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ON A NORM AND APPROXIMATE CHARACTERISTICS OF CLASSES OF MULTIVARIABLE FUNCTIONS

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Abstract. We introduce a space of quasicontinuous functions and study its approximate characteristics, i.e., ε-entropy and widths. We establish inequalities for norms of trigonometric polynomials in this space. In addition, we obtain exponents of the ε-entropy and widths of some classes of functions with low smoothness.

1. Introduction

In this paper, asymptotic characteristics of classes of multivariable functions are considered, namely, ε-entropy and the Kolmogorov widths. The classes of interest are defined by a limitation on mixed derivatives or by a Lipschitz-type condition on mixed differences. Such classes have been studied in approximation theory for the past forty years. Our interest in it is due to its important applications to numerical integration, to numerical solution of partial differential equations (sparse grid method), to the problem of the complexity of continuous algorithms, and to probability theory. There was also another incentive to begin this work. It is known that for the above-mentioned classes many approximation problems in the L_p-norm with constraint on the derivation or difference under the sign of the L_q-norm in the case where p or q equals 1 or ∞ have remained open to decades. Consequently, on the one hand we concentrate on these questions, and on the other we introduce a new norm close to the uniform one for which one can make progress in problems remaining unsolved with respect to the uniform norm. An important role in our investigation is played by studying trigonometric polynomials with harmonics from hyperbolic crosses.

The results we obtain are briefly listed below.

For any set $\Lambda \subset \mathbb{Z}^d$, denote by $\mathcal{T}(\Lambda)$ the collection of trigonometric polynomials of the form

$$t(x) = \sum_{k\in\Lambda} c_k e^{i(k,x)}, \qquad x \in \mathbb{T}^d;$$

in the case where Λ is symmetric about the origin ($\Lambda = -\Lambda$), put

$$\mathcal{T}_r(\Lambda) = \bigl\{t(x) \in \mathcal{T}(\Lambda) : c_k = \overline{c}_{-k},\ k \in \Lambda\bigr\}.$$

For even n and $d \geq 2$, put

$$Y^d_n = \bigl\{s = (2l_1, \dots, 2l_d),\ l_1 + \dots + l_d = n/2,\ l \in \mathbb{Z}^d_+\bigr\};$$

for $s \in \mathbb{Z}^d_+$, put

$$\rho(s) := \Bigl\{k = (k_1, \dots, k_d) \in \mathbb{Z}^d : \bigl[2^{s_j-1}\bigr] \leq |k_j| < 2^{s_j},\ j = 1, \dots, d\Bigr\}.$$

For $n = 1, 2, \dots$, set

$$Q_n \equiv \bigcup_{\|s\|_1\leq n} \rho(s), \qquad \Delta Q_{n+1} \equiv Q_{n+1} \setminus Q_n.$$

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Further, let μ be the normed Lebesgue measure on the unit circle. Given a function $f \in L^1(d\mu)$ with Fourier series

$$f \sim \sum_{s=0}^{\infty} \delta_s(f, x),$$

$$\delta_0 = \int f \, d\mu, \qquad \delta_s = \sum_{2^{s-1} \le |k| < 2^s} \widehat{f}(k) e^{ikx}, \quad s = 1, 2, \dots,$$

let us introduce the value

$$\|f\|_{QC} \equiv \int_0^1 \left\| \sum_{s=0}^{\infty} r_s(\omega) \delta_s(f, x) \right\|_{L^\infty(d\mu)} d\omega, \tag{1.1}$$

where $\{r_k(\omega)\}_{k=0}^{\infty}$ is Rademaher's system (see [8].) *The space of quasicontinuous functions* is defined as the closure of the set of trigonometric polynomials in norm (1.1).

Spaces of quasicontinuous functions can also be defined in the multidimensional case; there are several methods of doing this (see Sec. 5.) Next we consider one of them, namely, the closure of the set of trigonometric polynomials in d variables ($d = 2, 3, \dots$) in the norm

$$\|f\|_{QC} \equiv \left\| \|f(\cdot, x^1)\|_{QC} \right\|_{\infty}, \tag{1.2}$$

where, for $x = (x_1, \dots, x_d) \in \mathbb{T}^d$, we set by definition $x^1 = (x_2, \dots, x_d) \in \mathbb{T}^{d-1}$. In other words, the QC-norm in (1.2) is with respect to the variable x_1 and the sup-norm with respect to the remaining variables.

In [22], the following inequality for trigonometric polynomials in two variables ($d = 2$) was established treating approximation problems. For any $t_s \in \mathcal{T}(\rho(s))$,

$$\left\| \sum_{s \in Y_n^2} t_s \right\|_{\infty} \ge A \sum_{s \in Y_n^2} \|t_s\|_1, \qquad A > 0. \tag{1.3}$$

Let us add that an inequality similar to (1.3) for polynomials with respect to the Haar system had been obtained in [18] in connection with applications to Gaussian processes. V. N. Temlyakov proposed the conjecture that in the multidimensional case ($d \ge 3$) there is the inequality

$$\left\| \sum_{s \in Y_n^d} t_s \right\| \ge c(d) n^{-(d-2)/2} \sum_{s \in Y_n^d} \|t_s\|_1, \qquad c(d) > 0, \quad t_s \in \mathcal{T}(\rho(s)), \tag{1.4}$$

which is an open problem. In any case, in the present paper we prove an analogue of inequality (1.4) for the QC-norm. Moreover, in Sec. 5, we note that there is an analogue of (1.4) where the norm $\| \cdot \|_\infty$ is replaced by the next one

$$\|f\|_{QC,L} \equiv \left\| \|f(\cdot, x^1)\|_{QC} \right\|_1,$$

which is weaker than $\| \cdot \|_{QC}$. Treating the d-dimensional case is based on a one-dimensional inequality for the QC-norm.

Theorem 1.1. *Any real function $f \in L^1(d\mu)$ satisfies the inequality*

$$\|f\|_{QC} \ge \frac{1}{16} \sum_{s=0}^{\infty} \|\delta_s(f)\|_1.$$

Remark 1.1. From Theorem 1.1 and a result due to P. G. Grigoriev [5] it follows that

$$\sup_{t \in \mathcal{T}_r(2^k)} \frac{\|t\|_{QC}}{\|t\|_\infty} \ge c\sqrt{k}, \qquad c > 0, \quad \mathcal{T}_r(2^k) = \mathcal{T}_r([-2^k, 2^k]).$$

On the other hand, from results concerning lacunar series we have

$$\sup_{t\in\mathcal{T}_r(2^k)}\frac{\|t\|_\infty}{\|t\|_{QC}}\ge c_1\sqrt{k},\qquad c_1>0.$$

The corresponding example, for which we are grateful to K. I. Oskolkov, is given at the end of the article.

Section 2 is devoted to inequalities for trigonometric polynomials; in Sec. 3 we apply the results of Sec. 2 to estimate entropy numbers and Kolmogorov widths in the QC-norm for classes of functions with bounded derivative ($W^r_{q,\alpha}$) and classes of functions with constraint on mixed difference (H^r_q).

Let us recall the corresponding definitions.

Let $r>0$ and $\alpha\in\mathbb{R}$. Define the one-dimensional Bernoulli kernel by the formula

$$F_r(t,\alpha)=1+2\sum_{k=1}^{\infty}k^{-r}\cos\left(kt-\frac{\alpha\pi}{2}\right),\qquad t\in(0,2\pi),$$

and the multidimensional one by

$$F_r(x,\alpha)=\prod_{j=1}^{d}F_r(x_j,\alpha_j)$$

for $x=(x_1,\dots,x_d)$, $\alpha=(\alpha_1,\dots,\alpha_d)$.

Finally, put

$$W^r_{q,\alpha}\equiv\big\{f: f=F_r(\cdot,\alpha)*\varphi(\cdot),\ \|\varphi\|_q\le 1\big\},$$

where $*$ is the convolution.

For $r>0$, set $l=[r]+1$ and consider the difference operator $\Delta_h^{l,j}$ of order l with respect to the variable x_j and with step h. For a collection of positive integers $e\subset[1,d]$, let us define the mixed difference operator

$$\Delta_t^l(e)=\prod_{j\in e}\Delta_{t_j}^{l,j},\qquad t=(t_1,\dots,t_d),\quad \Delta_t^l(\varnothing)=\mathrm{Id}\,.$$

Then

$$H^r_q=\left\{f\in L_q(\mathbb{T}^d):\forall e\subset[1,d],\ \|\Delta_t^l(e)f\|_q\le\prod_{j\in e}|t_j|^r\right\}.$$

Recall two definitions essential for the following. Let K be a compact set in a Banach space X; let B_X be the unit ball in X. The values

$$d_m(K,X)=\inf_{\{u_i\}_{i=1}^m\subset X}\ \sup_{f\in K}\ \inf_{c_i}\left\|f-\sum_{i=1}^{m}c_iu_i\right\|_X,$$

$$\varepsilon_m(K,X)=\inf\left\{\varepsilon:\exists\{u_i\}_{i=1}^q\in X,\ q\le 2^{m-1},\ K\subset\bigcup_{i=1}^{q}\{u_i+\varepsilon B_X\}\right\}$$

($m=1,2,\dots$) are called the mth *Kolmogorov width* and the m-th *entropy number* of the set K in the space X respectively.

There are, in particular, the results below in Sec. 3.

Theorem 1.2. *For* $r>\max(1/q,1/2)$ *and* $1<q\le\infty$, *we have* ($d\ge 2$)

$$\varepsilon_m(H^r_q,QC)\asymp m^{-r}(\log m)^{r(d-1)+d/2},$$

$$\varepsilon_m(W^r_q,QC)\asymp m^{-r}(\log m)^{r(d-1)+1/2}.$$

Theorem 1.3. *For $r > 1/2$ and $2 \le q \le \infty$, we have* $(d \ge 2)$

$$d_m(H^r_q, QC) \asymp m^{-r}(\log m)^{r(d-1)+d/2},$$
$$d_m(W^r_q, QC) \asymp m^{-r}(\log m)^{r(d-1)+1/2}.$$

In Sec. 4, we consider the problem of equivalence between the uniform norm and the uniform discrete norm for trigonometric polynomials with harmonics from hyperbolic crosses. As is well known, for the space $\mathcal{T}(\Pi)$ of trigonometric polynomials in d variables with the spectrum in a parallelepiped Π there exists a finite set Ω of cardinality $|\Omega|$ whose exponent equals the dimension of $\mathcal{T}(\Pi)$ such that

$$\|t\|_\infty \asymp \|t\|_{\infty,\Omega} \equiv \max_{x\in\Omega} |t(x)|, \qquad t \in \mathcal{T}(\Pi).$$

The results in Sec. 4 show a striking difference in the d-dimensional case ($d = 2, 3, \dots$) for the spaces $\mathcal{T}(Q_n)$. The equivalence between the norm $\|t\|_\infty$ and $\|t\|_{\infty,\Omega}$ for all the polynomials in $\mathcal{T}(Q_n)$ may exist only if the cardinality of Ω much exceeds the dimension $\dim \mathcal{T}(Q_n) \asymp 2^n n^{d-1}$: $|\Omega| \ge 2^{(1+\gamma)n}$, $\gamma > 0$.

More precisely, from Theorem 4.1 proved in Sec. 4 follows the following corollary.

Corollary 1.1. *Assume that a set $\Omega \subset \mathbb{T}^d$, $d \ge 2$, has the following property: for any polynomial $t \in \mathcal{T}(Q_n)$,*

$$\|t\|_\infty \le bn^\alpha \|t\|_{\infty,\Omega}, \qquad 0 \le \alpha \le \frac{1}{2}.$$

Then

$$|\Omega| \ge c_1 |Q_n| \exp(cn^{1-2\alpha}), \qquad c = c(b, \alpha, d), \quad c_1 = c_1(d).$$

Finally, Sec. 5 offers some properties of the QC-norm.

The main results have been announced in [10].

2. Inequalities for Trigonometric Polynomials

The purpose of the present section is to prove two inequalities for the QC-norm (see Theorems 1.1 and 2.1), but first we give simple properties.

If $f \in L^1(d\mu)$ and

$$F(x, \omega) = \sum_{s=0}^{\infty} r_s(\omega)\delta_s(f, x), \tag{2.1}$$

then

$$\inf_\omega \|F(\cdot, \omega)\|_\infty \le \|f\|_{QC} \le \sup_\omega \|F(\cdot, \omega)\|_\infty, \tag{2.2}$$

$$\|f\|_{QC} \ge \left\| \int_0^1 |F(x, \omega)|\, d\omega \right\|_\infty \gg \left\| \left(\sum_s |\delta_s(f, x)|^2 \right)^{1/2} \right\|_\infty, \tag{2.3}$$

$$\|f\|_{QC} \le \sum_s \|\delta_s(f)\|_\infty \equiv \|f\|_{B_{\infty,1}}. \tag{2.4}$$

We now turn to the proof of Theorem 1.1 stated in the Introduction; it will be a consequence of the following lemma.

Lemma 2.1. *Let $P_n \subset \mathbb{Z}^+$ be an arithmetical progression of the form $4l + b$, $l = 0, \dots, n$, $b \in \{1, 2, 3, 4\}$. Then for any real trigonometric polynomial f we have*

$$\|f\|_{QC} \ge \frac{1}{4} \sum_{s \in P_n} \|\delta_s(f)\|_1 + |\widehat{f}(0)|.$$

Proof. Assume that $U_s = V_{2^s} - V_{2^{s-2}}$, $s \geq 2$, $U_1 = 2\cos x$, $U_0 \equiv 1$, where V_k is the de la Vallée-Poussin kernel

$$V_k(x) = \frac{1}{k} \sum_{l=k}^{2k-1} \sum_{|\nu| \leq l} e^{i\nu x}, \qquad k \geq 1.$$

Then $\deg U_s < 2^{s+1}$; for $s \geq 1$

$$\widehat{U}_s(k) = 1, \quad \text{for} \quad 2^{s-1} \leq |k| \leq 2^s,$$
$$\widehat{U}_s(k) = 0, \quad \text{for} \quad |k| \leq 2^{s-2}.$$

Further, using the well-known estimate $\|V_k\|_1 \leq 2$, we obtain $\|U_s\|_1 \leq 4$.

Without loss of generality, we assume that $\widehat{f}(0) \geq 0$ and for the function $f = \sum_s \delta_s(f)$ define the polynomials

$$g_s = \big(\operatorname{sign} \delta_s(f)\big) * U_s.$$

Then $\|g_s\|_\infty \leq 4$; using the notation

$$\langle f, g \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} fg\, dx \equiv \int fg\, d\mu,$$

we obtain

$$\langle \delta_s(f), g_s \rangle = \langle \delta_s(f), \operatorname{sign} \delta_s(f) \rangle = \|\delta_s(f)\|_1. \tag{2.5}$$

Consider a Riesz product

$$\Phi(x,t) = \prod_{s \in P_n} \left(1 + \frac{1}{4} g_s(x) r_s(t)\right).$$

It can be represented in the form

$$\Phi(x,t) = 1 + \frac{1}{4} \sum_{s \in P_n} g_s(x) r_s(t) + \sum_e w_e(x,t),$$

where $e \subset P_n$, $|e| \geq 2$, and

$$w_e(x,t) = \prod_{s \in e} \frac{1}{4} g_s(x) r_s(t). \tag{2.6}$$

Let us first consider $\Phi(x,t)$ as a function of x while t is fixed. By virtue of the inequality $\|g_s\|_\infty \leq 4$ $\Phi(\cdot, t)$ is nonnegative. Prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi(x,t)\, dx = 1.$$

From the definition of $g_s(x)$ it follows that for all $s \in P_n$, $\widehat{g}_s(0) = 0$.

We also note that for any e, $|e| \geq 2$, the zero Fourier coefficient of $w_e(x,t)$ as a function of x equals zero. Indeed, let $e = \{s_1 > s_2 > \cdots > s_m\} \subset P_n$. Then the Fourier series of $w_e(x,t)$ contains no frequencies with absolute values less than

$$2^{s_1 - 2} - 2^{s_2 + 1} - \cdots - 2^{s_m + 1} \geq 2^{s_1} \left(\frac{1}{4} - \frac{1}{6}\right) > 0.$$

Thus, for each t

$$\|\Phi(\cdot, t)\|_1 = 1.$$

Consider now $\Phi(x,t)$ as a function of t while x is fixed. From (2.6) it follows that for any constant x the functions $w_e(x,t)$, $e \subset P_n$, $|e| \geq 2$, of the variable t are orthogonal to the Rademacher functions

$r_s(t)$, $s = 0, 1, \dots$. Moreover, they are pairwise orthogonal as distinct Walsh functions. Hence (see also (2.1) and (2.5)) we have the inequality

$$I \equiv \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} F(x,t)\Phi(x,t)\,dx\,dt = \frac{1}{2\pi} \sum_s \int_0^1 r_s(t) \int_0^{2\pi} \delta_s(f,x)\Phi(x,t)\,dx\,dt$$

$$= \sum_s \frac{1}{2\pi} \int_0^{2\pi} \delta_s(f,x) \int_0^1 r_s(t)\Phi(x,t)\,dt\,dx = \widehat{f}(0) + \sum_{s \in P_n} \frac{1}{2\pi}\frac{1}{4} \int_0^{2\pi} \delta_s(f,x) g_s(x)\,dx$$

$$= |\widehat{f}(0)| + \frac{1}{4} \sum_{s \in P_n} \|\delta_s(f)\|_1.$$

Next for any t

$$\frac{1}{2\pi} \int_0^{2\pi} F(x,t)\Phi(x,t)\,dx \le \|F(\cdot,t)\|_\infty \|\Phi(\cdot,t)\|_1 = \|F(x,t)\|_\infty.$$

Therefore,

$$I \le \int_0^1 \|F(\cdot,t)\|_\infty\,dt;$$

therefore, Lemma 2.1 is proved. □

We now turn to the multidimensional case. For a function f depending on d variables ($d = 2, 3, \dots$) we define the norm

$$\|f\|_{QC} = \big\| \|f(\cdot, x^1)\|_{QC} \big\|_\infty,$$

i.e., the QC-norm is with respect to x_1 and the L_∞-norm is with respect to the others.

Theorem 2.1. *For any polynomial $f \in \mathcal{T}_r(\Delta Q_n)$ such that*

(1) $\|\delta_s(f)\|_4 \le 1$ *for all* s, $\|s\|_1 = n$, *where* $\delta_s = \sum_{k \in \rho(s)} \widehat{f}(k) e^{i(k,x)}$,

(2) $\|f\|_2 \gg n^{(d-1)/2}$,

the inequality

$$\|f\|_{QC} \gg n^{d/2}$$

holds.

Proof. Let us prove that $\|f\|_2^2$ is bounded by $\|f\|_{QC}$. We have

$$\|f\|_2^2 = \sum_{s_1=0}^{n} \Bigg\| \sum_{\|s^1\|_1 = n - s_1} \delta_s(f) \Bigg\|_2^2, \qquad s^1 = (s_2, \dots, s_d).$$

Using the inequality $\|g\|_2 \le \|g\|_1^{1/3} \|g\|_4^{2/3}$, we obtain

$$\le \sum_{s_1=0}^{n} \left(\frac{1}{2\pi}\right)^{d-1} \int_{\mathbb{T}^{d-1}} \Bigg\| \sum_{\|s^1\|_1 = n-s_1} \delta_s(f, \cdot, x^1) \Bigg\|_1^{2/3} \Bigg\| \sum_{\|s^1\|_1 = n-s_1} \delta_s(f, \cdot, x^1) \Bigg\|_4^{4/3} dx^1.$$

Let

$$f_{s_1} = \sum_{\|s^1\|_1 = n - s_1} \delta_s(f).$$

Employing the Hölder inequality corresponding to the exponents 3/2 and 3, we obtain

$$\sum_{s_1=0}^{n} \|f_{s_1}(\cdot,x^1)\|_1^{2/3}\|f_{s_1}(\cdot,x^1)\|_4^{4/3} \le \left(\sum_{s_1=0}^{n}\|f_{s_1}(\cdot,x^1)\|_1\right)^{2/3}\left(\sum_{s_1=0}^{n}\|f_{s_1}(\cdot,x^1)\|_4^4\right)^{1/3}.$$

Further, by Theorem 1.1, for any x^1

$$\sum_{s_1=0}^{n}\|f_{s_1}(\cdot,x^1)\|_1 \le 16\|f(\cdot,x^1)\|_{QC} \le 16\|f\|_{QC}.$$

Thus we obtain

$$\|f\|_2^2 \le \left(\frac{1}{2\pi}\right)^{d-1}\left(16\|f\|_{QC}\right)^{2/3}\int\limits_{\mathbb{T}^{d-1}}\left(\sum_{s_1=0}^{n}\|f_{s_1}(\cdot,x^1)\|_4^4\right)^{1/3}dx^1. \tag{2.7}$$

Let us estimate the second factor on the right-hand side of (2.7). By the inequality

$$\|g\|_1 \le \|g\|_3,$$

we obtain

$$A \equiv \int\limits_{\mathbb{T}^{d-1}}\left(\sum_{s_1=0}^{n}\|f_{s_1}(\cdot,x^1)\|_4^4\right)^{1/3}dx^1 \le \left(\int\limits_{\mathbb{T}^{d-1}}\sum_{s_1=0}^{n}\|f_{s_1}(\cdot,x^1)\|_4^4 dx^1\right)^{1/3}.$$

Using the following consequence of the Littlewood–Paley theorem

$$\|g\|_4 \le C\left(\sum_{s}\|\delta_s(g)\|_4^2\right)^{1/2},$$

we have

$$\int\limits_{\mathbb{T}^{d-1}}\|f_{s_1}(\cdot,x^1)\|_4^4 dx^1 = \frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_{\mathbb{T}^{d-1}}|f_{s_1}(x,x^1)|^4\,dx^1\,dx_1 \ll \int\limits_0^{2\pi}\left(\sum_{s^1:\|s^1\|_1=n-s_1}\|\delta_s(f,x_1,\cdot)\|_4^2\right)^2 dx_1$$

$$\ll \left\|\sum_{s^1}\|\delta_s(f,x_1,\cdot)\|_4^2\right\|_{2(x_1)}^2 \ll \left(\sum_{s^1}\left\|\|\delta_s(f,x_1,\cdot)\|_4^2\right\|_{2(x_1)}\right)^2 = C\left(\sum_{s^1}\|\delta_s(f)\|_4^2\right)^2.$$

Thus

$$A \le C\left(\sum_{s_1=0}^{n}\left(\sum_{s^1:\|s^1\|_1=n-s_1}\|\delta_s(f)\|_4^2\right)^2\right)^{1/3}.$$

Taking into account condition (1), we see that

$$A \le C\Big(n\big(n^{d-2}\big)^2\Big)^{1/3} = Cn^{2d/3-1}.$$

Finally, from (2.7) and condition (2) of Theorem 2.1, we obtain

$$\|f\|_{QC} \gg n^{d/2},$$

which was to be proved. $\square$

Next we prove another inequality for trigonometric polynomials in a single variable.

Theorem 2.2. *For any polynomial of the form*

$$f=\sum_{k=l+1}^{2l} p_k(x)\cos 4^k x,$$

where $p_k\in\mathcal{T}_r(2^l)$, $k=l+1,\dots,2l$, $l=1,2,\dots,$ *the inequality*

$$\|f\|_\infty\ge c\sum_{k=l+1}^{2l}\|p_k\|_1,\qquad c>0,$$

holds.

Proof. Consider the polynomials

$$g_k=(\operatorname{sign}p_k)*V_{2^l},\qquad k=l+1,\dots,2l.$$

We have

$$\langle p_k,g_k\rangle=\|p_k\|_1,\qquad \|g_k\|_\infty\le 2,$$

and the Riesz product

$$\Phi(x)=\prod_{k=l+1}^{2l}\left(1+\frac{1}{2}\,g_k(x)\cos 4^k x\right)$$

defines a nonnegative function. Prove that $\Phi(x)$ has the form

$$\Phi(x)=1+\frac{1}{2}\sum_{k=l+1}^{2l} g_k(x)\cos 4^k x+t(x), \tag{2.8}$$

where $t(x)$ is orthogonal to the subspace

$$L=\left\{a+\sum_{k=l+1}^{2l} t_k(x)\cos 4^k x,\ t_k\in\mathcal{T}_r(2^{l+1}),\ a\in\mathbb{R}\right\}.$$

Indeed, $t(x)$ contains summands of the form ($m\ge 2$)

$$w(x)=2^{-m}\prod_{i=1}^{m} g_{k_i}(x)\cos 4^{k_i}x=2^{-m}\Big(\frac{1}{2}\cos(4^{k_1}+4^{k_2})x+\frac{1}{2}\cos(4^{k_1}-4^{k_2})x\Big)\prod_{i=3}^{m}\cos 4^{k_i}x\prod_{i=1}^{m} g_{k_i}(x),$$

where $k_1>k_2>\dots>k_m>l$. The frequencies w corresponding to nonzero Fourier coefficients of $w(x)$ satisfy

$$|w-4^{k_1}-4^{k_2}|\le 4^{k_3}+\dots+4^{k_m}+m2^{l+1}$$

or

$$|w-4^{k_1}+4^{k_2}|\le 4^{k_3}+\dots+4^{k_m}+m2^{l+1}.$$

Therefore,

$$4^{k_2}-4^{k_3}-\dots-4^{k_m}-m2^{l+1}\le|w-4^{k_1}|\le 4^{k_2}+\dots+4^{k_m}+m2^{l+1},$$

i.e.,

$$4^{k_2}\frac{2}{3}-l2^{l+1}\le|w-4^{k_1}|\le 4^{k_2}\frac{4}{3}+l2^{l+1}.$$

The upper bound means that $w(x)$ is orthogonal to 1 and all the functions $t_k(x)\cos 4^k x$, $t_k\in\mathcal{T}_r(2^{l+1})$, $k\ne k_1$. The lower bound proves that $w(x)$ is orthogonal to $t_{k_1}(x)\cos 4^{k_1}x$, $t_{k_1}\in\mathcal{T}_r(2^{l+1})$ for $l\ge 3$. Thus, representation (2.8) is established. In particular, (2.8) implies that $\|\Phi\|_1=1$.

Taking into account (2.8), for the inner product $\langle f, \Phi \rangle$ we have

$$\langle f, \Phi \rangle = \frac{1}{2} \sum_{k=l+1}^{2l} \frac{1}{2\pi} \int_0^{2\pi} \cos^2 4^k x p_k(x) g_k(x)\, dx. \tag{2.9}$$

Moreover, $\cos^2 4^k x = \dfrac{1}{2}(1 + \cos(2 \cdot 4^k x))$ and

$$\frac{1}{2\pi} \int_0^{2\pi} p_k(x) g_k(x) \cos^2 4^k x\, dx = \frac{1}{4\pi} \int_0^{2\pi} p_k(x) g_k(x)\, dx = \frac{1}{2} \|p_k\|_1, \tag{2.10}$$

$k = l+1, \dots, 2l$. On the other hand,

$$\langle f, \Phi \rangle \le \|f\|_\infty \|\Phi\|_1 = \|f\|_\infty. \tag{2.11}$$

Combining (2.9)–(2.11), we obtain

$$\|f\|_\infty \ge \frac{1}{4} \sum_{k=l+1}^{2l} \|p_k\|_1.$$

□

3. Estimates of Entropy Numbers and Kolmogorov Widths

In this section, Theorems 1.2 and 1.3 stated in the introduction will be proved. In order to do this, we first find the exponents of entropy numbers and widths for classes of functions of a single variable which are of logarithmic smoothness. In the one-dimensional case, the proof is technically simpler yet in concept is close to the proofs of Theorems 1.2 and 1.3 (see also [22].)

We consider classes of functions LG^r which are defined by the condition on the uniform norm of binary blocks $\delta_s(f)$

$$LG^r = \left\{ f \in L^\infty(\mathbb{T}) : \|\delta_s(f)\|_\infty \le (s+1)^{-r},\ s = 0, 1, \dots \right\}.$$

Theorem 3.1. *For $r > 1$ we have*

$$\varepsilon_n(LG^r, L_p) \asymp d_n(LG^r, L_p) \asymp \begin{cases} (\log n)^{-r+1}, & \text{for} \quad p = \infty, \\ (\log n)^{-r+1/2}, & \text{for} \quad 1 \le p < \infty. \end{cases}$$

Before proving Theorem 3.1, we note that for functions in LG^r there are binary blocks, for example, $\delta_s(f)$, $n \le s \le 2n$, with the same bound; therefore, the interference between blocks should be considered. Such an effect occurs often in studying classes of multivariable functions with constraints on mixed derivation or difference; however, interfering blocks are of the "same value," i.e., $\delta_s(f)$ and $\delta_v(f)$ for $\|s\|_1 = \|v\|_1$. In the case of the classes LG^r, the sizes of the interfering blocks are quite different.

Theorem 3.1 implies, in particular, that the exponent of a width changes spasmodically as the L_p, $p < \infty$ metric changes to L_∞. Such a phenomenon occurs in the two-dimensional case for the classes H^r_∞ (see [22]).

Proof of Theorem 3.1. The upper bounds for widths present no problems. For $p = \infty$ and $r > 1$, a function $f \in LG^r \subset C(0, 2\pi)$ trivially satisfies the inequality

$$\|f - S_{2^m}(f)\|_\infty \le \sum_{s>m} \|\delta_s(f)\|_\infty \ll m^{-r+1}.$$

For $2 < p < \infty$, using the Littlewood–Paley theorem, we have for $f \in LG^r$

$$\|f - S_{2^m}(f)\|_p \le C(p) \left(\sum_{s>m} \|\delta_s(f)\|_p^2 \right)^{1/2} \ll m^{-r+1/2}.$$

To establish the lower bounds is a nontrivial part of the theorem, especially for $p = \infty$. Let us prove the lower bounds for entropy numbers. First consider the case $p = 1$; we shall use the following well-known assertion (see [6, 21]).

Lemma 3.1. *The inequalities*

$$\varepsilon_{2m+1}\left(\mathcal{T}_r(m)_\infty\right)_1 \geq c > 0, \qquad m = 1, 2, \ldots,$$

hold for entropy numbers of the unit L_∞-ball in the space of real trigonometric polynomials of degree m in the L_1-norm.

The lemma below readily follows from Lemma 3.1

Lemma 3.2. *There exists an absolute constant $c_0 > 0$ such that there are 2^m functions $t^1, \ldots, t^{2^m}$ in each subspace*

$$\mathcal{T}_r[N, N+m] \equiv \left\{ t = \sum_{N \leq |k| \leq N+m} c_k e^{ikx},\ c_k = \overline{c}_{-k} \right\}$$

satisfying the following conditions:

(1) $\|t^i\|_\infty \leq 1$ *for all* i;
(2) $\|t^{i_1} - t^{i_2}\|_1 \geq c_0,\ i_1 \neq i_2,\ i_1, i_2 \in [1, 2^m]$.

Now fix a number l and construct a special collection of functions. Applying Lemma 3.2 to the set $\mathcal{T}_r[2^j, 2^j + 2^l]$ for each $j = l+1, \ldots, 2l$ we obtain l collections $\{t^i_j\}_{i=1}^{2^{2^l}} \subset \mathcal{T}_r[2^j, 2^j + 2^l]$, $j = l+1, \ldots, 2l$, such that

(1) $\|t^i_j\|_\infty \leq 1$;
(2) $\|t^{i_1}_j - t^{i_2}_j\|_1 \geq c_0$ for all j, $i_1 \neq i_2$.

Let us introduce a set of functions as follows.
Assume that $I = (i_{l+1}, \ldots, i_{2l})$, $i_j \in [1, 2^{2^l}]$, and

$$f_I = \sum_{j=l+1}^{2l} t^{i_j}_j. \tag{3.1}$$

The cardinality of such a collection is 2^{l2^l}.
Let us make use of the following lemma, which is easy to verify.

Lemma 3.3. *Assume that we are given positive integers m, μ, $\mu < m$, and a "parallelepiped" $\Pi \subset \mathbb{Z}^m$,*

$$\Pi = \bigotimes_{j=1}^{m} [1, M_j],$$

with the property that for some $Q \in \mathbb{N}$, $M \in \mathbb{N}$, $Q \leq M$,

$$Q \leq M_j \leq M, \qquad j = 1, 2, \ldots, m.$$

Then there exists a set $\Omega \subset \Pi$ containing distinct points greater than or equal to $\left[\dfrac{Q^m - 1}{\binom{m}{\mu} M^\mu}\right]$ such that if $x = \{x_j\} \in \Omega$, $y = \{y_j\} \in \Omega$, $x \neq y$, then

$$\{j : x_j \neq y_j\} \geq \mu.$$

We now take for Π the cube $\bigotimes_{j=1}^{l}[1,M]$, $M=2^{2^l}$, $m=l$, $\mu=[l/3]$. Then

$$\left[\frac{M^m-1}{\binom{m}{\mu}M^\mu}\right]\geq 2^{2^{l-1}l}.$$

Let Ω be the set of collections I defined in Lemma 3.3 and $\mathcal{F}=\{f_I,\ I\in\Omega\}$ (see (3.1).) Then for $f\in\mathcal{F}$ we have

$$\begin{aligned}\|\delta_s(f)\|_\infty&\leq 1, &&s=l+1,\dots,2l;\\ \delta_s(f)&=0 &&\text{for other}\quad s.\end{aligned}\tag{3.2}$$

Therefore, by the Littlewood–Paley theorem,

$$\|f\|_4\leq Cl^{1/2};\tag{3.3}$$

using the inequality

$$\|f\|_2\leq\|f\|_1^{1/3}\|f\|_4^{2/3},$$

for any $f,g\in\mathcal{F}$, $f\neq g$ we have

$$\|f-g\|_1\geq C\left(l^{-1/3}\|f-g\|_2\right)^3\geq cl^{1/2}.\tag{3.4}$$

Thus, we have constructed the set $\mathcal{F}$ of functions satisfying conditions (3.2)–(3.4), which is of cardinality $\#\mathcal{F}\geq 2^{l2^{l-1}}$. Therefore

$$\varepsilon_{l2^{l-1}}(\mathcal{F},L_1)\gg l^{1/2}.$$

Further (3.2) implies that $(2l)^{-r}\mathcal{F}\subset LG^r$ and

$$\varepsilon_{2^l}(LG^r,L_1)\gg l^{-r+1/2}.$$

This completes the proof of the lower bound in the case where $1\leq p<\infty$. We now turn to the case $p=\infty$. We use Theorem 2.2; in other respects the proof is similar to the case $p=1$ considered above. We shall just point out the modifications to be made. Consider instead of f_I the functions

$$h_I=\sum_{k=l+1}^{2l}t^{i_k}\cos 4^k x,$$

where the collection of trigonometric polynomials of degree $2^l$and of cardinality 2^{2^l} satisfy the conditions of Lemma 3.2 for $N=0$, $m=2^l$. Then we choose the subset

$$H=\{h_I,\ I\in\Omega\}$$

from these polynomials as above (see Lemma 3.3.) Using Theorem 2.2 instead of estimate (3.4), we have

$$\|h-g\|_\infty\geq cl$$

for $h\in H$, $g\in H$, $h\neq g$, whence by the inclusion $(4l)^{-r}H\subset LG^r$ it follows that

$$\varepsilon_{2^l}(LG^r,L_\infty)\gg l^{-r+1}.$$

Thus, the upper bounds of widths and the lower bounds of entropy numbers, which are of the same exponent, are established. It remains to apply the following lemma, obtained from an inequality due to Charles (see [15].)

Lemma 3.4. *Let A be a compact subset of a Banach space X. Assume that a pair of numbers (r,b), where $r>0$, $b\in\mathbb{R}$ or $r=0$, $b<0$ satisfies the following*:

$$d_m(A,X)\ll m^{-r}(\log m)^b,$$
$$\varepsilon_m(A,X)\gg m^{-r}(\log m)^b.$$

Then

$$\varepsilon_m(A,X)\asymp d_m(A,X)\asymp m^{-r}(\log m)^b.$$

Theorem 3.1 is proved. □

Proof of Theorems 1.2 *and* 1.3. The following lemma is a corollary of [11, Theorem 6] (see also [9]).

Lemma 3.5. *Let A be a compact subset of a separable Banach space X. Assume that*

$$\varepsilon_m(A,X)\asymp m^{-r}(\log m)^a, \qquad r>0, \quad a\in\mathbb{R}$$

as $m\to\infty$. Then

$$d_m(A,X)\gg m^{-r}(\log m)^a.$$

From Lemma 3.5 it follows that to prove Theorem 1.2 it suffices to obtain the lower bounds of Theorem 1.3. We begin by verifying the lower bounds of Theorem 1.2, which is based on Theorem 2.2 but in other respects follows the proof of [22, Theorem 3.1].

Let $N_\varepsilon(F,X)$ be the minimal number of closed balls of radius ε in X that is sufficient to cover a compact set F. Let $M_\varepsilon(F,X)$ be the maximal number of points $x_i\in F$ such that $\|x_i-x_j\|>\varepsilon$, $i\neq j$. Then, as is well known,

$$N_\varepsilon(F,X)\le M_\varepsilon(F,X)\le N_{\varepsilon/2}(F,X). \tag{3.5}$$

In the case of the n-dimensional space $X=X_n$, it can easily be checked (see, e.g., [15]) that

$$N_\varepsilon(B_X,X)\ge \varepsilon^{-n}, \qquad 0<\varepsilon\le 1, \tag{3.6}$$

$$N_\varepsilon(B_Y,X)\ge \varepsilon^{-n}\frac{\mathrm{Vol}(B_Y)}{\mathrm{Vol}(B_X)}, \qquad 0<\varepsilon\le 1, \tag{3.7}$$

where B_X denotes the unit ball in the space X.

Assume that $D_n=\bigcup_{s\in Y_n^d}\rho(s)$, $\mathcal{T}_r(D_n)$ is the corresponding space of real trigonometric polynomials. Let us define for each n a collection of functions $\{f_i^n\}_{i=1}^{A_n}$, $f_i^n\in\mathcal{T}_r(D_n)$ such that

(1) $\|\delta_s(f_i^n)\|_\infty\le 1$, $s\in Y_n^d$, $i=1,\dots,A_n$;
(2) $\|f_i^n-f_j^n\|_{QC}\ge c(d)n^{d/2}$, $i\neq j$;
(3) $A_n\ge 2^{|D_n|/2}$.

For $s\in Y_n^d$, by $b(s)$ denote the d-dimensional vector $b(s)=(b_1(s),\dots,b_d(s))$, where $b_j(s)=2^{s_j-2}-1$ for $s_j\ge 2$ and $b_j(s)=0$ for $s=0,1$.

Let $\mathcal{T}_r(b(s))$ be the space of real trigonometric polynomials of the form

$$t(x)=\sum_{\substack{0\le k_j\le b_j(s),\\ 1\le j\le d}}\ \sum_{e\subset[1,d]} a_k^e\prod_{j\in e}\cos k_jx_j\prod_{j\in[1,d]\setminus e}\sin k_jx_j,$$

where $a_k^e\in\mathbb{R}$ are the Fourier coefficients of $t(x)$; e ranges over all subsets of the set $[1,\dots,d]$. We shall consider $\{a_k^e\}$ as a vector in $\mathbb{R}^{v(b(s))}$ where, for $b\in\mathbb{R}^d$, $v(b)\equiv\prod_{j=1}^d(2b_j+1)$ and $v(b(s))=\dim\mathcal{T}_r(b(s))$. Given $b\in\mathbb{Z}_+^d$ and $1\le p\le\infty$, define $A_p(b)\subset\mathbb{R}^{v(b)}$ to be the set of coefficients $\{a_k^e\}_{k,e}\subset\mathbb{R}^{v(b)}$ of polynomials $t\in\mathcal{T}_r(b)$ such that $\|t\|_p\le 1$.

With the use of inequality (3.7), we reproduce here a well-known estimate for the volume of $A_p(b)$ (see [6] for $d=1$ and [20, Lemma 1.2] for $d>1$).

Lemma 3.6. *The following estimate holds*:

$$\mathrm{Vol}(A_\infty(b)) \geq v(b)^{-v(b)/2}[c_2(d)]^{-v(b)}.$$

Using a formula for the volume of the unit ball $A_2(b)$ in $\mathbb{R}^{v(b)}$, which implies that

$$\mathrm{Vol}(A_2(b)) \leq v(b)^{-v(b)/2}[c_3(d)]^{-v(b)},$$

we easily derive the following assertion from (3.5), (3.7) and Lemma 3.6: *there exists a constant* $c_4(d) > 0$ *and an aggregate of polynomials* $\{t_i^b,\ i = 1, 2, \dots, 2^{v(b)}\} \subset \mathcal{T}_r(b)$ *such that*

$$\|t_i^b\|_\infty \leq 1, \qquad i = 1, \dots, 2^{v(b)}, \tag{3.8}$$

$$\|t_i^b - t_j^b\|_2 \geq c_4(d), \qquad i \neq j. \tag{3.9}$$

Further, for each collection $I = \{i(s),\ s \in Y_n^d\}$ $i(s) \in \{1, \dots, 2^{v(b(s))}\}$ define the function

$$f_I^n = \sum_{s \in Y_n^d} \left(\prod_{j=1}^d \cos k_j^s x \right) t_{i(s)}^{b(s)}, \tag{3.10}$$

where $k_j^s = 2^{s_j-1} + 2^{s_j-2}$ for $s \in Y_n^d$, if $s_j \geq 2$ and $k_j^s = s_j$, if $s_j = 0, 1$.

The total number of such functions is

$$N = \prod_{s \in Y_n^d} 2^{v(b(s))}.$$

Note that for all $s \in Y_n^d$

$$0 < c(d) \cdot \dim \mathcal{T}_r(\rho(s)) \leq v(b(s)) \leq \dim \mathcal{T}_r(\rho(s)) = 2^n.$$

In view of this, it is easy to prove by Lemma 3.3 that there exists a subset G_n, $|G_n| \geq 2^{2^n n^{d-1} c'(d)}$, of the set $\prod_{s \in Y_n^d} [1, 2^{v(b(s))}]$ of collections I with the additional property that each two distinct collection $I, J \in G_n$ possess at least $c''(d)|Y_n^d|$ distinct "coordinates" $i(s)$; $c'(d) > 0$, $c''(d) > 0$.

Let us verify that for each $I, J \in G_n$, $I \neq J$,

$$\|f_I^n - f_J^n\|_{QC} \gg n^{d/2}. \tag{3.11}$$

This follows from Theorem 2.2 and the simple inequality that for any polynomial $t(x) \in \mathcal{T}_r(b(s))$ we have

$$\left\| \left(\prod_{j=1}^d \cos k_j^s x \right) t(x) \right\|_2 \geq c\|t(x)\|_2, \qquad c = c(d) > 0 \tag{3.12}$$

where the numbers k_j^s and b_j^s are the same as in (3.10). We note also that for such k_j^s and b_j^s the polynomial

$$\left(\prod_{j=1}^d \cos k_j^s x \right) t(x)$$

belongs to the subspace $\mathcal{T}_r(\rho(s))$; therefore, for distinct s these polynomials have nonoverlapping spectrums.

The lower bound of $\varepsilon_m(H_q^r, QC)$ follows from (3.11) and the inclusion

$$\{f_I^n \cdot 2^{-rn}\}_{I \in G_n} \subset H_\infty^r \cdot C(r, d)$$

(see (3.10) and [19, Chap. 2, Theorem 1.1]).

The lower bound of $\varepsilon_m(W_q^r, QC)$ for arbitrary $r > 0$ follows from (3.11) and the inclusion

$$\{f_I^n \cdot 2^{-rn} n^{-(d-1)/2}\} \subset W_q^r \cdot C(r, d), \qquad 1 < q < \infty$$

obtained from (3.10), the Littlewood–Paley theorem, and [19, Chap. 1, Theorem 1.1].

The lower bound of $\varepsilon_m(W^r_\infty, QC)$ for $r > 1/2$ is a consequence of the above-proved lower bounds of $\varepsilon_m(W^r_q, QC)$, $q < \infty$, the upper bound of $\varepsilon_m(W^r_2, QC)$, which will be established below, and the following inequality:

$$\varepsilon_{2m}(W^r_4, QC) \le 2\varepsilon_m(W^r_2, QC)^{1/2} \cdot \varepsilon_m(W^r_\infty, QC)^{1/2}. \tag{3.13}$$

Estimate (3.13) is a special case of an estimate for entropy numbers of an operator acting on an "intermediate space"; see [14, Sec. 12.1.12]. Here we take for the operator the identity embedding $W^r_4 \hookrightarrow QC$.

Proof of the Upper Bounds of Theorems 1.2 *and* 1.3. In what follows, by $\mathcal{T}(\Delta Q_n)_q$ we denote the unit L_q-ball in the space $\mathcal{T}(\Delta Q_n)$ and by $\mathcal{T}(\Delta Q_n)_{B_{q,\infty}}$ the set of polynomials f in $\mathcal{T}(\Delta Q_n)$ such that $\|\delta_s(f)\|_q \le 1$, $\|s\|_1 = n$. Moreover, put

$$\gamma(q,a,b) = \begin{cases} \left(\dfrac{b}{a}\right)^{1/q} \left[\ln\left(1+\dfrac{b}{a}\right)\right]^{1/q-1/2}, & a \le b, \\ e^{-a/b}, & a > b. \end{cases}$$

Lemma 3.7. *For* $1 < q \le 2$ *the following relations are valid*:

$$\varepsilon_m(\mathcal{T}(\Delta Q_n)_q, QC) \ll n^{1/2}\gamma(q,m,K|\Delta Q_n|), \tag{3.14}$$

$$\varepsilon_m\left(\mathcal{T}(\Delta Q_n)_{B_{q,\infty}}, QC\right) \ll n^{d/2}\gamma(q,m,K|\Delta Q_n|), \tag{3.15}$$

$$d_m(\mathcal{T}(\Delta Q_n)_2, QC) \ll n^{1/2}(|\Delta Q_n|/m)^{1/2} \tag{3.16}$$

($K = K(d)$; *other constants in* (3.14)–(3.16) *are also independent of* m *and* n.)

The proof of Lemma 3.7 is based on the usual method of estimating entropy and widths. Let X be the space $\mathbb{R}^N$ with the norm $\|\cdot\|_X$. As usual, denote by B^N_2 and S^{N-1} the unit Euclidean ball and its boundary in $\mathbb{R}^N$ respectively. Further, let $\sigma = \sigma_N$ be the normed Lebesgue measure on S^{N-1}. The following value plays an important role in estimating ε-entropy and Kolmogorov widths (see [15] for details):

$$M_X := \int_{S^{N-1}} \|f\|_X \, d\sigma.$$

Lemma 3.8 (see [13])**.** *The relation*

$$\varepsilon_m(B^N_2, X) \ll \gamma(2,m,N)M_X$$

holds.

First, we prove estimate (3.14) in the particular case where $q = 2$. Consider the set $\mathcal{T}(\Delta Q_n)^e$ of polynomials with real coefficients in $\mathcal{T}(\Delta Q_n)_r$. Then $\mathcal{T}(\Delta Q_n)^e_2$ can be thought of as a Euclidean ball in $\mathbb{R}^N$, $N = |\Delta Q_n|/2$. Obviously it is sufficient to prove estimate (3.14), $q = 2$ for $\mathcal{T}(\Delta Q_n)^e_2$.

Let us represent the polynomial f in the form

$$f(x) = \sum_s \sum_{2^{s-1} \le |k_1| < 2^s} e^{ik_1x_1} f_{k_1}(x^1);$$

then, by the definition of the QC-norm,

$$\|f\|_{QC} = \int_0^1 \left\| \sum_s r_s(\omega) \sum_{2^{s-1} \le |k_1| < 2^s} e^{ik_1x_1} f_{k_1}(x^1) \right\|_\infty d\omega.. \tag{3.17}$$

From (3.17) it readily follows that in this case

$$M_{QC} = \int\limits_{S^{N-1}} \|f\|_{QC}\, d\sigma = \int\limits_{S^{N-1}} \|f\|_\infty\, d\sigma. \tag{3.18}$$

The latter value has been estimated in [2]:

$$\int\limits_{S^{N-1}} \|f\|_\infty\, d\sigma \ll n^{1/2} \tag{3.19}$$

(note that inequality (3.19) easily follows from the exponential estimate for polynomials with respect to the Rademacher system).

Lemma 3.8 and (3.18), (3.19) imply estimate (3.14) for $q = 2$.

We now turn to the general case $1 < q < 2$.

Lemma 3.9. *The inequalities*

$$\varepsilon_m(\mathcal{T}(\Delta Q_n)_q, L_2) \ll \begin{cases} \left(\dfrac{|\Delta Q_n|}{m}\right)^{1/q-1/2} \left[\log\left(\dfrac{|\Delta Q_n|}{m}+1\right)\right]^{1/q-1/2}, & m \le |\Delta Q_n|, \\ 2^{-mc/|\Delta Q_n|}, \quad c = c(d) > 0, & m > |\Delta Q_n| \end{cases}$$

hold.

This lemma can be derived in the usual way from the corresponding result on entropy in the l_p^N-metric of l_q-balls

$$B_q^N = \left\{ x \in \mathbb{R}^N : \left(\sum_{i=1}^N |x_i|^q\right)^{1/q} \le 1 \right\}.$$

Lemma 3.10 (see [16])**.** *For $1 \le q \le p \le \infty$,*

$$\varepsilon_m(B_q^N, l_p^N) \ll \begin{cases} \left(\dfrac{\log(\frac{N}{m}+1)}{m}\right)^{1/q-1/p}, & m \le N, \\ m^{1/p-1/q} \quad 2^{-m/N}, & m > N. \end{cases}$$

We also use the well-known Marcinkiewicz theorem on an equivalence between the usual L_q-norm and the L_q-grid norm of trigonometric polynomials. Namely, this theorem implies that for any $s \in \mathbb{Z}_+^d$, $\|s\|_1 = n$, and any polynomial

$$t(x) = \sum_{k \in \rho(s)} \widehat{t}(k) e^{i(k,x)}$$

for $1 < q < \infty$ the following inequalities are satisfied:

$$k_1(d,q) \left(\frac{1}{2^n} \sum_{x \in \Omega_s} |t(x)|^q\right)^{1/q} \le \|t\|_{L_q} \le k_2(d,q) \left(\frac{1}{2^n} \sum_{x \in \Omega_s} |t(x)|^q\right)^{1/q},$$

where $k_1(d,q) > 0$ and for $s = (s_1, \dots, s_d) \in \mathbb{Z}_+^d$

$$\Omega_s = \left\{ \left(\frac{2\pi l_1}{2^{s_1+1}+1}, \dots, \frac{2\pi l_d}{2^{s_d+1}+1}\right) \right\}, \qquad 0 \le l_j \le 2^{s_j+1}, \quad 1 \le j \le d.$$

To a polynomial $f \in \mathcal{T}_r(\Delta Q_n)$ of the form

$$f = \sum_{s:\|s\|_1 = n} \delta_s(f, x)$$

assign the vector

$$v(f) = \{\delta_s(f,x)\}_{\|s\|_1=n,\ x\in\Omega_s} \in \mathbb{R}^N, \qquad N \asymp 2^n n^{d-1}$$

the order of the components being arbitrary and the same for all f.

Using the Marcinkiewicz theorem and the inequality

$$\left(\sum_s \|\delta_s(f)\|_q^2\right)^{1/2} \ll \|f\|_{L_q}, \qquad 1 < q \le 2,$$

which is a consequence of the Littlewood–Paley theorem, let us estimate the l_q-norm of the vector $v(f)$, $1 < q < 2$:

$$\|v(f)\|_{l_q} \asymp 2^{n/q}\left(\sum_{s,\ \|s\|_1=n} \|\delta_s(f)\|_q^q\right)^{1/q} \ll 2^{n/q} n^{(d-1)(1/q-1/2)} \left(\sum_{s,\ \|s\|_1=n} \|\delta_s(f)\|_q^2\right)^{1/2} \ll 2^{n/q} n^{(d-1)(1/q-1/2)} \|f\|_q. \quad (3.20)$$

For $q=2$ we have

$$\|v(f)\|_{l_2} \asymp 2^{n/2}\|f\|_2. \qquad (3.21)$$

We also remark on another fact. Let $L \subset l_2^N$ be a subspace. Then

$$\varepsilon_m(B_q^N \cap L, l_2^N \cap L) \le \varepsilon_m(B_q^N, l_2^N). \qquad (3.22)$$

The proof of estimate (3.22) is evident; the required ε-net is formed by the orthoprojection of the ε-net for B_q^N.

We use (3.22) in the case where

$$L = \{v(f),\ f \in \mathcal{T}_r(\Delta Q_n)\}$$

and $(\dim L)/N > c > 0$. In view of (3.21) and (3.20), Lemmas 3.10 and (3.22) imply the conclusion of Lemma 3.9.

Let us complete the proof of estimate (3.14) for $1 < q < 2$. We have

$$\varepsilon_{2m}(\mathcal{T}_r(\Delta Q_n)_q, QC) \le \varepsilon_m(\mathcal{T}_r(\Delta Q_n)_q, L^2)\varepsilon_m(\mathcal{T}_r(\Delta Q_n)_2, QC). \qquad (3.23)$$

It remains to apply Lemma 3.9 and use estimate (3.14) proved above for $q=2$.

Estimate (3.15), being a bit easier, may be established similarly to (3.14). We prove the inequality

$$\varepsilon_m(\mathcal{T}_r(\Delta Q_n)_{B_{q,\infty}}, L^2) \ll n^{(d-1)/2} \begin{cases} \left(\dfrac{|\Delta Q_n|}{m}\right)^{1/q-1/2} \left[\log\left(\dfrac{\Delta Q_n}{m}+1\right)\right]^{1/q-1/2}, & m \le |\Delta Q_n|, \\ 2^{-mc/|\Delta Q_n|}, & m > |\Delta Q_n| \end{cases} \qquad (3.24)$$

instead of Lemma 3.9. Here we again use quantization based on the Marcinkiewicz theorem, Lemma 3.10, and the following inequality analogous to (3.23):

$$\varepsilon_{2m}(\mathcal{T}_r(\Delta Q_n)_{B_{q,\infty}}, QC) \le \varepsilon_m(\mathcal{T}_r(\Delta Q_n)_{B_{q,\infty}}, L^2)\varepsilon_m(\mathcal{T}_r(\Delta Q_n)_2, QC).$$

We now turn to proving estimate (3.16) the proof of which is analogous to that of (3.14) in the case $q=2$. The following lemma is used instead of Lemma 3.8.

Lemma 3.11 (see [13]). *We have*

$$d_m(B_2^N, X) \ll M_{X,2}\left(\frac{N}{m}\right)^{1/2},$$

where

$$M_{X,2} \equiv \left(\int\limits_{S^{N-1}} \|f\|_X^2 \, d\sigma \right)^{1/2}.$$

As in proving the estimate of M_{QC} (see (3.18)), by the inequality

$$M_{L^\infty,2} \ll n^{1/2}$$

(see [1]), we obtain

$$M_{QC,2} \le M_{L^\infty,2} \ll n^{1/2}. \tag{3.25}$$

Inequality (3.16) follows from Lemma 3.11 and (3.25).

Next we complete the proof of the upper bounds of Theorems 1.2 and 1.3. Obviously it is sufficient to prove Theorem 1.2 for $1 < q \le 2$, $r > 1/q$ and Theorem 1.3 for $q = 2$, $r > 1/2$. The proof is based on Lemma 3.7 and the following well-known properties of functions in the classes W_q^r and H_q^r. For any function $f \in W_q^r$, $1 < q \le 2$, we have (see [19, Chap. 2, Theorem 2.1])

$$\left\| \sum_{k \in \Delta Q_n} \widehat{f}(k) e^{i(k,x)} \right\|_q \ll 2^{-rn}. \tag{3.26}$$

For any $f \in H_q^r$, $1 < q < \infty$ (see [19, Chap. 2, Theorem 1.1]),

$$\|\delta_s(f)\|_q \ll 2^{-r\|s\|_1}. \tag{3.27}$$

To prove the first relation in Theorem 1.2 we use (3.27) and (3.15) in Lemma 3.7. To prove the second relation in Theorem 1.2 we use (3.26) and (3.16). To prove the second relation in Theorem 1.3 we apply (3.26) and (3.16). Finally, to prove the first relation in Theorem 1.3 we employ relation (3.16) and the following simple consequence of inequality (3.27): for any $f \in H_2^r$,

$$\left\| \sum_{\|s\|_1 = n} \delta_s(f) \right\|_2 \ll n^{(d-1)/2} 2^{-rn}.$$

All four cases are treated analogously. We give only the proof of the second relation in Theorem 1.2.

Assume that a sufficiently large m is given. Choose n such that

$$|\Delta Q_{n-1}| < m \le |\Delta Q_n|.$$

Then $m \asymp 2^n n^{d-1}$. Set $\sigma = \dfrac{1}{2} \min(r - 1/q, 1)$ and

$$\overline{m}_l = c_\sigma \begin{cases} [m 2^{-\frac{1}{2}(n-l)}], & l < n, \\ [m 2^{-\sigma(l-n)}], & l \ge n, \end{cases}$$

where $c_\sigma > 0$ is chosen so that

$$\sum_{l=0}^{\infty} \overline{m}_l \le m.$$

Let $m_l = [\overline{m}_l]$. Then $m_l = 0$, if $c_\sigma m 2^{-\sigma(l-n)} < 1$, i.e.,

$$l > n_1 \equiv n + \frac{1}{\sigma} \log c_\sigma m.$$

Put

$$S_{\Delta Q_l}(W_q^r) \equiv \left\{ g = \sum_{k \in \Delta Q_l} \widehat{f}(k) e^{i(k,x)}, \; f \in W_q^r \right\}$$

and

$$\|S_{\Delta Q_l}(W^r_q)\|_{QC} \equiv \sup_{g\in S_{\Delta Q_l}(W^r_q)} \|g\|_{QC}.$$

Then

$$\varepsilon_m(W^r_q, QC) \le \sum_{l=0}^{n_1} \varepsilon_{m_l}\left(S_{\Delta Q_l}(W^r_q), QC\right) + \sum_{l>n_1} \|S_{\Delta Q_l}(W^r_q)\|_{QC} = \sum\nolimits_1 + \sum\nolimits_2 .$$

Each term in $\sum_2$ can be estimated by (3.26) and the inequality

$$\|f\|_\infty \ll 2^{l/q} l^{(d-1)/(1-1/q)} \|f\|_q, \quad f \in \mathcal{T}(Q_l)$$

(see [19, Chap. 1, Theorem 2.1]):

$$\|S_{\Delta Q_l}(W^r_q)\|_{QC} \ll 2^{-l(r-1/q)} l^{(d-1)(1-1/q)}.$$

Summing over $l > n_1$ and taking into account the definition of n_1, we obtain

$$\sum\nolimits_2 \ll 2^{-rn}. \tag{3.28}$$

Further, by (3.26) and Lemma 3.7 we have

$$\sum_{l<n} \varepsilon_{m_l}(S_{\Delta Q_l}(W^r_q), QC) \ll \sum_{l<n} 2^{-rl} n^{1/2} \exp\left\{-\frac{m_l}{K|\Delta Q_l|}\right\} \ll 2^{-rn} n^{1/2} \tag{3.29}$$

and

$$\sum_{n\le l\le n_1} \varepsilon_{m_l}(S_{\Delta Q_l}(W^r_q), QC) \ll \sum_{n\le l\le n_1} 2^{-rl} n^{1/2} \left(\frac{|\Delta Q_l|}{m_l}\right)^{1/q} \left[\ln\left(1 + \frac{|\Delta Q_l|}{m_l}\right)\right]^{1/q-1/2} \ll 2^{-rn} n^{1/2}. \tag{3.30}$$

Combining inequalities (3.28)–(3.30) since $m \asymp 2^n n^{d-1}$, we complete the proof of the upper bound in the second relation in Theorem 1.2. □

□

4. On a Uniform Grid Norm of Polynomials in $\mathcal{T}(\Delta Q_n)$

This section contains the proof of the following theorem.

Theorem 4.1. *Assume that for some $n \ge 1$ and $y \ge 1$ a finite set $\Omega \subset \mathbb{T}^2$ has the property that for each polynomial $t \in \mathcal{T}(\Delta Q_n)$ we have*

$$\|t\|_\infty \le y\|t\|_{\infty,\Omega}. \tag{4.1}$$

Then the number of elements of Ω is bounded from below by

$$|\Omega| \ge c_1 |\Delta Q_n| \exp\left(\frac{c_2 n}{y^2}\right), \tag{4.2}$$

where $c_1 > 0$ and $c_2 > 0$ are absolute constants.

Proof. Without loss of generality we may assume that n is sufficiently large and $1 \le y \le c_3 n^{1/2}$, where $c_3 > 0$ is an arbitrary absolute constant. Moreover we assume that n is even; in the case of an odd n the argument is similar.

Let $g_k(\omega)$, $k \in \mathbb{Z}^2$, be a set of independent, normally distributed random variables with zero expected value and unit variance, indexed by points in $\mathbb{Z}^2$.

Consider the random process

$$P(x,\omega) = \sum_{s\in Y^2_n} \sum_{k=(k_1,k_2)\in\rho(s)\cap\mathbb{Z}^2_+} \lambda_k g_k(\omega) e^{i(k,x)} \equiv \sum_{k\in\Lambda_n} \lambda_k g_k(\omega) e^{i(k,x)}, \tag{4.3}$$

where for $k \in \rho(s)$

$$\lambda_k = \lambda_{(k_1,k_2)} = \begin{cases} 1, & \text{for} \quad [2^{s_1-1}] < k_1 < 2^{s_1}, \quad [2^{s_2-1}] < k_2 < 2^{s_2}, \\ 1/2, & \text{for} \quad k_1 = [2^{s_1-1}], \quad [2^{s_2-1}] < k_2 < 2^{s_2}, \\ 1/2, & \text{for} \quad k_2 = [2^{s_2-1}], \quad [2^{s_1-1}] < k_1 < 2^{s_1}, \\ 1/4, & \text{for} \quad k_1 = [2^{s_1-1}], \quad k_2 = [2^{s_2-1}], \end{cases}$$

and $\Lambda_n = \bigcup_{s\in Y_n^2} \rho(s) \cap \mathbb{Z}_+^2$. Note that $|\Lambda_n| \asymp n2^n$ and $|\Lambda_n| \le \left(\frac{n}{2}+1\right) 2^{n-2}$.

The assertion of Theorem 4.1 will be established by bounding the probability

$$\gamma(w) \equiv \mathrm{P}\left\{\|P(x,\omega)\|_{C_x} < w|\Lambda_n|^{1/2}\right\}$$

from below and from above for w in the interval $0 < w \le n^{1/2}$. Put

$$\delta_s(x,\omega) = \sum_{k\in\rho(s)\cap\mathbb{Z}_+^2} \lambda_k g_k(\omega) e^{ikx}, \qquad s \in Y_n^2.$$

By inequality (1.3) stated in [22],

$$\gamma(w) \le \mathrm{P}\left\{\sum_{s\in Y_n^2} \|\delta_s(x,\omega)\|_{L_1} < A^{-1} w|\Lambda_n|^{1/2}\right\}. \tag{4.4}$$

Owing to the Chebyshev inequality the first part of (4.4) is less than or equal to

$$\binom{m}{[m/2]} \max_{\{s^j,\ 1\le j\le m/2\}\subset Y_n^2} \mathrm{P}\left\{\|\delta_{s^j}\|_{L_1} < \frac{2A^{-1}w}{m}|\Lambda_n|^{1/2},\ 1 \le j \le m/2\right\}, \tag{4.5}$$

where $m = |Y_n^2| = n/2 + 1$.

From (4.4) and (4.5), using the independence of the random variables $\|\delta_s(x,\omega)\|_{L_1}$, $s \in Y_n^2$, we have

$$\gamma(w) \le 2^{n/2}\left[\max_{s\in Y_n^2} \mathrm{P}\left\{\|\delta_s(x,\omega)\| \le \frac{4A^{-1}w}{n}|\Lambda_n|^{1/2}\right\}\right]^{n/4}. \tag{4.6}$$

Fix $s = (s_1, n - s_1) \in Y_n^2$, $0 < s_1 < n$; let us bound from above the probability

$$\mathrm{P}\left\{\|\delta_s\|_{L_1} \le \frac{4A^{-1}w}{n}|\Lambda_n|^{1/2}\right\} \le \mathrm{P}\left\{\|\delta_s\|_{L_1} \le \frac{2^{n/2+1}}{n^{1/2}} A^{-1} w\right\}. \tag{4.7}$$

Isolating the real part for $s = (s_1, n - s_1) \in Y_n^2$, we have

$$\begin{aligned} |\delta_s(x,\omega)| &= \left|\sum_{k_1=0}^{2^{s_1-1}-1} \sum_{k_2=0}^{2^{n-s_1-1}-1} \lambda'_k g_k(\omega) e^{i(k_1x_1+k_2x_2)}\right| \\ &\ge \left|\sum_{k_1=0}^{2^{s_1-1}-1} \sum_{k_2=0}^{2^{n-s_1-1}-1} \lambda'_k g_k(\omega)\big(\cos(k_1x_1)\cos(k_2x_2) - \sin(k_1x_1)\sin(k_2x_2)\big)\right|, \end{aligned} \tag{4.8}$$

where $\lambda'_{(k_1,k_2)} \equiv \lambda_{(k_1+[2^{s_1-1}],k_2+[2^{n-s_1-1}])}$.

Since the cosine function is even and the sine is odd, the last relation implies

$$\mathrm{P}\left\{\|\delta_s\|_{L_1} \le \frac{2^{n/2+1}}{n^{1/2}} A^{-1} w\right\}$$
$$\le \mathrm{P}\left\{\left\|\sum_{k_1=0}^{2^{s_1-1}-1} \sum_{k_2=0}^{2^{n-s_1-1}-1} \lambda'_k g_k(\omega)\cos(k_1x_1)\cos(k_2x_2)\right\|_{L_1} \le \frac{2^{n/2+2}}{n^{1/2}} A^{-1} w\right\} \tag{4.9}$$
$$\equiv \mathrm{P}\left\{\|\delta'_s\|_{L_1} \le \frac{2^{n/2+2}}{n^{1/2}} A^{-1} w\right\}.$$

Consider the set

$$\Delta_s = \left\{\frac{2\pi j_1}{2^{s_1}-1}, \frac{2\pi j_2}{2^{n-s_1}-1}\right\}, \qquad 0 \le j_1 \le 2^{s_1}-2, \quad 0 \le j_2 \le 2^{n-s_1}-2,$$

for a given $s = (s_1, n - s_1)$.

By the Marcinkiewicz result (see [12])

$$\|\delta'_s\|_{L_1} \ge \frac{c}{2^n} \sum_{z\in\Delta_s} |\delta'_s(z)|, \qquad c > 0;$$

therefore,

$$\mathrm{P}\left\{\|\delta'_s\|_{L_1} \le \frac{2^{n/2+2}A^{-1}w}{n^{1/2}}\right\} \le \mathrm{P}\left\{\omega : \frac{1}{2^n}\sum_{z\in\Delta_s} |\delta_s(z,\omega)| \le \frac{2^{n/2}}{n^{1/2}}\, c_1 w\right\}. \tag{4.10}$$

Assume that $z = (z_1, z_2) \in \Delta_s$, $v = (v_1, v_2) \in \Delta_s$, and $0 < z_1, z_2, v_1, v_2 < \pi$. Then

$$\sum_{k_1=0}^{2^{s_1-1}-1} \sum_{k_2=0}^{2^{n-s_1-1}-1} \lambda'_k \cos(k_1z_1)\cos(k_2z_2)\cos(k_1v_1)\cos(k_2v_2)$$
$$= \begin{cases} 0, & \text{for} \quad z \ne v, \\ \dfrac{(2^{s_1-1}-1/2)(2^{n-s_1-1}-1/2)}{4}, & \text{for} \quad z = v. \end{cases} \tag{4.11}$$

Indeed, the left-hand side of (4.11) equals

$$\left(\frac{1}{2} + \sum_{k_1=1}^{2^{s_1-1}-1} \cos(k_1z_1)\cos(k_1v_1)\right)\left(\frac{1}{2} + \sum_{k_2=1}^{2^{n-s_1-1}-1} \cos(k_2z_2)\cos(k_2v_2)\right),$$

so it remains to note that for $z = 2\pi j/(2p+1)$, $v = 2\pi j'/(2p+1)$, $0 < z, v < \pi$,

$$\frac{1}{2} + \sum_{k=1}^{p} \cos(k_1z_1)\cos(k_1v_1) = \begin{cases} \dfrac{p+1/2}{2}, & z_1 = v_1, \\ 0, & z_1 \ne v_1. \end{cases}$$

From (4.11) and a well-known property of the normal distribution (see [8, Theorem 2.10]) we conclude that the random vector

$$\{\delta_s(z,\omega),\ z = (z_1, z_2) \in \Delta_s,\ 0 < z_1, z_2 < \pi\}$$

also has a normal distribution; its components are independent with zero expected value and variance $\geq c_3' 2^n$, $c_3' > 0$. Therefore (see also (4.9)),

$$\mathrm{P}\left\{\|\delta_s'\|_{L_1} \leq \frac{2^{n/2+2}A^{-1}w}{n^{1/2}}\right\} \leq \mathrm{P}\left\{\frac{1}{2^n}\sum_{k=1}^{2^{n-5}} |g_k(\omega)| \leq \frac{c_4 w}{n^{1/2}}\right\}$$

$$\leq 2^{2^{n-5}}\left[\mathrm{P}\left\{|g_1| \leq \frac{2c_4 w}{n^{1/2}}\right\}\right]^{2^{n-6}} \leq \left(\frac{c_5 w}{n^{1/2}}\right)^{2^{n-6}} \qquad (4.12)$$

for $w \leq c_6 n^{1/2}$, where the constant $c_6 > 0$ is sufficiently small. Finally (see also (4.6), (4.8)), we have

$$\gamma(w) \leq \left(\frac{c_7 w}{n^{1/2}}\right)^{n\cdot 2^{n-8}}, \qquad w \leq c_6 n^{1/2}. \qquad (4.13)$$

Assume that for some $\Omega \subset [0, 2\pi]^2$ inequality (4.1) holds. Then for each w

$$\gamma(w) \geq \mathrm{P}\left\{\max_{x\in\Omega} |P(x,\omega)| < |\Lambda_n|^{1/2}\frac{w}{y}\right\}. \qquad (4.14)$$

Let us bound the right-hand side of (4.14) from below. Define the random vector $\{r_x, r_x', \ x \in \Omega\}$ of dimension $2|\Omega|$ as follows:

$$r_x = \operatorname{Re} P(x,\omega), \qquad r_x' = \operatorname{Im} P(x,\omega).$$

Then

$$\mathrm{P}\left\{\max_{x\in\Omega} |P(x,\omega)| < |\Lambda_n|\frac{w}{y}\right\} \geq \mathrm{P}\left\{\max_{x\in\Omega} \max\big(|r_x|, |r_x'|\big) < \left(\frac{|\Lambda_n|}{2}\right)^{1/2}\frac{w}{y}\right\}. \qquad (4.15)$$

The vector $\{r_x, r_x', \ x \in \Omega\}$ has a normal distribution with zero expected value. By the Ŝidak theorem [17] (see also [4, Corollary 1]), the right-hand side of (4.15) is bounded from below by the value

$$\Pi = \prod_{x\in\Omega}\left(\mathrm{P}\left\{|r_x| < \left(\frac{|\Lambda_n|}{2}\right)^{1/2}\frac{w}{y}\right\}\mathrm{P}\left\{|r_x'| < \left(\frac{|\Lambda_n|}{2}\right)^{1/2}\frac{w}{y}\right\}\right). \qquad (4.16)$$

Let us bound product (4.16) from below for $w = c_6 n^{1/2}$. Since, for each $x \in \Omega$,

$$\big(\mathrm{E}|r_x|^2\big)^{1/2} \leq |\Lambda_n|^{1/2}, \qquad \big(\mathrm{E}|r_x'|^2\big)^{1/2} \leq |\Lambda_n|^{1/2},$$

$$\Pi \geq \left(1 - 2\int_{c_9 n^{1/2}/y}^{\infty} e^{-x^2/2}\,dx\right)^{2|\Omega|} \geq \exp\big[-4|\Omega|\exp(-c_9^2 n/(2y^2))\big]; \qquad (4.17)$$

here we use the inequality

$$1 - 2\int_z^{\infty} e^{-x^2/2}\,dx \geq \exp(-2\exp(-z^2/2)), \qquad z \geq 1,$$

and assume that $y < c_9 n^{1/2}$. Then (see (4.14)–(4.17)),

$$\gamma(c_6 n^{1/2}) \geq \exp\big[-4|\Omega|\exp(-c_{10} n/y^2)\big]. \qquad (4.18)$$

Comparing (4.18) and inequality (4.13) for $w = c_6 n^{1/2}$ we obtain

$$c_{11}^{n2^n} \geq \exp\big[-4|\Omega|\exp(-c_{10} n/y^2)\big],$$

i.e.,

$$|\Omega| \geq c_{12} n 2^n \exp(c_{10} n/y^2).$$

Thus, relation (4.2) is proved; therefore, Theorem 4.1 is also proved. □

The assertion of Corollary 1.1 stated in the introduction readily follows from Theorem 4.1 for $d = 2$.

For $d > 2$ and $\alpha = 1/2$ the assertion of Corollary 1.1 is evident; for $\alpha < 1/2$ it follows from the two-dimensional result since

$$|Q_n^d| \ll |Q_n^2| \exp(cn^\varepsilon)$$

for arbitrary $c > 0$, $\varepsilon > 0$ and the subspace of $\mathcal{T}(Q_n^d)$ formed by polynomials in two variables coincides with $\mathcal{T}(Q_n^2)$.

5. Concluding Remarks

(a) Let us construct polynomials $t_k \in \mathcal{T}_r(2^k)$, $k = 1, 2, \dots$ such that $\|t_k\|_\infty \geq c_1 k^{1/2} \|t_k\|_{QC}$, $c_1 > 0$, whose existence was mentioned in the introduction. Assume that for a given k

$$f(x) = \sum_{s=0}^{k-1} 2^{-s} \sum_{j=2^s}^{2^{s+1}-1} e^{ijx}$$

and $t_k = \operatorname{Re} f$. Obviously $f(0) = t_k(0) = k$. Let us show that $\|f\|_{QC} \ll k^{1/2}$; if it is so by the inequality $\|t_k\|_{QC} \leq \|f\|_{QC}$ we obtain $\|t_k\|_{QC} \ll k^{1/2}$.

Define the function

$$g_\omega(x) = \sum_{s=0}^{k-1} r_s(\omega) \chi_{[-2^{-s}, 2^{-s}]}(x).$$

Then for each ω

$$\|f_\omega(x) - g_\omega(x)\|_\infty \ll 1, \qquad f_\omega(x) = \sum_{s=0}^{\infty} r_s(\omega) \delta_s(f, x).$$

Indeed, let $2^{-l-1} < |x| \leq 2^{-l}$, $l \leq k$. We have

$$|f_\omega(x) - g_\omega(x)| \ll \sum_{s=0}^{l} 2^{-s} \sum_{j=2^s}^{2^{s+1}-1} |1 - e^{ijx}| + \sum_{s>l} \frac{2^{-s}}{x} \ll 1.$$

Similarly for $|x| < 2^{-k-1}$

$$|f_\omega(x) - g_\omega(x)| \leq \sum_{s=0}^{k} 2^{-s} \sum_{j=2^s}^{2^{s+1}-1} |1 - e^{ijx}| \ll 1.$$

Next, the estimate

$$\int_0^1 \|g_\omega(x)\|_\infty \, d\omega \ll \sqrt{k}$$

follows from the estimate for the majorant of partial sums of a polynomial with respect to the Rademacher system

$$\sum_{s=0}^{k-1} r_s(\omega)$$

(see [8, Theorem 2.9]).

(b) The multidimensional QC-norm was defined as

$$\|f(x_1, \dots, x_d)\|_{QC} = \big\| \|f(\cdot, x_2, \dots, x_d)\|_{QC} \big\|_\infty,$$

that is, the QC-norm with respect to the variable x_1.

There are other definitions where the sign average is also with respect to other variables. Consider two such methods:

$$\|f\|_{QC}^{T} \equiv \int\limits_{[0,1]^d} \left\| \sum_s r_{s_1}(\omega_1)\cdot\dots\cdot r_{s_d}(\omega_d)\delta_{(s_1,\dots,s_d)}(f,x) \right\|_\infty d\omega, \tag{5.1}$$

$$\|f\|_{QC}^{*} \equiv \int\limits_0^1 \left\| \sum_s r_{i(s)}(\omega)\delta_s(f) \right\|_\infty d\omega, \tag{5.2}$$

where i gives a one-to-one correspondence between $\mathbb{Z}_+^d$ and $\mathbb{N}$.

From the proofs of the upper bounds in Theorems 1.2 and 1.3, it follows that these estimates remain valid for norms (5.1), (5.2). The analogs of Theorem 2.2 in the cases of norms (5.1) and (5.2) may be easily derived; therefore, lower bounds for these norms are the same as the bounds for the QC-norm stated in Theorems 1.2 and 1.3.

Note that in some cases the $\|\cdot\|_{QC}^*$-norm is easier to treat than the $\|\cdot\|_{QC}$-norm.

(c) Now we give a result connected with Theorem 2.1. In Sec. 2 this theorem was derived from the one-dimensional result of Theorem 1.1 by means of a comparatively simple technique; it was the Littlewood–Paley theorem and the Hölder inequality. P. G. Grigoriev has used results due to S. V. Bochkarev [3] to study multivariable polynomials in his degree work. Similarly, using results of S. V. Bochkarev one can obtain the assertion below from Theorem 1.1.

Assertion 5.1. *Assume that*

$$\|f\|_{QC,L} \equiv \big\| \|f(\cdot,x^1)\|_{QC} \big\|_{L_1}$$

and

$$\rho^+(s) = \rho(s) \cap \mathbb{Z}_+^d.$$

Then, for any $t_s \in \mathcal{T}(\rho^+(s))$, the inequality

$$\left\| \sum_{s\in Y_n^d} t_s \right\|_{QC,L} \ge c(d) n^{-(d/2-1)} \sum_{s\in Y_n^d} \|t_s\|_1, \qquad c(d) > 0,$$

is satisfied.

(d) Corollary 1.1 shows that the properties of the subspace $\mathcal{T}(\Delta Q_n)$ in $\mathcal{T}([-2^n,2^n]^d)$ are analogous to the properties of a random subspace in $\mathcal{T}([-2^n,2^n]^d)$ of the same dimension (see also [7]).

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REFERENCES

1. E. S. Belinskii, "Asymptotic characteristics for classes of functions with condition on mixed derivative (mixed difference)," in: *Investigations on the Theory of Real Multivalued Functions*, Yaroslavl (1990), p. 22–37.
2. E. S. Belinskiy, "Estimates of entropy numbers and Gaussian measures for classes of functions with bounded mixed derivative," *J. Approx. Theory*, **93**, No. 1, 114–127 (1998).
3. S. V. Bochkarev, "De la Vallée-Poussin series in the BMO, L_1, $H^1(D)$ spaces and multiplicative inequalities," *Tr. Mat. Inst. Steklova*, **210**, 41–64 (1995).
4. E. D. Gluskin, "Extreme properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces," *Mat. Sb.*, **136(178)**, No. 1, 85–96 (1988).

5. P. G. Grigoriev, “On a sequence of trigonometric polynomials,” *Mat. Zametki*, **61**, No. 6, 935–938 (1997).
6. B. S. Kashin, “On some properties of a space of trigonometric polynomials with uniform norm,” *Tr. Mat. Inst. Steklova*, **145**, 111–116 (1980).
7. B. S. Kashin, “On properties of N-dimensional cube random sections,” *Vestn. MGU, Ser. Mat., Mech.*, No. 3, 8–11 (1983).
8. B. S. Kashin and A. A. Saakian, *Orthogonal Series* [in Russian], Nauka, Moscow (1984).
9. B. S. Kashin and V. N. Temlyakov, “On estimates for approximate characteristics of classes of functions with bounded mixed derivative,” *Mat. Zametki*, **58**, No. 6, 922–925 (1995).
10. B. S. Kashin and V. N. Temlyakov, “On a norm and related applications,” *Mat. Zametki*, **64**, No. 4, 637–640 (1998).
11. G. G. Lorentz, “Metric entropy and approximation,” *Bull. Amer. Math. Soc.*, **72**, 903–937 (1966).
12. J. Marcinkiewicz, *Collected Papers*, PWN, Warszawa (1964).
13. A. Pajor and N. Tomczak-Jaegermann, “Subspaces of small codimension of finite-dimensional Banach spaces,” *Proc. Amer. Math. Soc.*, **97**, No. 4, 637–642 (1986).
14. A. Pietsch, *Operator Ideals* [Russian translation], Mir, Moscow (1982).
15. G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Univ. Press, Cambridge (1989).
16. C. Schütt, “Entropy numbers of diagonal operators between symmetric Banach spaces,” *J. Approx. Theory*, **40**, 121–128 (1984).
17. Z. Ŝidak, “Rectangular confidence regions for the means of multivariate normal distributions,” *J. Amer. Statist. Assoc.*, **62**, 626–633 (1967).
18. M. Talagrand, “The small ball problem for the Brownian sheet,” *Ann. Probab.*, **22**, 1331–1354 (1994).
19. V. N. Temlyakov, “An approximation of functions with bounded mixed derivative,” *Tr. Mat. Inst. Steklova*, **178** (1986).
20. V. N. Temlyakov, “Estimates for asymptotic characteristics of classes of functions with bounded mixed derivative or difference,” *Tr. Mat. Inst. Steklova*, **189**, 138–168 (1989).
21. V. N. Temlyakov, “Estimates for the best bilinear approximations of functions and approximate numbers of integral operators,” *Mat. Zametki*, **51**, No. 5, 125–134 (1992).
22. V. N. Temlyakov, “An inequality for trigonometric polynomials and its application for estimating the entropy numbers,” *J. Complexity*, **11**, 293–307 (1995).
23. V. N. Temlyakov, “An inequality for trigonometric polynomials and its application for estimating the Kolmogorov widths,” *East J. Approx.*, **2**, No. 2, 253–262 (1996).

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