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Decomposing a Matrix into two Submatrices with Extremally Small (2,1)-Norm

B. S. Kashin[1,2*] **and I. V. Limonova**[3**]

[1]*Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, 119991 Russia*

[2]*Lomonosov Moscow State University, Moscow, 119991 Russia*

[3]*Laboratory "Multidimensional Approximation and Applications," Lomonosov Moscow State University, Moscow, 119991 Russia*

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Abstract—We consider conditions on a matrix A with unit operator $(2,1)$-norm ensuring the existence of a partition of this matrix into two submatrices with $(2,1)$-norms close to $1/2$.

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For an $N \times n$ matrix $A = \{a_{ij}\}$ viewed as an operator from $\mathbb{R}^n$ to $\mathbb{R}^N$, we define the norm

$$\|A\|_{(p,q)} = \sup_{\|x\|_{l_p^n} \le 1} \|Ax\|_{l_q^N}, \qquad 1 \le p, q \le \infty. \tag{1}$$

Estimates of the norms (1) of submatrices of a given matrix A find applications in various areas of mathematics. The first results on this topic were obtained in [1] and [2]. Moreover, the estimates in [2] (see also [3]) for the classical operator norm (i.e., $p = q = 2$) of submatrices are a consequence of the corresponding estimates for $(2,1)$-norms. The study of the $(2,1)$-norms of submatrices of a given matrix carried out in this paper is also of independent interest.

We use the following notation:

- $\langle N \rangle$ is the set of positive integers $1, 2, \dots, N$;
- the v_i, $i \in \langle N \rangle$, are the rows of A;
- the w_j, $j \in \langle n \rangle$, are the columns of A.

For $\Omega \subset \langle N \rangle$, $A(\Omega)$ is the submatrix of A formed by the rows v_i, $i \in \Omega$. We denote the inner product on $\mathbb{R}^n$ by $(\,\cdot\,,\cdot\,)$ and the norm of a vector x in the space l_p^n, $1 \le p \le \infty$, by $\|x\|_p$; $|\Lambda|$ is the number of elements in a finite set Λ.

Note that a duality argument and the definition of $(\infty,2)$-norms imply the relation

$$\|A\|_{(2,1)} = \sup_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{N} \varepsilon_i v_i \Big\|_2. \tag{2}$$

Marcus, Spielman, and Srivastava [4] established the following important result: let the columns of A form an orthonormal system in $\mathbb{R}^N$, and assume that the rows satisfy the estimate $\|v_i\|_2 \le \varepsilon$, $0 < \varepsilon < 1$. Then there exists a decomposition

$$\langle N \rangle = \Omega_1 \cup \Omega_2, \qquad \Omega_1 \cap \Omega_2 = \varnothing, \tag{3}$$

*E-mail: kashin@mi-ras.ru

**E-mail: limonova_irina@rambler.ru

such that

$$\|A(\Omega_k)\|_{(2,2)} \le \frac{1}{\sqrt{2}} + C\varepsilon, \qquad k = 1, 2,$$

where C is an absolute constant.

One of the authors of [4], Srivastava, restated this result in [5] in a form that applies to an arbitrary $N \times n$ matrix A: if

$$|(v_{i_0}, x)| \le \varepsilon \biggl(\sum_{i=1}^{N} |(v_i, x)|^2\biggr)^{1/2} \tag{4}$$

for any $x \in \mathbb{R}^n$ and $i_0 \in \langle N\rangle$, then there exists a partition (3) such that

$$\sum_{i \in \Omega_k} |(v_i, x)|^p \le \biggl(\frac{1}{2} + C\varepsilon\biggr) \sum_{i=1}^{N} |(v_i, x)|^p, \qquad p = 2, \quad k = 1, 2. \tag{5}$$

In [5], in connection with possible applications to algorithmic problems in graph theory and the geometry of normed spaces, the question was raised as to whether there exists a counterpart of relation (5) for $p = 1$. This question is directly related to the estimates of the $(2, 1)$-norms of submatrices and is considered in the present paper. As to the case of column-orthogonal matrices, a counterpart of the partition (3) with extremely small $(2, 1)$-norms of the corresponding submatrices was obtained in [6]. The approach in [6] is used below to prove Statements 3 and 4. However, Statement 2 below shows that the straightforward counterpart of property (4) of the matrix A for $p = 1$, i.e., the inequality

$$|(v_{i_0}, x)| \le \varepsilon \sum_{i=1}^{N} |(v_i, x)|, \qquad x \in \mathbb{R}^n, \quad i_0 \in \langle N\rangle, \tag{6}$$

does not guarantee the existence of a partition (3) with property (5) for $p = 1$. For its existence, it is necessary to assume in the general case that the parameter ε tends to zero as $n \to \infty$.

First, let us establish the following statement.

Statement 1. *Let A be an $N \times n$ matrix, and let*

$$\|A\|_{(2,1)} = 1, \qquad \|v_i\|_2 \le \varepsilon, \quad i \in \langle N\rangle.$$

Then there exists a partition (3) *with* $\bigl||\Omega_1^0| - |\Omega_2^0|\bigr| \le 1$ *such that*

$$\|A(\Omega_k^0)\|_{(2,1)} \le \frac{1}{\sqrt{2}} + 2\varepsilon, \qquad k = 1, 2. \tag{7}$$

Remark 1. The possibility of reducing the $(2, 1)$-norms of the submatrices of the partition under the assumptions of Statement 1 was shown in [7]. The estimate (7) refines the result obtained in [7].

Proof. Without loss of generality, we assume that N is even. (Otherwise, we supplement the matrix with a zero row.) Consider an arbitrary partition of the form (3) with $|\Omega_1| = |\Omega_2| = N/2$. Relation (2) and the parallelogram identity for the Euclidean norm imply that

$$\|A(\Omega_1)\|_{(2,1)}^2 + \|A(\Omega_2)\|_{(2,1)}^2 \le \|A\|_{(2,1)}^2 = 1. \tag{8}$$

It follows from (8) that the $(2, 1)$-norm of one of the submatrices $A(\Omega_k)$, $k = 1, 2$ (to be definite, let it be $A(\Omega_1)$), does not exceed $1/\sqrt{2}$. If, moreover,

$$\|A(\Omega_2)\|_{(2,1)} \le \frac{1}{\sqrt{2}} + 2\varepsilon,$$

then there is nothing to prove. Otherwise, we mark the point z_0 of the square $[0, 1]^2$ with coordinates $(\|A(\Omega_1)\|_{(2,1)}, \|A(\Omega_2)\|_{(2,1)})$. Let us consecutively transform the sets Ω_k, $k = 1, 2$, as follows. At the νth step ($\nu = 1, 2, \dots, N/2$), we replace one element of Ω_1 by an element of Ω_2 so that $T_{N/2}(\Omega_1)$

$T_{N/2}(\Omega_1)=\Omega_2$. (Here $T_\nu(\Omega_k)$ is the set obtained after ν steps of the transformation of the set Ω_k; in particular, $T_0(\Omega_k)=\Omega_k$.) It is clear that, for $\nu=N/2$, the point

$$z_\nu=(\|A(T_\nu(\Omega_1))\|_{(2,1)},\|A(T_\nu(\Omega_2))\|_{(2,1)})$$

is symmetric to the point z_0 with respect to the diagonal of the square passing through the point $(0,0)$. Moreover, the norm of the matrices changes by at most 2ε at each step of the construction: for $\nu=1,\dots,N/2-1$ and $k=1,2$, we have

$$\big|\|A(T_{\nu+1}(\Omega_k))\|_{(2,1)}-\|A(T_\nu(\Omega_k))\|_{(2,1)}\big|\le 2\varepsilon.$$

Therefore, if the first transition of the point through the above-mentioned diagonal occurs at the ν_0th step, then

$$\max_{k=1,2}\|A(T_{\nu_0}(\Omega_k))\|_{(2,1)}\le\frac{1}{\sqrt2}+2\varepsilon.$$

Setting $\Omega_k^0=T_{\nu_0}(\Omega_k)$, $k=1,2$, we obtain Statement 1. □

Statement 2. *Let $n=2^s$, $s=1,2,\dots$, and let $n^{-1/2}\le\varepsilon\le1$. There exists a $2n\times n$ matrix $A=A(n,\varepsilon)$ such that the estimate* (6) *holds for any $x\in\mathbb{R}^n$ and $i_0\in\langle 2n\rangle$ but, nevertheless, the inequality*

$$M\equiv\max(\|A(\Omega_1)\|_{(2,1)},\|A(\Omega_2)\|_{(2,1)})\ge\frac{1}{\sqrt2}\biggl(\frac{1}{1+(\varepsilon n^{1/2})^{-1}}\biggr)\|A\|_{(2,1)}\tag{9}$$

holds for any partition (3) *(with $N=2n$).*

Proof. The entries a_{ij} of the matrix A are defined as

$$a_{ij}=\begin{cases}\begin{cases}\varepsilon & \text{if } i=j,\\ 0 & \text{if } i\ne j,\end{cases} & \text{for}\quad 1\le i\le n,\\ \dfrac1n w_{j-1}\biggl(\dfrac{i-n-1/2}{n}\biggr) & \text{for}\quad n+1\le i\le 2n,\quad j=1,\dots,n,\end{cases}$$

where $\{w_s(x)\}_{s=0}^\infty$ is the Walsh function system orthonormal in $L^2(0,1)$ (see [8]). It is clear from the definition that $|a_{ij}|=n^{-1}$ for $i>n$ and also that the rows and columns of the matrix

$$\biggl\{w_{j-1}\biggl(\frac{i-n-1/2}{n}\biggr)\biggr\},\qquad i=n+1,\dots,2n,\quad j=1,\dots,n,$$

are pairwise orthogonal. Therefore,

$$\biggl\|\sum_{i=n+1}^{2n}\varepsilon_i v_i\biggr\|_2=\biggl(\sum_{i=n+1}^{2n}\|v_i\|_2^2\biggr)^{1/2}=1$$

for any $\varepsilon_i=\pm1$, $i=n+1,\dots,2n$.

Since the $(2,1)$-norm of the identity operator in $\mathbb{R}^n$ is $n^{1/2}$, we obtain, taking into account (2),

$$\|A\|_{(2,1)}\le\varepsilon n^{1/2}+1.$$

Considering the rows of A with numbers $1\le i\le n$, we see that, for any partition of the rows into two groups, the submatrix generated by the larger part has $(2,1)$-norm $\ge\varepsilon(n/2)^{1/2}$. Thus, the number M defined in (9) admits the lower bound

$$M\ge\frac{\varepsilon(n/2)^{1/2}}{\varepsilon n^{1/2}+1}\|A\|_{(2,1)}\ge\frac{1}{\sqrt2}\biggl(\frac{1}{1+(\varepsilon n^{1/2})^{-1}}\biggr)\|A\|_{(2,1)}$$

for any partition (3) (with $N = 2n$), which proves (9). Now let us verify that the matrix A satisfies relation (6). Let $x = \{x_j\} \in \mathbb{R}^n$ and $i_0 \in \langle 2n\rangle$ be fixed. If $i_0 > n$, then, by the construction of A,

$$|(v_{i_0}, x)| \le \frac{1}{n}\|x\|_1, \qquad \sum_{i=1}^{n} |(v_i, x)| = \varepsilon\|x\|_1,$$

whence, in view of the inequality $\varepsilon \ge n^{-1/2}$, we obtain (6) for this i_0. If, on the contrary, $i_0 \le n$, then

$$|(v_{i_0}, x)| = \varepsilon|x_{i_0}| \tag{10}$$

and

$$|(v_i, x)| = \frac{1}{n}\bigg|\sum_{j=1}^{n} x_j w_{j-1}\Big(\frac{i-n-1/2}{n}\Big)\bigg|$$

for $i > n$. Therefore,

$$\sum_{i=n+1}^{2n} |(v_i, x)| = \frac{1}{n}\sum_{i=n+1}^{2n}\bigg|\sum_{j=1}^{n} x_j w_{j-1}\Big(\frac{i-n-1/2}{n}\Big)\bigg| = \|f\|_{L^1(0,1)}, \tag{11}$$

where the function f constant on the intervals $((\nu-1)/n, \nu/n)$, $1 \le \nu \le n$, is given by

$$f\Big(\frac{\nu-1/2}{n}\Big) = \sum_{j=1}^{n} x_j w_{j-1}\Big(\frac{\nu-1/2}{n}\Big).$$

It follows from the formula for the Fourier–Walsh coefficients of the function f that

$$\|f\|_{L^1(0,1)} = \int_0^1 |f(x)w_{j-1}(x)|\,dx \ge \bigg|\int_0^1 f(x)w_{j-1}(x)\,dx\bigg| = |x_j|$$

for $j \in \langle n\rangle$, so that

$$\|f\|_{L^1(0,1)} \ge \|x\|_\infty,$$

whence inequality (6) for $i_0 \in \langle n\rangle$ readily follows as well (see (10) and (11)). The proof of Statement 2 is complete. □

Statement 3. *Assume that* (6) *holds for an* $N \times n$ *matrix* A *for some* $0 < \varepsilon \le 1/n$. *Then there exists a partition* (3) *such that*

$$\|A(\Omega_k)\|_{(2,1)} \le \Big(\frac{1}{2} + 2\phi(n,\varepsilon)\Big)\|A\|_{(2,1)}, \qquad k = 1, 2,$$

where

$$\phi(n,\varepsilon) = \Big(n\varepsilon \ln\frac{8}{n\varepsilon}\Big)^{1/3}. \tag{12}$$

Remark 2. By a slight modification of (12), one can additionally ensure that the numbers $|\Omega_1|$ and $|\Omega_2|$ are close to $N/2$.

Remark 3. For $1/n < \varepsilon < 1/\sqrt{n}$, the problem on the existence of a partition (3) with the $(2,1)$-norms of submatrices close to $(1/2)\|A\|_{(2,1)}$ remains open. It seems to be likely that the desired partition exists for $\varepsilon = \overline{\overline{o}}(1/\sqrt{n})$.

Proof. Assume that $\varepsilon n < 1/64$; otherwise, Statement 3 is obvious. Let $\delta = (\varepsilon n)^{1/3}$, and let $\mathbb{Y}$ be a δ-net on the Euclidean sphere $\mathbf{S}^{n-1} \subset \mathbb{R}^n$ with at most $(3/\delta)^n$ elements. For $\omega \subset \langle N\rangle$, we set $\overline{\omega} = \langle N\rangle \setminus \omega$. For each pair $(\omega, \overline{\omega})$, we find vectors $z_1, z_2 \in \mathbf{S}^{n-1}$ such that

$$\|A(\omega)\|_{(2,1)} = \|A(\omega)z_1\|_1, \qquad \|A(\overline{\omega})\|_{(2,1)} = \|A(\overline{\omega})z_2\|_1.$$

If $\|A(\omega)\|_{(2,1)} \ge \|A(\overline{\omega})\|_{(2,1)}$, then we set $\omega' = \omega$ and $z_{\omega'} = z_1$. Otherwise, we set $\omega' = \overline{\omega}$ and $z_{\omega'} = z_2$. Take $y_{\omega'}$ to be one of the vectors in $\mathbb{Y}$ closest to $z_{\omega'}$. In all, there exist $2^{N-1} - 1$ distinct partitions of the set $\langle N \rangle$ into two nonempty parts. Therefore, there exists a $y_0 \in \mathbb{Y}$ such that the set $K = \{\omega' : y_0 = y_{\omega'}\}$ is sufficiently large:

$$|K| \ge (2^{N-1} - 1)\left(\frac{\delta}{3}\right)^n. \tag{13}$$

Let

$$\beta = \min_{w \subset \langle N \rangle} \max\left\{\frac{\|A(\omega)\|_{(2,1)}}{\|A\|_{(2,1)}}, \frac{\|A(\overline{\omega})\|_{(2,1)}}{\|A\|_{(2,1)}}\right\}.$$

Note that

$$\begin{aligned}\beta\|A\|_{(2,1)} &\le \|A(\omega')\|_{(2,1)} = \|A(\omega')z_{\omega'}\|_1 \\ &= \|A(\omega')((z_{\omega'} - y_0) + y_0)\|_1 \le \|A(\omega')(z_{\omega'} - y_0)\|_1 + \|A(\omega')y_0\|_1 \\ &\le \|A(\omega')\|_{(2,1)}\delta + \sum_{i \in \omega'} |(v_i, y_0)| \le \|A\|_{(2,1)}\delta + \sum_{i \in \omega'} |(v_i, y_0)| \qquad \text{for} \quad \omega' \in K.\end{aligned}$$

Thus,

$$\sum_{i \in \omega'} |(v_i, y_0)| > (\beta - \delta)\|A\|_{(2,1)} \qquad \text{for} \quad \omega' \in K. \tag{14}$$

Therefore, to prove Statement 3, it suffices to verify that, for $\beta = 1/2 + 2\phi(n, \varepsilon)$, the number R of subsets $\omega' \subset \langle N \rangle$ such that inequality (14) is true is less than the right-hand side of (13).

Let σ be a permutation of the set $\langle N \rangle$ such that the numbers $g_\nu = |(v_{\sigma(\nu)}, y_0)|$ form a nonincreasing sequence. We set $g_\nu = 0$ for $\nu > N$. Further, let

$$S = \sum_{\nu=1}^{N} g_\nu.$$

Let us verify that

$$\sum_{\nu=q+1}^{N} g_\nu^2 \le \frac{S^2}{q} \tag{15}$$

for each q.

Indeed, since the sequence g_ν is monotone, we have $g_{q+1} \le S/q$, and therefore,

$$\sum_{\nu=q+1}^{N} g_\nu^2 \le g_{q+1} \sum_{\nu=q+1}^{N} g_\nu \le \frac{S}{q} \cdot S = \frac{S^2}{q}.$$

It is clear from the definition of $(2,1)$-norm that $\|A\|_{(2,1)} \ge S$; therefore (see (14)), if $\omega' \in K$, then

$$\sum_{i \in \omega'} |(v_i, y_0)| > (\beta - \delta)S.$$

It follows from the last relation that

$$\begin{aligned}R &\le 2^N m\left\{t \in (0,1) : \sum_{\nu=1}^{N} g_\nu r_\nu(t) > (2(\beta - \delta) - 1)S\right\} \\ &= 2^N m\left\{t \in (0,1) : \sum_{\nu=1}^{N} g_\nu r_\nu(t) > (4\phi(n,\varepsilon) - 2\delta)S\right\},\end{aligned} \tag{16}$$

where m is the Lebesgue measure on $(0,1)$ and $\{r_\nu(t)\}_{\nu=1}^\infty$ is the Rademacher function system (see [8]). Using property (6) of the matrix A for $x = y_0$, we obtain

$$\sum_{\nu=1}^{N} g_\nu r_\nu(t) \le \sum_{\nu\ge q+1} g_\nu r_\nu(t) + \varepsilon q S. \tag{17}$$

It follows from (16) and (17) that

$$R \le 2^N m\biggl\{t\in(0,1) : \sum_{\nu\ge q+1} g_\nu r_\nu(t) > (4\phi(n,\varepsilon) - 2\delta - \varepsilon q)S\biggr\}. \tag{18}$$

Fix $q = [2n^{1/3}\ln^{1/3}(2/\delta)\varepsilon^{-2/3}]$, where $[x]$ is the integer part of x. It is clear that

$$q > n^{1/3}\ln^{1/3}\biggl(\frac{2}{\delta}\biggr)\varepsilon^{-2/3}.$$

Let us estimate $4\phi(n,\varepsilon) - 2\delta - \varepsilon q$ for the selected q:

$$\begin{aligned}4\phi(n,\varepsilon) - 2\delta - \varepsilon q &\ge 4\cdot 3^{1/3} n^{1/3}\varepsilon^{1/3}\ln^{1/3}\frac{2}{\delta} - 2n^{1/3}\varepsilon^{1/3}\frac{\ln^{1/3}(2/\delta)}{\ln^{1/3}8} - 2n^{1/3}\varepsilon^{1/3}\ln^{1/3}\frac{2}{\delta}\\ &\ge (5,7-1,6-2)n^{1/3}\varepsilon^{1/3}\ln^{1/3}\frac{2}{\delta} \ge 2n^{1/3}\varepsilon^{1/3}\ln^{1/3}\frac{2}{\delta}\,,\end{aligned} \tag{19}$$

where we used the fact that $\delta < 1/4$. In particular, $4\phi(n,\varepsilon) - 2\delta - \varepsilon q > 0$. In what follows, we assume that $\sum_{\nu=q+1}^N g_\nu^2 \ne 0$, because otherwise $R \le 0$ and Statement 3 holds. To estimate the measure on the right-hand side in (18), we use (15), (19), and the classical exponential estimate for the distribution function of a polynomial over the Rademacher system (see [8]):

$$\begin{aligned}R &\le 2^N \exp\biggl\{\frac{-S^2 4n^{2/3}\varepsilon^{2/3}\ln^{2/3}(2/\delta)}{2\sum_{\nu=q+1}^N g_\nu^2}\biggr\} \le 2^N \exp\biggl\{-2n^{2/3}\varepsilon^{2/3}\biggl(\ln^{2/3}\frac{2}{\delta}\biggr)q\biggr\}\\ &\le 2^N\exp\biggl\{-2n\biggl(\ln\frac{2}{\delta}\biggr)\biggr\} < (2^{N-1}-1)\biggl(\frac{\delta}{3}\biggr)^n,\end{aligned}$$

which contradicts inequality (13). The proof of Statement 3 is complete. □

Statement 3 only estimates the norm of the matrices $A(\Omega_k)$, $k = 1, 2$. For a pointwise estimate similar to (5), we need to impose additional conditions on the matrix A.

Statement 4. *Assume that an $N\times n$ matrix A satisfies (6) with $0 < \varepsilon < 1/n$ and*

$$0 < b\|x\|_2 \le \|Ax\|_1 \le B\|x\|_2$$

for any $x\in\mathbb{R}^n$, $x\ne 0$. Then there exists a partition (3) such that

$$\|A(\Omega_k)x\|_1 \le \gamma\|Ax\|_1, \qquad \gamma = \frac{1}{2} + 4\biggl(n\varepsilon\ln\frac{2B}{b\varepsilon^{1/3}n^{1/3}}\biggr)^{1/3} \tag{20}$$

for any $x\in\mathbb{R}^n$ and $k = 1, 2$.

Proof. Assume that $\gamma < 1$; otherwise, the estimate (20) is obvious. In this case, $(n\varepsilon)^{1/3} < 1/4$. Assume that, on the contrary, for any partition (3), either there exists a vector $x_1 \in \mathbf{S}^{n-1}$ such that

$$\|A(\Omega_1)x_1\|_1 > \gamma\|Ax_1\|_1$$

(in this case, we set $\omega' = \Omega_1$ and $x_{\omega'} = x_1$) or

$$\|A(\Omega_2)x_2\|_1 > \gamma\|Ax_2\|_1$$

for some $x_2 \in \mathbf{S}^{n-1}$ (in this case, we set $\omega' = \Omega_2$ and $x_{\omega'} = x_2$). Let

$$\delta = \frac{b}{B}(\varepsilon n)^{1/3}.$$

Further, arguing as in the proof of Statement 3, we can show that there exists a $y_0 \in \mathbf{S}^{n-1}$ and at least $(2^{N-1}-1)(\delta/3)^n$ subsets of $\omega' \subset \langle N \rangle$ such that

$$\|A(\omega')x_{\omega'}\|_1 > \gamma\|Ax_{\omega'}\|_1, \qquad \|y_0 - x_{\omega'}\|_2 < \delta.$$

Let K be the collection of such subsets. Note that

$$\|A(\omega)x\|_1 \le \|A(\omega)\|_{(2,1)} \le \|A\|_{(2,1)}$$

for $x \in \mathbf{S}^{n-1}$ and $\omega \subset \langle N \rangle$. For $\omega' \in K$, given that $\gamma < 1$, we have

$$\begin{aligned}
\|A(\omega')y_0\|_1 &\ge \|A(\omega')x_{\omega'}\|_1 - \|A(\omega')(x_{\omega'} - y_0)\|_1 \\
&> \gamma\|Ax_{\omega'}\|_1 - \delta\left\|A(\omega')\left\{\frac{x_{\omega'} - y_0}{\|x_{\omega'} - y_0\|_2}\right\}\right\|_1 \\
&\ge \gamma\|Ay_0\|_1 - \gamma\|A(x_{\omega'} - y_0)\|_1 - \delta B \\
&\ge \gamma\|Ay_0\|_1 - 2\delta B \ge \gamma\|Ay_0\|_1 - 2\delta\frac{\|Ay_0\|_1}{b}B = \|Ay_0\|_1\left(\gamma - 2\delta\frac{B}{b}\right).
\end{aligned} \tag{21}$$

Again, by analogy with the proof of Statement 3, we set

$$g_\nu = |(v_{\sigma(\nu)}, y_0)|,$$

where the permutation σ of the set $\langle N \rangle$ is chosen so that the sequence g_ν is nonincreasing; further, let $g_\nu = 0$ for $\nu = N+1, N+2, \dots$, and let

$$S = \|Ay_0\|_1 = \sum_{\nu=1}^{N} g_\nu = \sum_{\nu=1}^{\infty} g_\nu.$$

Let R be the number of subsets $\omega \subset \langle N \rangle$ such that

$$\|A(\omega)y_0\|_1 \ge \left(\gamma - 2\delta\frac{B}{b}\right)\|Ay_0\|_1. \tag{22}$$

Let us show that $R < (2^{N-1}-1)(\delta/3)^n$, thus arriving at a contradiction and completing the proof of Statement 4. By analogy with (16), we have

$$\begin{aligned}
R &\le 2^N m\left\{t \in (0,1) : \sum_{\nu=1}^{N} g_\nu r_\nu(t) \ge \left(1 + 8\left(n\varepsilon \ln\frac{2}{\delta}\right)^{1/3} - 4\delta\frac{B}{b} - 1\right)S\right\} \\
&\le 2^N m\left\{t \in (0,1) : \sum_{\nu=1}^{N} g_\nu r_\nu(t) \ge 4\left(n\varepsilon \ln\frac{2}{\delta}\right)^{1/3} S\right\}.
\end{aligned} \tag{23}$$

Let

$$q = \left[2n^{1/3}\varepsilon^{-2/3}\ln^{1/3}\left(\frac{2}{\delta}\right)\right] > n^{1/3}\varepsilon^{-2/3}\ln^{1/3}\left(\frac{2}{\delta}\right).$$

Continuing the estimate (23) and using property (6) of the matrix A, we find that

$$\begin{aligned}
R &\le 2^N m\left\{t \in (0,1) : \sum_{\nu>q} g_\nu r_\nu(t) + \varepsilon q S > 4\left(n\varepsilon \ln\frac{2}{\delta}\right)^{1/3} S\right\} \\
&= 2^N m\left\{t \in (0,1) : \sum_{\nu>q} g_\nu r_\nu(t) > \left(4\left(n\varepsilon \ln\frac{2}{\delta}\right)^{1/3} - \varepsilon q\right)S\right\}.
\end{aligned} \tag{24}$$

Let us estimate $4(n\varepsilon\ln(2/\delta))^{1/3} - \varepsilon q$ from below. The choice of q ensures that this number is not less than

$$4\left(n\varepsilon\ln\frac{2}{\delta}\right)^{1/3} - 2\left(n\varepsilon\ln\frac{2}{\delta}\right)^{1/3} \geq 2\left(n\varepsilon\ln\frac{2}{\delta}\right)^{1/3}. \tag{25}$$

Again using the exponential estimate for the distribution function of a polynomial in the Rademacher system and (25), from (24) we obtain

$$R \leq 2^N \exp\left\{\frac{-4S^2(n^{1/3}\varepsilon^{1/3}\ln^{1/3}(2/\delta))^2}{2\sum_{\nu>q} g_\nu^2}\right\}; \tag{26}$$

since $\sum_{\nu>q} g_\nu{}^2 \leq S^2/q$ (see the proof of Statement 3), we see that the right-hand side can be estimated from above as

$$R \leq 2^N \exp\left\{-2\left(n^{1/3}\varepsilon^{1/3}\ln^{1/3}\frac{2}{\delta}\right)^2 q\right\} \leq 2^N \exp\left\{-\left(2n\ln\frac{2}{\delta}\right)\right\}$$
$$\leq 2^N\left(\frac{\delta}{2}\right)^{2n} < (2^{N-1}-1)\left(\frac{\delta}{3}\right)^n,$$

where the last inequality follows from the fact that $\delta < 1/4$. The proof of Statement 4 is complete. ☐

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