Decomposing an orthogonal matrix into two submatrices with extremally small (2, 1)-norm

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This note complements [1] and is concerned with the properties of $N \times n$ matrices, regarded as operators from l_2^n to l_1^N. We use the notation from [1]. Modifying an approach proposed by Lunin [2], we establish a result similar to the theorem in [3] relating to the case when the matrices are treated as operators between Euclidean spaces.

Let I_n, $n = 1, 2, \dots$, be the matrix of the identity operator in $\mathbb{R}^n$ and let

$$\varphi(\varepsilon) \equiv 5\varepsilon^{1/2} \log^{1/4}(3/\varepsilon), \qquad 0 < \varepsilon < 1. \tag{1}$$

Theorem 1. *For some constant $\varepsilon_0 > 0$ and each $N \times n$ matrix A such that $A^*A = I_n$ and the rows V_j, $j = 1, \dots, N$, of A satisfy $\|V_j\|_2 \leqslant \varepsilon$, $j = 1, \dots, N$, with $\varepsilon \leqslant \varepsilon_0$ there exists a decomposition*

$$\langle N \rangle = \Omega_1 \cup \Omega_2, \qquad \Omega_1 \cap \Omega_2 = \varnothing,$$

such that both the norms $\|A(\Omega_1)\|_{(2,1)}$ and $\|A(\Omega_2)\|_{(2,1)}$ do not exceed $(1/2+\varphi(\varepsilon))\|A\|_{(2,1)}$.

As in [1], here $\langle N \rangle = \{1, \dots, N\}$ and $A(\Omega)$ is the submatrix of A formed by the rows V_j with $j \in \Omega \subset \langle N \rangle$.

Remark. The question of the extent to which the orthogonality condition on the columns of A can be relaxed must be treated separately. It was noted in [4] that any $N \times n$ matrix A with norm $\|A\|_{(2,1)} = 1$ such that its rows have l_2-norm at most ε can be decomposed into two submatrices with (2, 1)-norms $\leqslant \sqrt{3}/2 + \varepsilon/(2\sqrt{3})$. However, [4] uses an elementary method which does not allow one to approach 1/2 (the extremally small value of the norms of the submatrices in the decomposition) or just to overcome the natural barrier of $1/\sqrt{2}$.

Scheme of the proof. By assumption and in view of the relation

$$\biggl(\sum_{j=1}^{N} \|V_j\|_2\biggr) \max_j \|V_j\|_2 \geqslant \sum_{j=1}^{N} \|V_j\|_2^2 = n,$$

we have $\sum_{j=1}^{N} \|V_j\|_2 \geqslant n/\varepsilon$. Taking account of this estimate and the well-known inequality

$$\int_{\mathbb{S}^{n-1}} |(y, v)|\, d\mu \geqslant \sqrt{\frac{2}{\pi n}}\, \|v\|_2, \qquad v \in \mathbb{R}^n$$

(see [5]; here μ is the normalized Lebesgue measure on the sphere $\mathbb{S}^{n-1}$), we obtain

$$\|A\|_{(2,1)} \geqslant \int_{\mathbb{S}^{n-1}} \|Ay\|_1\, d\mu = \sum_{j=1}^{N} \int_{\mathbb{S}^{n-1}} |(y, V_j)|\, d\mu \geqslant \frac{n^{1/2}}{\varepsilon} \sqrt{\frac{2}{\pi}}\,. \tag{2}$$

Also, note that under the assumptions of the theorem, for each $\omega \subset \langle N \rangle$ with $\sharp\omega = q$ we have

$$\|A(\omega)\|_{(2,1)} \leqslant q^{1/2} \|A(\omega)\|_{(2,2)} \leqslant q^{1/2}. \tag{3}$$

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Following [1] and using the notation $\overline{\omega} = \langle N\rangle \setminus \omega$, we set

$$\beta = \min_{\omega\subset\langle N\rangle} \max\biggl\{ \frac{\|A(\omega)\|_{(2,1)}}{\|A\|_{(2,1)}}, \frac{\|A(\overline{\omega})\|_{(2,1)}}{\|A\|_{(2,1)}} \biggr\}.$$

We can now repeat the arguments in [1] and establish the existence of a vector $y_0 \in \mathbb{S}^{n-1}$ and a system $\Psi_0 = \{\omega\colon \omega \subset \langle N\rangle\}$ of subsets with

$$\sharp\Psi_0 \geqslant (2^{N-1} - 1)(\varepsilon/3)^{2n} \tag{4}$$

which has the following properties: if σ is a rearrangement of the tuple $\langle N\rangle$ such that the quantities $g_\nu = |(V_{\sigma(\nu)}, y_0)|$, $\nu = 1, \dots, N$, form a non-increasing sequence, then for any $\omega \in \Psi_0$, $\omega = (j_1, \dots, j_s)$, we have

$$\sum_{j\in\omega} g_{\sigma^{-1}(j)} > \beta \cos\varepsilon\, \|A\|_{(2,1)}. \tag{5}$$

It is sufficient to show that for $\beta \geqslant 1/2 + \varphi(\varepsilon)$ the number R of subsets ω of $\langle N\rangle$ such that (5) holds is certainly less than $\sharp\Psi_0$ (see (4)). Note that $S \equiv \sum_{\nu=1}^N g_\nu \geqslant (1/3)\|A\|_{(2,1)} \geqslant (1/3)\sqrt{2/\pi}\, n^{1/2}/\varepsilon$ (see (3)). Furthermore,

$$\begin{aligned} R &\leqslant \sharp\biggl\{\omega\colon \sum_{\nu\in\omega} g_\nu > \biggl[\biggl(\frac{1}{2} + \varphi(\varepsilon)\biggr)\cos\varepsilon\biggr] S\biggr\} \\ &= 2^N \mathsf{m}\biggl\{t \in (0,1)\colon \sum_{\nu=1}^N g_\nu r_\nu(t) > \bigl[(1 + 2\varphi(\varepsilon))\cos\varepsilon - 1\bigr] S\biggr\}, \end{aligned} \tag{6}$$

where $\{r_\nu\}$ are the Rademacher functions and m is Lebesgue measure on $(0,1)$. For the required upper estimate of the measure in (6) we only need the classical exponential estimate for Rademacher polynomials (see [6]), provided that we bear in mind the following:

(a) we have

$$\sum_{\nu=1}^N g_\nu r_\nu(t) > \sum_{\nu=q+1}^N g_\nu r_\nu(t) - \sum_{\nu=1}^q g_\nu > \sum_{\nu=q+1}^N g_\nu r_\nu(t) - \sqrt{q}, \qquad 1 \leqslant q < N$$

(see also (3));

(b) for $q = \lfloor(\varphi(\varepsilon)/\varepsilon)^2 n/100\rfloor$ we have $S < 2\sum_{\nu=q+1}^N g_\nu$ and

$$\sum_{\nu=q+1}^N g_\nu^2 \leqslant \bigl(\max_{\nu>q} g_\nu\bigr) \sum_{\nu=q+1}^N g_\nu \leqslant \frac{S}{q} \sum_{\nu=q+1}^N g_\nu \leqslant \frac{2}{q}\biggl(\sum_{\nu=q+1}^N g_\nu\biggr)^2.$$

The choice of the function (1) ensures the required estimate for the right-hand side of (6).

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