\omega'')(\delta_i(\omega) - \delta_i'(\omega')) \Big|, \quad (23)$$

where $\{\varepsilon_i(\omega''),\ i = 1, \ldots, n\}$ is a family of independent random variables defined on (Ω'', μ'') and taking the values $+1$ and -1 with equal probability $1/2$. More precisely, we have used the fact that the independent random variables $\delta_i(\omega) - \delta_i'(\omega')$, $i = 1, \ldots, n$, on $\Omega \times \Omega'$ and $\varepsilon_i(\omega'')(\delta_i(\omega) - \delta_i'(\omega'))$ on $\Omega \times \Omega' \times \Omega''$ have the same distribution.

Clearly, the quantity in (23) has the estimate

$$2\mathsf{E}_\omega \mathsf{E}_{\omega''} \sup_{a,b\in F_\rho} \Big| \sum_{i=1}^n \{|a_i|^2 - |b_i|^2\}\varepsilon_i(\omega'')\delta_i(\omega) \Big|. \quad (24)$$

As a result (see (22) and (24)) we see that

$$J^{(1)} \leqslant 2\mathsf{E}_\omega \mathsf{E}_{\omega''} \sup_{a,b\in F_\rho} \Big| \sum_{i=1}^n \{|a_i|^2 - |b_i|^2\}\varepsilon_i(\omega'')\delta_i(\omega) \Big|$$
$$= 4\mathsf{E}_\omega \mathsf{E}_{\omega''} \sup_{a\in F_\rho} \sum_{i=1}^n |a_i|^2 \varepsilon_i(\omega'')\delta_i(\omega) \equiv J^{(2)} \quad (25)$$

(to verify the last equality we have used the fact that the stochastic process in question is symmetric; see also [12], Lemma 1.2.8). The quantity $J^{(2)}$ can be estimated using Dudley's inequality (see below), which is one of the main tools in the chaining method.

For fixed $\omega \in \Omega$ we look at the stochastic process

$$Y_{\omega''}(a) = \sum_{i=1}^{n} |a_i|^2 \varepsilon_i(\omega'')\delta_i(\omega).$$

It is subgaussian with respect to the following pseudometric on F_ρ:

$$\gamma(a,b) = \gamma_\omega(a,b) = \biggl(\sum_{i=1}^{n} \delta_i(\omega)\Bigl(\sum |a_i|^2 - |b_i|^2\Bigr)\biggr)^{1/2},$$

that is, for $y > 0$ we have

$$\mathbb{P}\{|Y_{\omega''}(a) - Y_{\omega''}(b)| > y\} \leqslant 2\exp\biggl(-\frac{y^2}{2\gamma^2(a,b)}\biggr). \tag{26}$$

Relation (26) is a direct consequence of the following well-known estimate for polynomials in the Rademacher system (see [15], Ch. 2):

$$\mu\biggl\{t \in (0,1) : \Bigl|\sum a_i r_i\Bigr| > y\Bigl(\sum |a_i|^2\Bigr)^{1/2}\biggr\} \leqslant 2\exp\biggl(-\frac{y^2}{2}\biggr). \tag{27}$$

Dudley's inequality (see [14], § 1, inequality (1.16)) for the process $Y_{\omega''}(a)$ reduces to the estimate

$$\mathsf{E}_{\omega''} \sup_{a\in F_\rho} Y_{\omega''}(a) \leqslant C \sum_{s\geqslant 0} 2^{s/2} e_{2^s}(F_\rho, \gamma_\omega), \tag{28}$$

so that (see (25)),

$$J^{(2)} \leqslant C\mathsf{E}_\omega\biggl(\sum_{s\geqslant 0} 2^{s/2} e_{2^s}(F_\rho,\gamma_\omega)\biggr).$$

For $\omega \in \Omega$ we set $H_\omega = \{i : \delta_i(\omega) = 1\}$. Then

$$\begin{aligned}\gamma_\omega(a,b) &\leqslant \biggl(\sum_{i\in H_\omega} (|a_i| - |b_i|)^2(|a_i| + |b_i|)^2\biggr)^{1/2} \\ &\leqslant 2\|a-b\|_{l_\infty(H_\omega)} \cdot \biggl(\sup_{a\in F_\rho} \sum_{i\in H_\omega} |a_i|^2\biggr)^{1/2} = 2\|a-b\|_{l_\infty(H_\omega)} Z_\omega^{1/2},\end{aligned} \tag{29}$$

where $Z_\omega = \sup_{a\in F_\rho} \sum_{i=1}^{n} |a_i|^2 \delta_i(\omega)$.

From (29) we obtain an inequality for the entropy numbers:

$$e_{2^s}(F_\rho,\gamma_\omega) \leqslant e_{2^s}(F_\rho \|\cdot\|_{l_\infty(H_\omega)}) \cdot Z_w^{1/2} \leqslant e_{2^s}(F_\rho, \|\cdot\|_{l_\infty^n}) \cdot Z_w^{1/2}, \tag{30}$$

so setting

$$\Sigma = \sum_{s\geqslant 0} 2^{s/2} e_{2^s}(F_\rho, \|\cdot\|_{l_\infty^n}) \quad \text{and} \quad Z = \mathsf{E}_\omega Z_\omega, \tag{31}$$

from (28) and (30) we obtain

$$J^{(2)} \leqslant C\Sigma \mathsf{E}_\omega Z_\omega^{1/2} \leqslant C\Sigma Z^{1/2}. \tag{32}$$

On the other hand (see (20), (22), (25))

$$J^{(2)} \geqslant J^{(1)} \geqslant \mathsf{E}_\omega \sup_{a\in F_\rho} |X_\omega(a) - X_\omega(a^0)| = \mathsf{E}_\omega \sup_{a\in F_\rho} \Bigl|\sum \delta_i(\omega)(|a_i|^2 - |a_i^0|^2)\Bigr|$$

$$\geqslant \mathsf{E}_\omega \sup_{a\in F_\rho} \sum_{i=1}^n \delta_i(\omega)|a_i|^2 - \mathsf{E}_\omega\biggl(\sum_{i=1}^n \delta_i(\omega)|a_i^0|^2\biggr) \geqslant Z - \rho^2\frac{k}{n}. \tag{33}$$

Thus (see (32) and (33))

$$Z - \rho^2\frac{k}{n} \leqslant C\Sigma Z^{1/2},$$

which means that

$$Z \leqslant \max\biggl\{2\rho^2\frac{k}{n}, 2C\Sigma Z^{1/2}\biggr\}. \tag{34}$$

Hence (see also (32), (20) and (25))

$$J \leqslant J^{(2)} \leqslant C \max\biggl\{\Sigma\rho\sqrt{\frac{k}{n}}, \Sigma^2\biggr\}. \tag{35}$$

Finally we find an estimate for the sum Σ. Since $F_\rho \subset B_*$, where B_* is the unit ball in $(\mathbb{C}^n, \|\cdot\|_*)$, we have

$$e_q(F_\rho, l_\infty^n) \leqslant e_q(B_*, l_\infty^n). \tag{36}$$

It is easy to see that if $\|x\|_{l_\infty^n} \geqslant 1$, then $\|x\|_* \geqslant 1/K$, that is,

$$B_* \subset KB_\infty^n \equiv K\cdot\{x\in\mathbb{C}^n : \|x\|_{l_\infty^n} \leqslant 1\}. \tag{37}$$

Hence $e_q(B_*, l_\infty^n) \leqslant K$, $q = 0, 1, \dots$, and the entropy numbers $e_q(B_*, l_\infty^n)$ are exponentially decreasing for $q > n$:

$$e_q(B_*, l_\infty^n) \leqslant CK\exp\biggl(-\frac{1}{10}\frac{q}{n}\biggr). \tag{38}$$

Another estimate for the entropy numbers is obtained using duality. Let P be the identity map in $\mathbb{C}^n$ regarded as an operator

$$P\colon (\mathbb{C}^n, \|\cdot\|_*) \to (\mathbb{C}^n, \|\cdot\|_{l_\infty^n}).$$

Clearly,

$$e_q(B_*, l_\infty^n) = e_q(P), \qquad q = 0, 1, \dots. \tag{39}$$

We look at the conjugate operator

$$P^*\colon (\mathbb{C}^n, \|\cdot\|_{l_1^n}) \to (\mathbb{C}^n, |||\cdot|||),$$

where the norm $|||\cdot|||$ is defined in (10). It is also clear that $\|P^*\| \leqslant K$ and $\|P\| \leqslant K$. For the entropy numbers of P^* we have Carl's inequality [16]

$$e_q(P^*) \leqslant \frac{C\|P^*\| \cdot T_2(\mathbb{C}^n, |||\cdot|||)}{q^{1/2}} \log^{1/2}\biggl(1+\frac{n}{q}\biggr), \quad 1 \leqslant q \leqslant C'n,$$

where $T_2(\mathbb{C}^n, |||\cdot|||)$ is the type-2 constant of the space $(\mathbb{C}^n, |||\cdot|||)$. Taking (10′) into account, from the last inequality we obtain

$$e_q(P^*) \leqslant \frac{C\cdot K(\log N)^{1/2}(\log n)^{1/2}}{q^{1/2}}, \qquad 1 \leqslant q \leqslant C'n. \tag{40}$$

By Theorem 6 in [9], for any operator $U\colon X \to Y$ such that $\|U\| \leqslant 1$, $\operatorname{rank} U \leqslant n$, and for $\varepsilon \in (0,1)$ the covering numbers $N(U,\varepsilon)$ have the estimate

$$\log N(U,\varepsilon) \leqslant C\biggl(\log\frac{n}{\varepsilon}\biggr)\cdot \log N\biggl(U^*, \frac{C^{-1}\varepsilon}{\log(n/\varepsilon)}\biggr) \tag{41}$$

with an absolute constant C.

If $\log 1/\varepsilon \leqslant C\log n$, then it follows from (41) that

$$\log N(U, C\varepsilon\log n) \leqslant C'\log n \,\log N(U^*,\varepsilon). \tag{42}$$

Going over to entropy numbers, for the operator $U = P/K$ we obtain from (42)

$$e_q(P) \leqslant C\log n\biggl(e_{[cq/\log n]}(P^*) + \frac{K}{n}\biggr). \tag{43}$$

In view of (40), we see from (43) that

$$e_q(P) \leqslant \frac{CK(\log N)^{1/2}(\log n)^2}{q^{1/2}} \leqslant \frac{CK(\log N)^{5/2}}{q^{1/2}} \tag{44}$$

for $q = 1,2,\dots,q(n)$, $q(n) \equiv [C''n\log n]$.

Using (44) and (38) (see also (36) and (39)) we find an estimate for Σ defined in (31):

$$\Sigma \leqslant \sum_{s\colon 2^s \leqslant q(n)} CK(\log N)^{5/2} + \sum_{s\colon 2^s > q(n)} K2^{s/2}\exp\biggl(-\frac{2^s}{10n}\biggr) \leqslant C'K(\log N)^{7/2}. \tag{45}$$

Hence (see (35)),

$$J \leqslant C'\max\biggl\{K(\log N)^{7/2}\rho\sqrt{\frac{k}{n}},\ K^2(\log N)^7\biggr\}. \tag{46}$$

Thus if ρ is defined in (13) and the absolute constant C in (13) is sufficiently large, then we have the required inequality (15): $J \leqslant k/(3n)\rho^2$. Indeed, in this case

$$\frac{C^2}{3}K^2(\log N)^7 \geqslant \max\bigl\{C'K(\log N)^{7/2}\cdot CK(\log N)^{7/2},\, C'K(\log N)^7\bigr\}.$$

Lemma 1 is proved.

In deducing Proposition 1 from Lemma 1 we can assume that the vectors φ_i, $i = 1, \dots, n$, are linearly independent. Otherwise we can go over to a system of linearly independent vectors $\{\widetilde{\varphi}_i\}_{c=1}^n \subset L_2^{N+n}$ such that the projection of $\widetilde{\varphi}_i$ onto the first N coordinates is equal to φ_i. We can also assume that $k \geqslant C(\log N)^{3/2}$ since otherwise (6) is trivial in view of the inequality

$$\operatorname{dist}_{L_\infty^N}\biggl(\sum_{i=1}^n a_i\varphi_i, 0\biggr) = \biggl\|\sum_{i=1}^n a_i\varphi_i\biggr\|_{L_\infty^N} \leqslant K\sum_{i=1}^n |a_i| \leqslant K\sqrt{n}\,.$$

Now using Lemma 1 for $\widetilde{k} = [0.9k] + 1$ we find $I \subset \{1, \dots, n\}$ with $|I| = n - k$ such that $\|a\|_{l_2^n} \leqslant CK(\log N)^{7/2}\sqrt{n/k}\,\|a\|_*$ if $\operatorname{supp} a \subset I$. Then $\Lambda = \{1, \dots, n\} \setminus I$ satisfies the assumptions of Proposition 1.

In fact, making the assumption that inequality (6) fails for some vector $\sum_{i=1}^n a_i\varphi_i$ with $\sum |a_i|^2 \leqslant 1$ we can find $f \in l_1^N$ such that

$$\sum_{i=1}^n a_i(\varphi_i, f) = 1, \quad \|f\|_{l_1^N} < C^{-1}K^{-1}(\log N)^{-7/2}\sqrt{\frac{k}{n}},$$

$$(\varphi_i, f) = 0 \quad \text{for} \quad i \in \Lambda$$

(we have used the fact that l_1^N and l_∞^N are dual spaces; the symbol $(\,\cdot\,,\,\cdot\,)$ denotes the standard scalar product in $\mathbb{C}^N$). Hence from Lemma 1 we obtain

$$1 = \sum_{i\in I} a_i(\varphi_i, f) \leqslant \biggl(\sum_{i\in I} |(f, \varphi_i)|^2\biggr)^{1/2} \leqslant CK(\log N)^{7/2}\sqrt{\frac{n}{k}}\|v\|_*,$$

where v is the restriction of the vector $\{(f, \varphi_i)\}$ to I:

$$v_i = \begin{cases} (f, \varphi_i) & \text{if } i \in I \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of the norm $\|\cdot\|_*$

$$\|v\|_* = \sup_{\|\sum b_i\varphi_i\|_{l_\infty^N} \leqslant 1} \biggl|\sum_{i\in I} (f, \varphi_i)\bar{b}_i\biggr| = \sup_{\|\sum b_i\varphi_i\|_{l_\infty^N} \leqslant 1} \biggl|\biggl(f, \sum_{i=1}^n b_i\varphi_i\biggr)\biggr| \leqslant \|f\|_{l_1^N}$$

which leads to a contradiction. Proposition 1 and the standard discretization procedure enable us to prove the following result.

Proposition 2. *Let $\{\varphi_i\}_{i=1}^n$ be C^1 functions on the interval $[T_1, T_2] \subset \mathbb{R}$ such that for $i = 1, 2, \dots, n$*

$$\|\varphi_i\|_{L^\infty(T_1,T_2)} \leqslant K, \qquad \|\varphi_i'\|_{L^\infty(T_1,T_2)} \leqslant D. \tag{47}$$

Then for each integer k, $1 \leqslant k \leqslant n$, there exists a set $\Lambda \subset \{1, \dots, n\}$ such that

$$|\Lambda| \leqslant k \tag{48}$$

and for each set of numbers $\{a_i\}_{i=1}^n$ such that $(\sum_{i=1}^n |a_i|^2)^{1/2} \leqslant 1$,

$$\operatorname{dist}_{L^\infty(T_1,T_2)}\biggl(\sum_{i=1}^n a_i\varphi_i, \operatorname{span}\{\varphi_i, i \in \Lambda\}\biggr) \leqslant C(K+1)\bigl[\log(n+Dn(T_2-T_1))\bigr]^{7/2}\sqrt{\frac{n}{k}}.$$

Proof. For $n > 1$ let $Q = [nD] + 1$ and

$$E = \left\{ \frac{r}{Q} : r \in \mathbb{Z},\ \frac{r}{Q} \in [T_1, T_2] \right\}.$$

Then $|E| \leqslant Q[T_2 - T_1] + 1$. Also let $E^* = E \cup V$, where $V = \{v_j\}_{j=1}^n$ is an n-point set, $V \cap E = \varnothing$. Consider a system of functions $\{\widetilde{\varphi}_i\}_{i=1}^n$ on E^* such that

$$\widetilde{\varphi}_i(x) = \varphi_i(x) \quad \text{for } x \in E, \qquad \widetilde{\varphi}_i(\widetilde{v}_j) = \delta_{ij}, \quad 1 \leqslant j \leqslant n$$

(δ_{ij} is the Kronecker delta). Applying Proposition 1 to $\{\widetilde{\varphi}_i\} \subset L^\infty(E^*)$ we find a set $\Lambda \subset \{1, \dots, n\}$ such that $|\Lambda| = k$ and (6) holds. Hence for each set $\{a_i\}_{i=1}^n$, $(\sum |a_i|^2)^{1/2} \leqslant 1$, there exists a set $b = \{b_i\}_{i=1}^n$ such that $\operatorname{supp} b \subset \Lambda$ and

$$\begin{aligned}\left\| \sum_{i=1}^n a_i \widetilde{\varphi}_i - \sum_{i=1}^n b_i \widetilde{\varphi}_i \right\|_{L^\infty(E^*)} &\leqslant \max_{x \in E} \left| \sum_{i=1}^n (a_i - b_i)\varphi_i(x) \right| + \max_{1 \leqslant i \leqslant n} |a_i - b_i| \\ &\leqslant C(K+1) \log^{7/2} |E^*| \sqrt{\frac{n}{k}} = S. \end{aligned} \tag{49}$$

Since for any $t \in [T_1, T_2]$ there exists $x \in E$ with $|t - x| \leqslant 1/Q$, from (47) and (49) we obtain

$$\begin{aligned}\left| \sum_{i=1}^n (a_i - b_i)\varphi_i(t) \right| &\leqslant \left| \sum_{i=1}^n (a_i - b_i)\varphi_i(x) \right| + \frac{1}{Q} \sum_{i=1}^n |a_i - b_i| \|\varphi_i'\|_{L^\infty(T_1,T_2)} \\ &\leqslant S + \frac{n}{Q} SD = 2S, \end{aligned}$$

as required.

§ 4. The proof of Theorem 1

Lemma 2. *For $J = 1, 2, \dots$ there exists a nonnegative even trigonometric polynomial*

$$K(\theta) = K_J(\theta) = \sum_{|j| \leqslant J} \widehat{k}(j) e^{2\pi i j\theta} \tag{50}$$

such that

$$\int_{-1/2}^{1/2} K(\theta) = 1$$

and

$$K(\theta) \leqslant CJ \exp(-10(J|\theta|)^{1/2}), \qquad -\frac{1}{2} \leqslant \theta \leqslant \frac{1}{2}. \tag{51}$$

The result of Lemma 2 follows easily from Theorem 1 in [17].

Lemma 3. *Let $\lambda \in \mathbb{R}$, $|\lambda| < 10$, and $J = 1, 2, \dots$. Then there exists a trigonometric polynomial*

$$f(x) = \sum_{|j| \leqslant J} \widehat{f}(j) e^{2\pi i j x}, \tag{52}$$

such that $f(0) = 1$ and

a) $|f(x) - e^{i\lambda x}| < Ce^{-J^{1/2}}$ *for* $|x| \leqslant 1/4$,
b) $\|f'\|_{L^\infty(-1/2,1/2)} \leqslant C$,

and therefore

c) $\sum |\widehat{f}(j)| \leqslant C'$.

Proof. Let $\gamma = \int_{-1/2}^{1/2} e^{-i\lambda u} K(u)\, du$, where $K = K_J(u)$ is the polynomial in Lemma 2.

Then we readily verify that

$$\gamma = 1 + O\left(\frac{1}{J}\right). \tag{53}$$

Also let $\varphi\colon \mathbb{R} \to [0,1]$ be a smooth function with $\operatorname{supp}\varphi \subset [1/2, 1/2]$ such that $\varphi = 1$ on $[-3/8, 3/8]$ and let

$$f_1(x) = \gamma^{-1} \int_{-1/2}^{1/2} e^{i\lambda s} K(x-s)\varphi(s)\, ds.$$

Clearly, $f(x)$ has the form (52). Using (51), for $|x| \leqslant 1/4$ we obtain

$$\begin{aligned} |e^{i\lambda x} - f_1(x)| &\leqslant 2\left|\gamma - \int_{-1/2}^{1/2} e^{-i\lambda(x-s)} K(x-s)\varphi(s)\, ds\right| \\ &= 2\left|\gamma - \int e^{-i\lambda u} K(u)\varphi(x-u)\, du\right| \leqslant Ce^{-\sqrt{J}}. \end{aligned}$$

Furthermore, integrating by parts we see that

$$\begin{aligned} \sum_{j\in\mathbb{Z}} |\widehat{f}_1(j)| &\leqslant C\|f_1\|_{L^\infty} + \|f_1'\|_{L^\infty} \\ &\leqslant C\left(\|K\|_{L^1} + \left\|\int e^{i\lambda s} K'(x-s)\varphi(s)\, ds\right\|_{L^\infty}\right) \leqslant C'\|K\|_1. \end{aligned}$$

The function $f(x) = f_1(x) - f_1(0) + 1$ satisfies all the assumptions of Lemma 3.

In what follows A denotes the space of functions with absolutely convergent Fourier series on $[-1/2\ 1/2]$, with the norm

$$\|f\|_A \equiv \|\widehat{f}\|_1 \equiv \sum_{j\in\mathbb{Z}} |\widehat{f}(j)|,$$

where $\widehat{f} = \{\widehat{f}(j)\}_{j\in\mathbb{Z}}$ and the $\widehat{f}(j) = \int_{-1/2}^{1/2} e^{-2\pi ijt} f(t)\, dt$ are the Fourier coefficients of f.

Lemma 4. *For fixed* $\delta > 0$ *there exists a function* $v(x) \in A$ *such that*

1) $e^{2\pi ix} - 1 = v(x)$ *for* $|x| \leqslant \delta$;
2) $\|v\|_A \leqslant C\delta$;
3) $|\widehat{v}(j)| \leqslant C\exp(-(\delta|j|)^{1/2})$, $j \in \mathbb{Z}$.

Proof. For $\delta > 1/8$ it is sufficient to set $v(x) = e^{2\pi ix} - 1$.

For $\delta \leqslant 1/8$ let φ_δ be a smooth function with $\|\varphi_\delta\|_{L^\infty} \leqslant 1.5$ such that

a) $\varphi_\delta(x) = 1$ if $|x| \leqslant \delta$;

b) $\varphi_\delta(x) = 0$ if $|x| \geqslant 2\delta$;

c) $|\widehat{\varphi_\delta}(j)| \leqslant C\exp(-(\delta|j|)^{1/2})$.

(To construct φ_δ we can take a function $\varphi_{1/8}$ with properties a) and b) in the Gevrey class $J_{3/2}$. Then c) follows from the well-known estimates for the Fourier coefficients of functions in the Gevrey class; see [18]. Finally, we set $\varphi_\delta(t) = \varphi_{1/8}(t/(8\delta))$.)

Let

$$v(x) = \varphi_\delta(x)(e^{2\pi ix} - 1).$$

Then $v(x)$ satisfies conditions 1) and 3) in the lemma. Moreover, $\|v\|_{L^2} \leqslant C\delta^{3/2}$ and $\|v'\|_{L^2} \leqslant C\delta^{1/2}$. Hence (see [19], Appendix 2)

$$\|\widehat{v}\|_1 \leqslant C\|v\|_{L^2}^{1/2} \cdot \|v'\|_{L^2}^{1/2} \leqslant C\delta.$$

The proof of the lemma is complete.

Lemma 5. *Fix a trigonometric polynomial*

$$f(x) = \sum_{|j|\leqslant J} \widehat{f}(j)e^{2\pi ijx}$$

and $\delta' \leqslant 1/J$. Then there exists a function $g \in A$ such that

1) $f(x) - f(0) = g(x)$ *for* $|x| \leqslant \delta'$;

2) $\|\widehat{g}\|_1 \leqslant C\delta' J\|\widehat{f}\|_1$;

3) $|\widehat{g}(n)| \leqslant C\|\widehat{f}\|_1 e^{-(\delta' n)^{1/2}}$, $n \in \mathbb{Z}$.

Proof. Let $\delta = J\delta'$ and let $v = v_\delta$ be the function in Lemma 4. Then

$$e^{2\pi ijx} - 1 = v(jx), \qquad |j| \leqslant J, \quad |x| \leqslant \delta'.$$

Hence

$$f(x) - f(0) = \sum_{|j|\leqslant J} \widehat{f}(j)v(jx) \equiv g(x)$$

for $|x| \leqslant \delta'$.

By property 2) in Lemma 4

$$\|\widehat{g}\|_1 \leqslant \sum_{|j|\leqslant J} |\widehat{f}(j)|\, \|\widehat{v}\|_1 \leqslant C\delta' J\|\widehat{f}\|_1,$$

and by property 3)

$$|\widehat{g}(n)| \leqslant \sum_{j|n,\ |j|\leqslant n} |\widehat{f}(j)| \left|\widehat{v}\left(\frac{n}{j}\right)\right| \leqslant C\|\widehat{f}\|_1 \exp(-(\delta'|n|)^{1/2}), \qquad n \in \mathbb{Z}.$$

The proof is complete.

Lemma 6. *Let J be a positive integer and M and T real numbers such that $M \geqslant JT$ and $T \geqslant J$. Then for each $n = 1, 2, \dots$ there exists a trigonometric polynomial*

$$\sigma_n(t) = \sum \widehat{\sigma}_n(k)e^{2\pi ikt}, \qquad \sigma_n(0) = 1,$$

such that

1) $\operatorname{supp}\sigma_n \equiv \{k : \widehat{\sigma}_n(k) \neq 0\} \subset \{k : |k - (M\log n)/(2\pi)| \leqslant J\}$;
2) $\|\widehat{\sigma}_n\|_1 \leqslant C$;
3) $|n^{it} - \sigma_n(t/M)| \leqslant Ce^{-\sqrt{J}}$, $|t| \leqslant T$.

Proof. If J is small the assertion is obvious, so we shall assume below that $J > 5$. We have

$$n^{it} = e^{(2\pi i k_0 t)/M} \cdot e^{i\lambda(t/M)}, \tag{54}$$

where $k_0 = [(M\log n)/(2\pi)]$ and $\lambda = 2\pi\{(M\log n)/(2\pi)\}$ ($\{x\}$ is the fractional part of the number x). Let $f(t)$ be the polynomial of degree $J-1$ provided by Lemma 3. We set

$$\sigma_n(t) = e^{2\pi i k_0 t} f(t).$$

Then property 1) is obvious, while 3) follows from the bound c) in Lemma 3. Now,

$$\sigma_n\biggl(\frac{t}{M}\biggr) = e^{(2\pi i t k_0)/M} f\biggl(\frac{t}{M}\biggr),$$

so that (see (54))

$$\biggl|n^{it} - \sigma_n\biggl(\frac{t}{M}\biggr)\biggr| \leqslant Ce^{-\sqrt{J}}$$

for $|t/M| \leqslant 1/4$, that is, for $|t| \leqslant M/4$, and therefore for $|t| \leqslant T$. The proof is complete.

The result of the following lemma is converse to Lemma 6 in a certain sense.

Lemma 7. *Let $J \in \mathbb{Z}$, $M \in \mathbb{R}$ and $T \in \mathbb{R}$ be positive numbers such that $M \geqslant C_0 JT$ (here C_0 is a sufficiently large absolute constant), $T \geqslant J^3$ and $J \geqslant \log^2 M/T$. Also fix $k_0 \in \mathbb{Z}$, $k_0 \geqslant (M\log M)/(2\pi)$. Then there exists a Dirichlet polynomial*

$$P(t) = \sum_{n\in\Lambda} c_n n^{it}$$

such that

$$|e^{2\pi i k_0 (t/M)} - P(t)| < Ce^{-\sqrt{J}}, \qquad |t| \leqslant T,$$

and

a) $\Lambda \subset \biggl\{n : \biggl|k_0 - \dfrac{M\log n}{2\pi}\biggr| \leqslant CJ^3(M/T)\biggr\}$;
b) $|\Lambda| \leqslant CJ^3(M/T)$;
c) $\sum_{n\in\Lambda} |c_n| \leqslant 2$.

Proof. We can assume that J is sufficiently large. Fix a positive integer $n = n_{(k_0)}$ such that

$$\biggl[\frac{M\log n}{2\pi}\biggr] = k_0. \tag{55}$$

Then by Lemma 6 there exists a polynomial $f(t) = \sum_{|j|\leqslant J} \widehat{f}(j) e^{2\pi i j t}$ such that $f(0) = 1$ and

$$n^{it} = e^{(2\pi i t k_0)/M}\biggl(f\biggl(\frac{t}{M}\biggr) + \Delta(t)\biggr), \tag{56}$$

where $|\Delta(t)| \leqslant C\exp(-J^{1/2})$ for $|t| \leqslant T$. We set

$$\sigma_n(t) = e^{2\pi i k_0 t} \cdot f(t).$$

Now we use Lemma 5, which states that for $\delta' = T/M \leqslant 1/J$ there exists a function $g(t) \in A$ such that

1) $f(t) = 1 + g(t)$ for $|t| \leqslant \delta'$;

2) $\|\widehat{g}\|_1 \leqslant C\delta' J \|\widehat{f}\|_1 < 1/10$

(provided that the constant C_0 in the statement of Lemma 7 is sufficiently large);

3) $|\widehat{g}(s)| \leqslant C\|\widehat{f}\|_1 \exp -(\delta' \cdot |s|)^{1/2}$, $s \in \mathbb{Z}$.

Let

$$\widetilde{g}(t) = \sum_{|s|<CJ(M/T)} \widehat{g}(s) e^{2\pi i s t},$$

where we choose a sufficiently large C so that

$$|g(t) - \widetilde{g}(t)| \leqslant C \sum_{|s|>CJ(M/T)} \exp\biggl(-\Bigl(\frac{T}{M}|s|\Bigr)^{1/2}\biggr) \leqslant e^{-J^{1/2}} \tag{57}$$

(here, apart from the rapid decay of the coefficients $\widehat{g}(s)$, we have taken account of the inequality $J \geqslant \log^2 M/T$).

Hence

$$f(t) = 1 + \widetilde{g}(t) + \Delta(t), \qquad |\Delta(t)| \leqslant e^{-J^{1/2}}, \quad |t| \leqslant \frac{T}{M},$$

and

$$e^{2\pi i k_0 t} = \frac{\sigma_n(t)}{f(t)} = \frac{\sigma_n(t)}{1+\widetilde{g}(t)} + \Delta'(t) = \sigma_n(t) - \frac{\widetilde{g}(t)}{1+\widetilde{g}(t)}\sigma_n(t) + \Delta'(t), \tag{58}$$

where $|\Delta'(t)| \leqslant 2\exp(-J^{1/2})$ for $|t| \leqslant T/M$.

Looking at the expansion

$$\frac{\widetilde{g}(t)}{1+\widetilde{g}(t)} = \sum_{l\geqslant 0} (-1)^{l+1} (g(t))^{l+1}$$

and bearing in mind that

$$\|\widetilde{g}\|_{L^\infty}^{l+1} \leqslant \|\widehat{\widetilde{g}}\|_1^{l+1} \leqslant \|\widehat{g}\|_1^{l+1} \leqslant 10^{-(l+1)},$$

we see that the trigonometric polynomial

$$g_1(t) = \sum_{l=0}^{J-1} (-1)^l (\widetilde{g}(t))^{l+1}$$

satisfies

$$\biggl\| \frac{\widetilde{g}(t)}{1+\widetilde{g}(t)} - g_1(t) \biggr\|_{L^\infty} \leqslant 10^{-J}.$$

Moreover,

$$\|\widehat{g}_1\|_1 \leqslant 2\|\widehat{g}\|_1 \leqslant C\frac{T}{M}J < \frac{1}{10} \tag{59}$$

and

$$\operatorname{supp}\widehat{g}_1 \subset \left[-CJ^2\frac{M}{T}, CJ^2\frac{M}{T}\right].$$

Hence setting

$$g_{(k_0)} = \sigma_n \cdot g_1$$

we obtain

$$e^{2\pi i k_0 t} = \sigma_n(t) - g_{(k_0)}(t) + \Delta(t), \qquad |\Delta(t)| < 3e^{-\sqrt{J}}, \quad |t| \leqslant \frac{T}{M}$$

(see (58) and (59)). That is,

$$e^{2\pi i k_0 (t/M)} = \sigma_n\left(\frac{t}{M}\right) - g_{(k_0)}\left(\frac{t}{M}\right) + \Delta'(t), \qquad |\Delta'(t)| < 3e^{-\sqrt{J}}, \quad |t| \leqslant T.$$

By (56) it follows from the last relation that

$$e^{2\pi i k_0 (t/M)} = n^{it} - g_{(k_0)}\left(\frac{t}{M}\right) + \Delta(t), \qquad |\Delta(t)| \leqslant Ce^{-\sqrt{J}}, \quad |t| \leqslant T, \tag{60}$$

and we have

$$\|\widehat{g}_{k_0}\|_1 \leqslant \|\widehat{\sigma}_n\|_1 \cdot \|\widehat{g}_1\|_1 \leqslant C\frac{T}{M}J < \frac{1}{10} \tag{61}$$

(recall that $(T/M)J$ is sufficiently small by assumption) and

$$\widehat{g}_{k_0}(s) = 0 \quad \text{for } |k_0 - s| > C'J^2\frac{M}{T}. \tag{62}$$

Now let

$$F(t) = \sum_{k\in I} \widehat{F}(k)e^{2\pi i k t}, \qquad \|F\|_A \leqslant 1,$$

be a fixed polynomial, where $I = \{k \in \mathbb{Z} : B \leqslant k \leqslant B'\}$, $M\log M < B \leqslant B'$. We use the representation (60) for each frequency in the spectrum of F. Then for $|t| \leqslant T$

$$\begin{aligned} F\left(\frac{t}{M}\right) &= \sum_{k\in I} \widehat{F}(k) n_{(k)}^{it} - \sum_{k\in I} \widehat{F}(k) g_{(k)}\left(\frac{t}{M}\right) + \Delta(t) \\ &\equiv \sum_{k\in I} c_{n(k)} n_{(k)}^{it} + F_1\left(\frac{t}{M}\right) + \Delta(t), \end{aligned} \tag{63}$$

where $|\Delta(t)| \leqslant C\exp -J^{1/2}$, $\sum |c_{n(k)}| = \|F\|_A \leqslant 1$ and $c_n = 0$ if $[(M\log n)/(2\pi)] \notin I$, and the polynomial

$$F_1(t) = -\sum_{k\in I} \widehat{F}(k) g_{(k)}(t)$$

satisfies

$$\|F_1\|_A < \frac{1}{10}, \qquad \operatorname{supp} \widehat{F}_1 \subset I + \left[-C'J^2\frac{M}{T}, C'J^2\frac{M}{T}\right].$$

Now we iterate the expansion (63). At the νth step we obtain the representation

$$F\left(\frac{t}{M}\right) = \sum_{n\in\Lambda} c_{n(k)} n^{it} + F_\nu\left(\frac{t}{M}\right) + \Delta_\nu(t), \qquad |t| \leqslant T,$$

where $\Lambda = \Lambda(\nu)$, $c_n = c_n(\nu)$, we have

$$\sum |c_n| \leqslant 1 + \sum_{l=1}^{\nu} (10)^{-l} < 2,$$

$$|\Delta(t)| \leqslant C \exp(-J^{1/2}),$$

$$|\Lambda| \leqslant |I| + \frac{\nu 2C'J^2M}{T},$$

$$\Lambda \subset \left\{n : \left[\frac{M\log n}{2\pi}\right] \subset I + \nu\left[-\frac{C'J^2M}{T}, \frac{C'J^2M}{T}\right]\right\},$$

and the trigonometric polynomial $F_\nu(t)$ has the following properties: $\|F_\nu\|_A \leqslant 10^{-\nu}$ and

$$\operatorname{supp} \widehat{F_\nu} \subset I + \nu\left[-\frac{C'J^2M}{T}, \frac{CJ^2M}{T}\right].$$

After J steps of the iteration starting from $F(t) = e^{2\pi i k_0 t}$, we obtain the estimate

$$\left|e^{2\pi i k_0 (t/M)} - \sum_{n\in\Lambda} c_n n^{it}\right| \leqslant C \exp -J^{1/2}, \qquad |t| \leqslant T,$$

where

$$\sum_{n\in\Lambda} |c_n| < 2, \qquad |\Lambda| \leqslant CJ^3\frac{M}{T} \quad \textit{and} \quad c_n = 0 \quad \text{for} \left|\frac{M\log n}{2\pi} - k_0\right| > CJ^3\frac{M}{T}.$$

Here we have taken account of the fact that the relations $k_0 \geqslant (M\log M)/(2\pi)$ and $J^3 \leqslant T$ ensure that the inequality $k \geqslant (M\log M)/(4\pi)$ is preserved for the points in the spectrum of the polynomials F_ν, $\nu = 1, 2, \dots, J$. Lemma 7 is proved.

The following result in a consequence of Theorem A.

Lemma 8. *Let n_0, n and m be integers, $1 \leqslant m \leqslant n$. Then there exists a set*

$$I \subset \mathbb{Z} \cap [n_0, n_0 + n]$$

such that $|I| = m$ and

$$\operatorname{dist}_{L^\infty(0,1)}(f, \operatorname{span}\{e^{2\pi ikt}, k \in I\}) \leqslant C'(\log n)^3 \sqrt{\frac{n}{m}}$$

for each trigonometric polynomial

$$f(t) = \sum_{k=n_0}^{n_0+n} b_k e^{2\pi ikt}, \qquad \sum_{k=n_0}^{n_0+n} |b_k|^2 \leqslant 1.$$

Proof. Obviously, we can set $n_0 = 0$. Let $\Lambda \subset \{0, \dots, n\}$, $|\Lambda| = n + 1 - m$, be the set defined by Theorem A for the discrete trigonometric system $\{e^{2\pi ikt}\}_{k=0}^{n}$, let $t \in \Omega_{n+1} = \{s/(n+1), s = 0, \dots, n\}$ and $I = \{0, \dots, n\} \setminus \Lambda$.

Making the assumption that Lemma 8 fails and using duality arguments we can find a function $F \in L^1(0, 1)$ such that

$$\int_0^1 f\overline{F}\,dt = 1, \qquad \|F\|_{L^1(0,1)} < (C')^{-1}(\log n)^{-3}\sqrt{\frac{m}{n}},$$
$$\widehat{F}(k) = 0 \quad \text{if } k \in I.$$

Then using the central inequality in Theorem A and classical facts about the trigonometric system we obtain

$$1 = \left|\sum_{k\in\Lambda} b_k\widehat{F}(k)\right| \leqslant \left(\sum_{k\in\Lambda}|\widehat{F}(k)|^2\right)^2 \leqslant C(\log n)^2\sqrt{\frac{n}{m}}\cdot\frac{1}{n+1}\sum_{s=0}^{n}\left|F\left(\frac{2\pi\cdot s}{n+1}\right)\right|$$
$$\leqslant C'(\log n)^3\sqrt{\frac{n}{m}}\|F\|_{L^1(0,1)} < 1$$

which is a contradiction.

Remark. In Lemma 8 we could have used Proposition 1 established above in place of Theorem A.

Lemma 9. *Let* $1 \leqslant T < N_0 < (1/2)N_1$, $(\log N_1)^{10} < T < N_0/(\log N_1)^{10}$ *and* $\rho \in (0, 1)$. *Then there exists a subset* Λ_0 *of the interval of the natural number sequence*

$$\mathbb{Z} \cap \left[N_0\left(1 - \frac{(\log N_1)^7}{T}\right), N_1\left(1 + \frac{(\log N_1)^7}{T}\right)\right] \tag{64}$$

such that

$$|\Lambda_0| \leqslant \rho T$$

and for each Dirichlet polynomial

$$f = \sum_{N_0\leqslant n\leqslant N_1} a_n n^{it} \tag{65}$$

with

$$|a_n| \leqslant 1, \qquad N_0 \leqslant n \leqslant N_1, \tag{66}$$
$$\frac{1}{2T}\int_{-T}^{T}|f|^2\,dt \leqslant 1 \tag{67}$$

the following inequality holds:

$$\operatorname{dist}_{L^\infty(-T/2,T/2)}(f, \operatorname{span}\{n^{it}; n \in \Lambda_0\}) \leqslant C(\log N_1)^{9.1}\rho^{-1/2}. \tag{68}$$

Remark. If $\rho T < 1$, then Λ_0 is the empty set and the left-hand side of (68) is $\|f\|_{L^\infty(-T/2,T/2)}$.

Proof. The proof is based on Lemmas 6, 7 and 8. We can assume that N_1 (and therefore also N_0) is sufficiently large, otherwise Lemma 9 is obvious. Let $J = ([\log N_1])^{2.2}$ and $M = C_0JT$, where C_0 is the absolute constant in Lemma 7. Let f be an arbitrary polynomial of the form (65) with properties (66) and (67). Let σ_n be the trigonometric polynomials constructed in Lemma 6, which approximates the function n^{it} uniformly on $[-T, T]$, and let

$$F(\theta) = \sum_{N_0 \leqslant n \leqslant N_1} a_n \sigma_n(\theta). \tag{69}$$

By property 1) in Lemma 6

$$\operatorname{supp} \widehat{F} \subset \mathbb{Z} \cap \left[\frac{M \log N_0}{2\pi} - J, \frac{M \log N_1}{2\pi} + J\right], \tag{70}$$

and the estimate 3) in Lemma 6 (see also (66)) yields

$$\left|f(t) - F\left(\frac{t}{M}\right)\right| \leqslant \sum_{N_0 \leqslant n \leqslant N_1} |a_n| \left|n^{it} - \sigma_n\left(\frac{t}{M}\right)\right| \leqslant N_1 C e^{-J^{1/2}} < 1 \tag{71}$$

for $t \in [-T, T]$. By (71) and (67),

$$\int_{-(C_0J)^{-1}}^{(C_0J)^{-1}} |F(\theta)|^2 \, d\theta = \int_{-T/M}^{T/M} |F(\theta)|^2 \, d\theta \leqslant 4\frac{T}{M} = \frac{4}{C_0 J} < \frac{1}{10}. \tag{72}$$

Let $\psi = \psi(\theta) = \psi(\theta, J)$ be a trigonometric polynomial of degree $\deg \psi \leqslant (C_0J)^2$ such that $\|\psi\|_{L^\infty} \leqslant 2$, $|\psi(\theta) - 1| < e^{-\sqrt{J}}$ for $|\theta| < 1/(2C_0J)$ and $|\psi(\theta)| < e^{-\sqrt{J}}$ for $|\theta| > 1/(C_0J)$. We can take the partial sum of order $(C_0J)^2$ of the Fourier series of the function φ_δ in Lemma 4 for $\psi(\theta)$, where $\delta = 1/(C_0J)$.

We set

$$F_1 = F \cdot \psi. \tag{73}$$

Then (see (70))

$$\operatorname{supp} \widehat{F_1} \subset \mathbb{Z} \cap \left[\frac{M \log N_0}{2\pi} - J - (C_0J)^2, \frac{M \log N_1}{2\pi} + J + (C_0J)^2\right] \equiv \{r_0, \dots, r_1\}. \tag{74}$$

Furthermore,

$$\begin{aligned} \|F_1\|_2^2 &\leqslant 4\int_{-(C_0J)^{-1}}^{(C_0J)^{-1}} |F(\theta)|^2 \, d\theta + \|F\|_{L^\infty}^2 \int_{|\theta| \geqslant (C_0J)^{-1}} |\psi(\theta)|^2 \, d\theta \\ &\leqslant \frac{4}{10} + CN_1^2 \exp(-2J^{1/2}) < 1. \end{aligned} \tag{75}$$

In addition, for $t \in [-T/2, T/2]$, using (71) we obtain

$$\left|f(t) - F_1\left(\frac{t}{M}\right)\right| \leqslant 1 + \|F\|_{L_\infty} \left|1 - \psi\left(\frac{t}{M}\right)\right| \leqslant 1 + CN_1 \exp(-\sqrt{J}) < 2. \tag{76}$$

Now we use Lemma 8 for $n_0 = r_0$, $n + n_0 = r_1$, where r_0 and r_1, $r_1 - r_0 \geqslant cM$, are the end-points of the segment of the natural number sequence defined in (74), and $m = [\rho_1 T]$, where $\rho_1 = \rho(\log N_1)^{-10}$. Let $I \subset \mathbb{Z} \cap [r_0, r_1]$, $|I| = m$, be the set of frequencies defined in Lemma 8. We can find a polynomial $F_2 \in \operatorname{span}\{e^{2\pi ikt}, k \in I\}$ such that

$$\|F_1 - F_2\|_{L^\infty} \leqslant C(\log N_1)^3 \left(\frac{r_2 - r_1}{\rho_1 T}\right)^{1/2} \leqslant C(\log N_1)^{4.1} \rho_1^{-1/2}. \tag{77}$$

From the last inequality and (76) we obtain

$$\left| f(t) - F_2\left(\frac{t}{M}\right)\right| \leqslant C(\log N_1)^{4.1}\rho_1^{-1/2} = C(\log N_1)^{9.1}\rho^{-1/2}, \qquad |t| \leqslant \frac{T}{2}. \tag{78}$$

It remains to approximate $F_2(t/M)$ by a Dirichlet polynomial with small spectrum. The coefficients of the polynomial $F_2(t) = \sum_{k\in I} \widehat{F}_2(k)e^{2\pi ik\theta}$ have the following estimates:

$$\sum |\widehat{F}_2(k)| \leqslant |I|^{1/2} \cdot \|F_2\|_2 \leqslant (\rho_1 T)^{1/2}(\log N_1)^{4.1} C\rho_1^{-1/2} \leqslant N_1^{1/2}. \tag{79}$$

The choice of J and M and the hypotheses of Lemma 9 ensure that $k > \dfrac{M \log M}{2\pi}$, $k \in I$. Hence by Lemma 7, for each $k \in I$ we have

$$\operatorname{dist}_{L^\infty(-T,T)}(e^{2\pi ikt/M}, \operatorname{span}\{n^{it}, n \in \Lambda(k)\}) \leqslant Ce^{-\sqrt{J}}.$$

so that, in view of (79),

$$\operatorname{dist}_{L^\infty(-T,T)}\left(F_2\left(\frac{t}{M}\right), \operatorname{span}\{n^{it}, n \in \Lambda_0\}\right) \leqslant N_1^{1/2} Ce^{-\sqrt{J}} < 1, \tag{80}$$

where $\Lambda_0 \equiv \bigcup_{k\in I_1} \Lambda(k)$.

It follows from (78) and (80) that

$$\operatorname{dist}_{L^\infty(-T/2,T/2)}(f, \operatorname{span}\{n^{it}, n \in \Lambda_0\}) \leqslant C(\log N_1)^{9.1}\rho^{-1/2}.$$

By relation b) in Lemma 7

$$|\Lambda_0| \leqslant |I| \cdot C\frac{M}{T}J^3 \leqslant \rho_1 TC'J^4 \leqslant C'\rho T(\log N_1)^{-10}(\log N_1)^{8.8} \leqslant \rho T.$$

Furthermore, for each $n \in \Lambda_0$

$$r_0 - C\frac{M}{T}J^3 \leqslant \frac{M\log n}{2\pi} \leqslant r_1 + C\frac{M}{T}J^3 \tag{81}$$

(see (74) and relation a) in Lemma 7). It follows from (81) and the definition of J that each element n of the subset Λ_0 lies in the segment (64) of the natural number sequence. The proof of Lemma 9 is complete.

Lemma 10. *Let $T \in \mathbb{R}$, $T \geqslant 1$, $N \in \mathbb{Z}$ and $k \in \mathbb{Z}$, $1 \leqslant k \leqslant N$, be fixed numbers. Then there exists a set $\Lambda' \subset \{1, \dots, N\}$, $|\Lambda'| \leqslant k$, such that for each Dirichlet polynomial*

$$f(t) = \sum_{n=1}^{N} a_n n^{it}, \qquad \sum_{n=1}^{N} |a_n|^2 \leqslant 1,$$

there exists a polynomial

$$g(t) \in \operatorname{span}\{n^{it}, n \in \Lambda'\}$$

such that

$$\|f(t) - g(t)\|_{L^\infty(-T,T)} \leqslant C[\log(N+T)]^{7/2} \sqrt{\frac{N}{k}}.$$

The result of Lemma 10 is obtained by applying Proposition 2 to the system of functions $\{n^{it}\}_{n=1}^{N}$, $t \in [-T, T]$, while taking account of the bound

$$\|(n^{it})'\|_{L^\infty(-T,T)} \leqslant \log N$$

and the inequality

$$\log(N + 2TN \log N) \leqslant C \log(N+T).$$

Now we turn to the proof of Theorem 1, which is based on Lemmas 9 and 10. Note that the estimate in Lemma 9 is used for N much greater than T.

Let $m \in \mathbb{Z}$ be defined by the relation

$$m \leqslant \rho \min\{T, N\} < m+1$$

and assume initially that $m \geqslant 1$. If $N \leqslant 2T(\log T)^{10}$, then it is sufficient to use Lemma 10 with $k = m$ bearing in mind that in this case

$$\log^{7/2}(N+T) \cdot \sqrt{\frac{N}{m}} \leqslant 2\log^{7/2}(N+T) \sqrt{\frac{N}{\rho \min\{T, N\}}} \leqslant C\rho^{-1/2}[\log(N+T)]^{8.5}.$$

If $N > 2T(\log T)^{10}$, then we set $N_0 = 1 + T(\log N)^{10}$ and for an arbitrary polynomial f of the form (1) look at the expansion

$$f = \sum_{1 \leqslant n \leqslant N_0} a_n n^{it} + \sum_{n > N_0} a_n n^{it} \equiv f_0 + f_1.$$

Applying Lemma 10 to $\{n^{it}\}_{n=1}^{N_0}$ with $k = [m/2]$ we find a subset $\Lambda' \subset \{1, \dots, N_0\}$, $|\Lambda'| \leqslant m/2$, and a polynomial $g' = \sum_{n \in \Lambda'} b_n n^{it}$ such that

$$\|f_0 - g'\|_{L^\infty(-T,T)} \leqslant C \log^{7/2}(N+T) \sqrt{\frac{N_0}{m}} \leqslant C \log^{8.5}(N+T) \rho^{-1/2}. \tag{82}$$

Next, applying Lemma 9 to $\{n^{it}\}_{n=N_0+1}^{N}$ we find a set Λ_0, $|\Lambda_0| \leqslant m/2$, such that (68) holds. Then for $\Lambda = \Lambda' \cup \Lambda_0$ we obviously obtain

$$|\Lambda| \leqslant \rho \min\{T, N\}.$$

In addition, taking (2) into account we see that

$$\frac{1}{2T}\int_{-T}^{T}|f_0(t)|^2 \leqslant 2\sum_{n=1}^{N}|a_n|^2 + 2\sum_{1\leqslant n_1<n_2\leqslant N_0}|a_{n_1}||a_{n_2}|\frac{1}{T\ln(n_2/n_1)} \leqslant C(\log N)^{11} \tag{83}$$

(see [3], Ch. 5, § 1 for details). It follows from (83) and (2) that

$$\frac{1}{2T}\int_{-T}^{T}|f_1^{(t)}|^2 \leqslant C(\log N)^{11}$$

and we can apply Lemma 9 to the polynomial $f_1(\log N)^{-5.6}$, which produces $g_0(t) = \sum_{n\in\Lambda_0} b_n n^{it}$ such that

$$\|f_1 - g_0\|_{L^\infty(-T/2,T/2)} \leqslant C\log^{15}(N+T)\rho^{-1/2}. \tag{84}$$

As a result, for $g = g_1 + g_2 \in \operatorname{span}\{n^{it}, n\in\Lambda\}$ we have (see (82) and (84))

$$\|f - g\|_{L^\infty(-T/2,T/2)} \leqslant C\log^{15}(N+T)\rho^{-1/2},$$

as required.

Finally, if $\rho\min\{T,N\} < 1$, then the assertion of Theorem 1 reduces to estimating $\|f\|_{L^\infty(-T/2,T/2)}$ for each polynomial of the form (1), (2). If $N \leqslant 2T(\log N)^{10}$, then the required estimate follows from the obvious inequality

$$|f(t)| \leqslant \sum_{n=1}^{N}|a_n| \leqslant N^{1/2}\biggl(\sum_{n=1}^{N}|a_n|^2\biggr)^{1/2} \leqslant N^{1/2}.$$

For $N > 2T(\log N)^{10}$, arguing as for $\rho\min\{T,N\} \geqslant 1$ we must consider the expansion

$$f = \sum_{1\leqslant n\leqslant N_0} a_n n^{it} + \sum_{N_0<n\leqslant N} a_n n^{it} = f_0 + f_1,$$

where $N_0 = T(\log N)^{10}+1$. Here we can use (74)–(76) and the inequality $\|F_1\|_{L^\infty} \leqslant (r_1-r_0)^{1/2}\|F_1\|_{L^2}$ (see (74)) to estimate $\|f_1\|_{L^\infty(-T/2,T/2)}$. The proof of Theorem 1 is complete.

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