Decomposing a Matrix into two Submatrices with Extremally Small (2,1)-Norm

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Abstract—We consider conditions on a matrix $A$ with unit operator $(2,1)$-norm ensuring the existence of a partition of this matrix into two submatrices with $(2,1)$-norms close to $1/2$.

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For an $N \times n$ matrix $A = \{a_{ij}\}$ viewed as an operator from $\mathbb{R}^n$ to $\mathbb{R}^N$, we define the norm

$$\|A\|_{(p,q)} = \sup_{\|x\|_p \leq 1} \|Ax\|_{l_N^q}, \quad 1 \leq p, q \leq \infty. \tag{1}$$

Estimates of the norms (1) of submatrices of a given matrix $A$ find applications in various areas of mathematics. The first results on this topic were obtained in [1] and [2]. Moreover, the estimates in [2] (see also [3]) for the classical operator norm (i.e., $p = q = 2$) of submatrices are a consequence of the corresponding estimates for $(2,1)$-norms. The study of the $(2,1)$-norms of submatrices of a given matrix carried out in this paper is also of independent interest.

We use the following notation:

- $\langle N \rangle$ is the set of positive integers $1, 2, \ldots, N$;
- the $v_i, i \in \langle N \rangle$, are the rows of $A$;
- the $w_j, j \in \langle n \rangle$, are the columns of $A$.

For $\Omega \subset \langle N \rangle$, $A(\Omega)$ is the submatrix of $A$ formed by the rows $v_i, i \in \Omega$. We denote the inner product on $\mathbb{R}^n$ by $(\cdot, \cdot)$ and the norm of a vector $x$ in the space $l_p^n$, $1 \leq p \leq \infty$, by $\|x\|_p$; $|\Lambda|$ is the number of elements in a finite set $\Lambda$.

Note that a duality argument and the definition of $(\infty, 2)$-norms imply the relation

$$\|A\|_{(2,1)} = \sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{N} \varepsilon_i v_i \right\|_2. \tag{2}$$

Marcus, Spielman, and Srivastava [4] established the following important result: let the columns of $A$ form an orthonormal system in $\mathbb{R}^N$, and assume that the rows satisfy the estimate $\|v_i\|_2 \leq \varepsilon$, $0 < \varepsilon < 1$. Then there exists a decomposition

$$\langle N \rangle = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \emptyset, \tag{3}$$

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such that
\[ \|A(\Omega_k)\|_{(2,2)} \leq \frac{1}{\sqrt{2}} + C\varepsilon, \quad k = 1, 2, \]
where \( C \) is an absolute constant.

One of the authors of [4], Srivastava, restated this result in [5] in a form that applies to an arbitrary \( N \times n \) matrix \( A \): if
\[ |(v_{i_0}, x)| \leq \varepsilon \left( \sum_{i=1}^{N} |(v_i, x)|^2 \right)^{1/2} \]  
for any \( x \in \mathbb{R}^n \) and \( i_0 \in \langle N \rangle \), then there exists a partition (3) such that
\[ \sum_{i \in \Omega_k} \|(v_i, x)\|^p \leq \left( \frac{1}{2} + C\varepsilon \right) \sum_{i=1}^{N} |(v_i, x)|^p, \quad p = 2, \quad k = 1, 2. \]  
(5)

In [5], in connection with possible applications to algorithmic problems in graph theory and the geometry of normed spaces, the question was raised as to whether there exists a counterpart of relation (5) for \( p = 1 \). This question is directly related to the estimates of the \( (2,1) \)-norms of submatrices and is considered in the present paper. As to the case of column-orthogonal matrices, a counterpart of the partition (3) with extremely small \( (2,1) \)-norms of the corresponding submatrices was obtained in [6]. The approach in [6] is used below to prove Statements 3 and 4. However, Statement 2 below shows that the straightforward counterpart of property (4) of the matrix \( A \) for \( p = 1 \), i.e., the inequality
\[ |(v_{i_0}, x)| \leq \varepsilon \sum_{i=1}^{N} |(v_i, x)|, \quad x \in \mathbb{R}^n, \quad i_0 \in \langle N \rangle, \]  
(6)
does not guarantee the existence of a partition (3) with property (5) for \( p = 1 \). For its existence, it is necessary to assume in the general case that the parameter \( \varepsilon \) tends to zero as \( n \to \infty \).

First, let us establish the following statement.

**Statement 1.** Let \( A \) be an \( N \times n \) matrix, and let
\[ \|A\|_{(2,1)} = 1, \quad \|v_i\|_2 \leq \varepsilon, \quad i \in \langle N \rangle. \]
Then there exists a partition (3) with \( |\Omega_1^0| - |\Omega_2^0| \leq 1 \) such that
\[ \|A(\Omega_k^0)\|_{(2,1)} \leq \frac{1}{\sqrt{2}} + 2\varepsilon, \quad k = 1, 2. \]  
(7)

**Remark 1.** The possibility of reducing the \( (2,1) \)-norms of the submatrices of the partition under the assumptions of Statement 1 was shown in [7]. The estimate (7) refines the result obtained in [7].

**Proof.** Without loss of generality, we assume that \( N \) is even. (Otherwise, we supplement the matrix with a zero row.) Consider an arbitrary partition of the form (3) with \( |\Omega_1| = |\Omega_2| = N/2 \). Relation (2) and the parallelogram identity for the Euclidean norm imply that
\[ \|A(\Omega_1)\|_{(2,1)}^2 + \|A(\Omega_2)\|_{(2,1)}^2 \leq \|A\|_{(2,1)}^2 = 1. \]  
(8)

It follows from (8) that the \( (2,1) \)-norm of one of the submatrices \( A(\Omega_k), k = 1, 2 \) (to be definite, let it be \( A(\Omega_1) \)), does not exceed \( 1/\sqrt{2} \). If, moreover,
\[ \|A(\Omega_2)\|_{(2,1)} \leq \frac{1}{\sqrt{2}} + 2\varepsilon, \]
there is nothing to prove. Otherwise, we mark the point \( z_0 \) of the square \([0, 1]^2\) with coordinates \((\|A(\Omega_1)\|_{(2,1)},\|A(\Omega_2)\|_{(2,1)})\). Let us consecutively transform the sets \( \Omega_k, k = 1, 2 \), as follows. At the \( \nu \)th step (\( \nu = 1, 2, \ldots, N/2 \)), we replace one element of \( \Omega_1 \) by an element of \( \Omega_2 \) so that \( T_{N/2}(\Omega_1) \)

\[ \|A(\Omega_2)\|_{(2,1)} \leq \frac{1}{\sqrt{2}} + 2\varepsilon, \]
holds for any partition
\( \nu (2 \text{ groups}) \) for every

\[ \| A(T_{\nu}(1)) \|_{(2,1)}, \| A(T_{\nu}(2)) \|_{(2,1)} \]

is symmetric to the point \( z_0 \) with respect to the diagonal of the square passing through the point \((0,0)\). Moreover, the norm of the matrices changes by at most \( 2\varepsilon \) at each step of the construction: for \( \nu = 1, \ldots, N/2 - 1 \) and \( k = 1, 2 \), we have

\[ \| A(T_{\nu+1}(\Omega_k)) \|_{(2,1)} - \| A(T_{\nu}(\Omega_k)) \|_{(2,1)} \leq 2\varepsilon. \]

Therefore, if the first transition of the point through the above-mentioned diagonal occurs at the \( \nu_0 \)th step, then

\[ \max_{k=1,2} \| A(T_{\nu_0}(\Omega_k)) \|_{(2,1)} \leq \frac{1}{\sqrt{2}} + 2\varepsilon. \]

Setting \( \Omega_k^0 = T_{\nu_0}(\Omega_k), k = 1, 2 \), we obtain Statement 1.

**Statement 2.** Let \( n = 2^s, s = 1, 2, \ldots, \) and let \( n^{-1/2} \leq \varepsilon \leq 1 \). There exists a \( 2n \times n \) matrix

\[ A = A(n, \varepsilon) \]

such that the estimate (6) holds for any \( x \in \mathbb{R}^n \) and \( i_0 \in (2n) \) but, nevertheless, the inequality

\[ M \equiv \max(\| A(\Omega_1) \|_{(2,1)}, \| A(\Omega_2) \|_{(2,1)} \) \geq \frac{1}{\sqrt{2}} \left( \frac{1}{1 + (\varepsilon n^{1/2})^{-1}} \right) \| A \|_{(2,1)} \]

(9)

holds for any partition (3) (with \( N = 2n \)).

**Proof.** The entries \( a_{ij} \) of the matrix \( A \) are defined as

\[ a_{ij} = \begin{cases} 
\varepsilon & \text{if } i = j, \\
0 & \text{if } i \neq j, \\
\frac{1}{n}w_{j-1} \left( \frac{i - n - 1/2}{n} \right) & \text{for } n + 1 \leq i \leq 2n, \quad j = 1, \ldots, n,
\end{cases} \]

where \( \{w_d(x)\}_{d=0}^{\infty} \) is the Walsh function system orthonormal in \( L^2(0,1) \) (see [8]). It is clear from the definition that \( |a_{ij}| = n^{-1} \) for \( i > n \) and also that the rows and columns of the matrix

\[ \left\{ w_{j-1} \left( \frac{i - n - 1/2}{n} \right) \right\}, \quad i = n + 1, \ldots, 2n, \quad j = 1, \ldots, n, \]

are pairwise orthogonal. Therefore,

\[ \left\| \sum_{i=n+1}^{2n} \varepsilon_i v_i \right\|_2 = \left( \sum_{i=n+1}^{2n} \| v_i \|_2 \right)^{1/2} = 1 \]

for any \( \varepsilon_i = \pm 1, i = n + 1, \ldots, 2n. \)

Since the \((2,1)\)-norm of the identity operator in \( \mathbb{R}^n \) is \( n^{1/2} \), we obtain, taking into account (2),

\[ \| A \|_{(2,1)} \leq n^{1/2} + 1. \]

Considering the rows of \( A \) with numbers \( 1 \leq i \leq n \), we see that, for any partition of the rows into two groups, the submatrix generated by the larger part has \((2,1)\)-norm \( \geq \varepsilon(n/2)^{1/2} \). Thus, the number \( M \) defined in (9) admits the lower bound

\[ M \geq \frac{\varepsilon(n/2)^{1/2}}{\varepsilon n^{1/2} + 1} \| A \|_{(2,1)} \geq \frac{1}{\sqrt{2}} \left( \frac{1}{1 + (\varepsilon n^{1/2})^{-1}} \right) \| A \|_{(2,1)} \]
for any partition (3) (with $N = 2n$), which proves (9). Now let us verify that the matrix $A$ satisfies relation (6). Let $x = \{x_j\} \in \mathbb{R}^n$ and $i_0 \in \{2n\}$ be fixed. If $i_0 > n$, then, by the construction of $A$,

$$|(v_{i_0}, x)| \leq \frac{1}{n} \|x\|_1, \quad \sum_{i=1}^{n} |(v_i, x)| = \varepsilon \|x\|_1,$$

whence, in view of the inequality $\varepsilon \geq n^{-1/2}$, we obtain (6) for this $i_0$. If, on the contrary, $i_0 \leq n$, then

$$|(v_{i_0}, x)| = \varepsilon |x_{i_0}|$$

and

$$|(v_i, x)| = \frac{1}{n} \sum_{j=1}^{n} x_j w_{j-1} \left( \frac{i - n - 1/2}{n} \right)$$

for $i > n$. Therefore,

$$\sum_{i=n+1}^{2n} |(v_i, x)| = \frac{1}{n} \sum_{i=n+1}^{2n} \left| \sum_{j=1}^{n} x_j w_{j-1} \left( \frac{i - n - 1/2}{n} \right) \right| = \|f\|_{L^1(0, 1)},$$

where the function $f$ constant on the intervals $((\nu - 1)/n, \nu/n)$, $1 \leq \nu \leq n$, is given by

$$f \left( \frac{\nu - 1/2}{n} \right) = \sum_{j=1}^{n} x_j w_{j-1} \left( \frac{\nu - 1/2}{n} \right).$$

It follows from the formula for the Fourier–Walsh coefficients of the function $f$ that

$$\|f\|_{L^1(0, 1)} = \int_{0}^{1} |f(x)w_{j-1}(x)| \ dx \geq \left| \int_{0}^{1} f(x)w_{j-1}(x) \ dx \right| = |x_j|$$

for $j \in \langle n \rangle$, so that

$$\|f\|_{L^1(0, 1)} \geq \|x\|_{\infty},$$

whence inequality (6) for $i_0 \in \langle n \rangle$ readily follows as well (see (10) and (11)). The proof of Statement 2 is complete.

**Statement 3.** Assume that (6) holds for an $N \times n$ matrix $A$ for some $0 < \varepsilon \leq 1/n$. Then there exists a partition (3) such that

$$\|A(\Omega_k)\|_{(2, 1)} \leq \left( \frac{1}{2} + 2 \phi(n, \varepsilon) \right) \|A\|_{(2, 1)}, \quad k = 1, 2,$$

where

$$\phi(n, \varepsilon) = \left( n \varepsilon \ln \frac{8}{n \varepsilon} \right)^{1/3}.$$  \hspace{1cm} (12)

**Remark 2.** By a slight modification of (12), one can additionally ensure that the numbers $|\Omega_1|$ and $|\Omega_2|$ are close to $N/2$.

**Remark 3.** For $1/n < \varepsilon < 1/\sqrt{n}$, the problem on the existence of a partition (3) with the $(2, 1)$-norms of submatrices close to $(1/2)\|A\|_{(2, 1)}$ remains open. It seems to be likely that the desired partition exists for $\varepsilon = \overline{\delta}(1/\sqrt{n})$.

**Proof.** Assume that $\varepsilon n < 1/64$; otherwise, Statement 3 is obvious. Let $\delta = (\varepsilon n)^{1/3}$, and let $\mathcal{Y}$ be a $\delta$-net on the Euclidean sphere $S^{n-1} \subset \mathbb{R}^n$ with at most $(3/\delta)^n$ elements. For $\omega \subset \langle N \rangle$, we set $\overline{\omega} = \langle N \rangle \setminus \omega$. For each pair $(\omega, \overline{\omega})$, we find vectors $z_1, z_2 \in S^{n-1}$ such that

$$\|A(\omega)\|_{(2, 1)} = \|A(\omega)z_1\|_1, \quad \|A(\overline{\omega})\|_{(2, 1)} = \|A(\overline{\omega})z_2\|_1.$$
If \( \|A(\omega)\|_{(2,1)} \geq \|A(\overline{\omega})\|_{(2,1)} \), then we set \( \omega' = \omega \) and \( z_{\omega'} = z_1 \). Otherwise, we set \( \omega' = \overline{\omega} \) and \( z_{\omega'} = z_2 \). Take \( y_{\omega'} \) to be one of the vectors in \( Y \) closest to \( z_{\omega'} \). In all, there exist \( 2^{N-1} - 1 \) distinct partitions of the set \( \langle N \rangle \) into two nonempty parts. Therefore, there exists a \( y_0 \in Y \) such that the set \( K = \{ \omega' : y_0 = y_{\omega'} \} \) is sufficiently large:

\[
|K| \geq (2^{N-1} - 1) \left( \frac{\delta}{3} \right)^n. \tag{13}
\]

Let

\[
\beta = \min_{\omega \subseteq \langle N \rangle} \max \left\{ \frac{\|A(\omega)\|_{(2,1)}}{\|A\|_{(2,1)}}, \frac{\|A(\overline{\omega})\|_{(2,1)}}{\|A\|_{(2,1)}} \right\}.
\]

Note that

\[
\beta \|A\|_{(2,1)} \leq \|A(\omega')\|_{(2,1)} = \|A(\omega')z_{\omega'}\|_1
\]

\[
= \|A(\omega')(z_{\omega'} - y_0) + y_0\|_1 \leq \|A(\omega')(z_{\omega'} - y_0)\|_1 + \|A(\omega')y_0\|_1
\]

\[
\leq \|A(\omega')\|_{(2,1)} \delta + \sum_{i \in \omega'} |(v_i, y_0)| \leq \|A\|_{(2,1)} \delta + \sum_{i \in \omega'} |(v_i, y_0)| \text{ for } \omega' \in K.
\]

Thus,

\[
\sum_{i \in \omega'} |(v_i, y_0)| > (\beta - \delta) \|A\|_{(2,1)} \text{ for } \omega' \in K. \tag{14}
\]

Therefore, to prove Statement 3, it suffices to verify that, for \( \beta = 1/2 + 2\phi(n, \varepsilon) \), the number \( R \) of subsets \( \omega' \subseteq \langle N \rangle \) such that inequality (14) is true is less than the right-hand side of (13).

Let \( \sigma \) be a permutation of the set \( \langle N \rangle \) such that the numbers \( g_\nu = |(v_{\sigma(\nu)}, y_0)| \) form a nonincreasing sequence. We set \( g_\nu = 0 \) for \( \nu > N \). Further, let

\[
S = \sum_{\nu=1}^{N} g_\nu.
\]

Let us verify that

\[
\sum_{\nu=q+1}^{N} g_\nu^2 \leq \frac{S^2}{q} \tag{15}
\]

for each \( q \).

Indeed, since the sequence \( g_\nu \) is monotone, we have \( g_{q+1} \leq S/q \), and therefore,

\[
\sum_{\nu=q+1}^{N} g_\nu^2 \leq g_{q+1} \sum_{\nu=q+1}^{N} g_\nu \leq \frac{S}{q} \cdot S = \frac{S^2}{q}.
\]

It is clear from the definition of \((2,1)\)-norm that \( \|A\|_{(2,1)} \geq S \); therefore (see (14)), if \( \omega' \in K \), then

\[
\sum_{i \in \omega'} |(v_i, y_0)| > (\beta - \delta)S.
\]

It follows from the last relation that

\[
R \leq 2^N m \left\{ t \in (0, 1) : \sum_{\nu=1}^{N} g_\nu \tau_\nu(t) > (2(\beta - \delta) - 1)S \right\}
\]

\[
= 2^N m \left\{ t \in (0, 1) : \sum_{\nu=1}^{N} g_\nu \tau_\nu(t) > (4\phi(n, \varepsilon) - 2\delta)S \right\}. \tag{16}
\]
where \( m \) is the Lebesgue measure on \((0, 1)\) and \( \{ r_\nu(t) \}_{\nu=1}^\infty \) is the Rademacher function system (see [8]).

Using property (6) of the matrix \( A \) for \( x = y_0 \), we obtain

\[
\sum_{\nu=1}^N g_\nu r_\nu(t) \leq \sum_{\nu \geq q+1} g_\nu r_\nu(t) + \varepsilon q S. \tag{17}
\]

It follows from (16) and (17) that

\[
R \leq 2^N m \left\{ t \in (0, 1) : \sum_{\nu \geq q+1} g_\nu r_\nu(t) > (4\phi(n, \varepsilon) - 2\delta - \varepsilon q) S \right\}. \tag{18}
\]

Fix \( q = [2n^{1/3} \ln^{1/3}(2/\delta) \varepsilon^{-2/3}] \), where \([x]\) is the integer part of \( x \). It is clear that

\[
q > n^{1/3} \ln^{1/3} \left( \frac{2}{\delta} \right) \varepsilon^{-2/3}.
\]

Let us estimate \( 4\phi(n, \varepsilon) - 2\delta - \varepsilon q \) for the selected \( q \):

\[
4\phi(n, \varepsilon) - 2\delta - \varepsilon q \geq 4 \cdot 3^{1/3} n^{1/3} \varepsilon^{1/3} \ln^{1/3} \frac{2}{\delta} - 2n^{1/3} \varepsilon^{1/3} \frac{\ln^{1/3}(2/\delta)}{\ln^{1/3} 8} - 2n^{1/3} \varepsilon^{1/3} \ln^{1/3} \frac{2}{\delta} 
\geq (5, 7 - 1, 6 - 2)n^{1/3} \varepsilon^{1/3} \ln^{1/3} \frac{2}{\delta} \geq 2n^{1/3} \varepsilon^{1/3} \ln^{1/3} \frac{2}{\delta}, \tag{19}
\]

where we used the fact that \( \delta < 1/4 \). In particular, \( 4\phi(n, \varepsilon) - 2\delta - \varepsilon q > 0 \). In what follows, we assume that \( \sum_{\nu=q+1}^N g_\nu^2 \neq 0 \), because otherwise \( R \leq 0 \) and Statement 3 holds. To estimate the measure on the right-hand side in (18), we use (15), (19), and the classical exponential estimate for the distribution function of a polynomial over the Rademacher system (see [8]):

\[
R \leq 2^N \exp \left\{ -S^2 4\ln^{2/3} \varepsilon^{2/3} \ln^{2/3} \frac{2}{\delta} \frac{n^{1/3}}{2} \sum_{\nu=q+1}^N g_\nu^2 \right\} \leq 2^N \exp \left\{ -2n^{2/3} \varepsilon^{2/3} \left( \ln^{2/3} \frac{2}{\delta} \right) q \right\}
\leq 2^N \exp \left\{ -2n \ln \frac{2}{\delta} \right\} < (2^{N-1} - 1) \left( \frac{\delta}{3} \right)^n,
\]

which contradicts inequality (13). The proof of Statement 3 is complete. \( \square \)

Statement 3 only estimates the norm of the matrices \( A(\Omega_k), k = 1, 2 \). For a pointwise estimate similar to (5), we need to impose additional conditions on the matrix \( A \).

**Statement 4.** Assume that an \( N \times n \) matrix \( A \) satisfies (6) with \( 0 < \varepsilon < 1/n \) and

\[
0 < b \| x \|_2 \leq \| Ax \|_1 \leq B \| x \|_2
\]

for any \( x \in \mathbb{R}^n, x \neq 0 \). Then there exists a partition (3) such that

\[
\| A(\Omega_k)x \|_1 \leq \gamma \| Ax \|_1, \quad \gamma = \frac{1}{2} + 4 \left( n \varepsilon \ln \frac{2B}{b^{1/3} \varepsilon^{1/3} n^{1/3}} \right)^{1/3} \tag{20}
\]

for any \( x \in \mathbb{R}^n \) and \( k = 1, 2 \).

**Proof.** Assume that \( \gamma < 1 \); otherwise, the estimate (20) is obvious. In this case, \((n \varepsilon)^{1/3} < 1/4\).

Assume that, on the contrary, for any partition (3), either there exists a vector \( x_1 \in S^{n-1} \) such that

\[
\| A(\Omega_1)x_1 \|_1 > \gamma \| Ax_1 \|_1
\]

(in this case, we set \( \omega' = \Omega_1 \) and \( x_{\omega'} = x_1 \)) or

\[
\| A(\Omega_2)x_2 \|_1 > \gamma \| Ax_2 \|_1
\]
for some $x_2 \in S^{n-1}$ (in this case, we set $\omega' = \Omega_2$ and $x_{\omega'} = x_2$). Let

$$\delta = \frac{b}{B} (\varepsilon n)^{1/3}.$$ 

Further, arguing as in the proof of Statement 3, we can show that there exists a $y_0 \in S^{n-1}$ and at least 

$$(2^{N-1} - 1)(\delta/3)^n$$ 

subsets of $\omega' \subset (N)$ such that

$$\|A(\omega') x_{\omega'} \|_1 > \gamma \|Ax_{\omega'}\|_1, \quad \|y_0 - x_{\omega'}\|_2 < \delta.$$ 

Let $K$ be the collection of such subsets. Note that

$$\|A(\omega) x\|_1 \leq \|A(\omega)\|_{(2,1)} \leq \|A\|_{(2,1)}$$ 

for $x \in S^{n-1}$ and $\omega \subset (N)$. For $\omega' \in K$, given that $\gamma < 1$, we have

$$\|A(\omega')y_0\|_1 \geq \|A(\omega')\|_{x_{\omega'} - x_0} - \|A(\omega')\|_{x_{\omega'} - y_0}\|_1$$

$$\geq \gamma \|A y_0\|_1 - \gamma \|A(x_{\omega'} - y_0)\|_1 - \delta B$$

$$\geq \gamma \|A y_0\|_1 - 2\delta B \geq \gamma \|A y_0\|_1 - 2\delta \frac{\|A y_0\|_1}{b} = \|A y_0\|_1 \left(\gamma - 2\delta \frac{B}{b}\right).$$

(21)

Again, by analogy with the proof of Statement 3, we set

$$g_\nu = |(v_{\sigma(\nu)}, y_0)|,$$ 

where the permutation $\sigma$ of the set $(N)$ is chosen so that the sequence $g_\nu$ is nonincreasing; further, let $g_\nu = 0$ for $\nu = N + 1, N + 2, \ldots$, and let

$$S = \|A y_0\|_1 = \sum_{\nu=1}^{N} g_\nu = \sum_{\nu=1}^{\infty} g_\nu.$$ 

Let $R$ be the number of subsets $\omega \subset (N)$ such that

$$\|A(\omega)y_0\|_1 \geq \left(\gamma - 2\delta \frac{B}{b}\right) \|A y_0\|_1.$$ 

(22)

Let us show that $R < (2^{N-1} - 1)(\delta/3)^n$, thus arriving at a contradiction and completing the proof of Statement 4. By analogy with (16), we have

$$R \leq 2^N \left\{ t \in (0, 1) : \sum_{\nu=1}^{N} g_\nu r_\nu(t) \geq \left(1 + 8 \left(n \varepsilon \ln \frac{2}{\delta}\right)^{1/3} - 4\delta \frac{B}{b} - 1\right) S \right\}$$

$$\leq 2^N \left\{ t \in (0, 1) : \sum_{\nu=1}^{N} g_\nu r_\nu(t) \geq 4 \left(n \varepsilon \ln \frac{2}{\delta}\right)^{1/3} S \right\}. \quad (23)$$

Let

$$q = \left[2n^{1/3} \varepsilon^{-2/3} \ln^{1/3} \left(\frac{2}{\delta}\right)\right] > n^{1/3} \varepsilon^{-2/3} \ln^{1/3} \left(\frac{2}{\delta}\right).$$ 

Continuing the estimate (23) and using property (6) of the matrix $A$, we find that

$$R \leq 2^N \left\{ t \in (0, 1) : \sum_{\nu > q} g_\nu r_\nu(t) + \varepsilon q S > 4 \left(n \varepsilon \ln \frac{2}{\delta}\right)^{1/3} S \right\}$$

$$= 2^N \left\{ t \in (0, 1) : \sum_{\nu > q} g_\nu r_\nu(t) > \left(4 \left(n \varepsilon \ln \frac{2}{\delta}\right)^{1/3} - \varepsilon q\right) S \right\}. \quad (24)$$
Let us estimate \(4(n \varepsilon \ln(2/\delta))^{1/3} - \varepsilon q\) from below. The choice of \(q\) ensures that this number is not less than

\[
4 \left( n \varepsilon \ln \frac{2}{\delta} \right)^{1/3} - 2 \left( n \varepsilon \ln \frac{2}{\delta} \right)^{1/3} \geq 2 \left( n \varepsilon \ln \frac{2}{\delta} \right)^{1/3}.
\]

Again using the exponential estimate for the distribution function of a polynomial in the Rademacher system and (25), from (24) we obtain

\[
R \leq 2^N \exp \left\{ -4S^2(n^{1/3} \varepsilon^{1/3} \ln^{1/3}(2/\delta))^2 \right\} \sum_{\nu > q} g^2_{\nu} \tag{26}
\]

since \(\sum_{\nu > q} g_{\nu}^2 \leq S^2/q\) (see the proof of Statement 3), we see that the right-hand side can be estimated from above as

\[
R \leq 2^N \exp \left\{ -2 \left( n^{1/3} \varepsilon^{1/3} \ln^{1/3} \frac{2}{\delta} \right)^2 q \right\} \leq 2^N \exp \left\{ - \left( 2n \ln \frac{2}{\delta} \right) \right\}
\]

\[
\leq 2^N \left( \frac{\delta}{2} \right)^{2n} < (2^{N-1} - 1) \left( \frac{\delta}{3} \right)^{n},
\]

where the last inequality follows from the fact that \(\delta < 1/4\). The proof of Statement 4 is complete. \(\square\)

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