ISSN 0001-4346, Mathematical Notes, 2014, Vol. 95, No. 6, pp. 775–779. © Pleiades Publishing, Ltd., 2014.
Original Russian Text © B. S. Kashin, A. V. Meleshkina, 2014, published in Matematicheskie Zametki, 2014, Vol. 95, No. 6, pp. 830–835.

On n-Term Approximations with Respect to Frames Bounded in $L^p(0,1)$, $2 < p < \infty$

B. S. Kashin[1*] and A. V. Meleshkina[2]**

[1]*Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, Russia*

[2]*Moscow State University, Moscow, Russia*

Received December 26, 2013

Abstract—In this paper, best canonical n-term approximations in the norm of the spaces $L^2(0,1)$ of the family $\mathbb{I}$ of characteristic functions of intervals are studied.

DOI: 10.1134/S0001434614050216

Keywords: *best canonical n-term approximation, tight frame, Haar system, Bessel's inequality, Rademacher function, Khinchine's inequality.*

In this paper, we study best canonical n-term approximations in the norm of the spaces $L^2(0,1)$ of the family $\mathbb{I}$ of characteristic functions of intervals lying in the interval $(0,1)$:

$$\mathbb{I} = \{I_\omega, \omega \subset (0,1)\},$$

where $\omega = (\alpha, \beta) \subset (0,1)$ and

$$I_\omega(x) = \begin{cases} 1 & \text{if } x \in \omega; \\ 0 & \text{if } x \notin \omega. \end{cases}$$

We consider polynomial approximation with respect to particular systems of functions which are *tight frames* $\Phi = \{\varphi_k\}_{k=1}^\infty \subset L^2(0,1)$ possessing the additional property

$$\|\varphi_k\|_{L^p(0,1)} \le D, \qquad k = 1,2,3,\dots, \quad 2 < p \le \infty. \tag{1}$$

By the *canonical decomposition* of a function $f \in L^2(0,1)$ with respect to a tight frame Φ we mean the following series convergent in the norm of $L^2(0,1)$:

$$f = \sum_{k=1}^\infty \langle f, \varphi_k\rangle \varphi_k. \tag{2}$$

Here and elsewhere, $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product in $L^2(0,1)$. In addition, we denote $\|f\|_p = \|f\|_{L^p(0,1)}$, $\#\Lambda$ is the number of elements in a finite set Λ, and $[x]$ is the integer part of a real number x.

By the *best canonical n-term approximation* of a function $f \in L^2(0,1)$ with respect to a tight frame Φ we mean the quantity

$$\sigma_n(f,\Phi) = \inf_{\Lambda\subset\mathbb{N},\, \#\Lambda \le n} \Big\| f - \sum_{k\in\Lambda} \langle f,\varphi_k\rangle\varphi_k \Big\|_2. \tag{3}$$

Further, if F is a subset of $L^2(0,1)$, then the quantity

$$\sigma_n(F,\Phi) = \sup_{f\in F} \sigma_n(f,\Phi) \tag{4}$$

*E-mail: kashin@mi.ras.ru
**E-mail: meleshkina-anna@mail.ru

is called the *best canonical n-term approximation* of F.

In the case where the tight frame is a complete orthonormal system, the quantities (3), (4) coincide with the usual best approximations by n-term polynomials. For more details on frames, see, for example, [1].

The example of the Haar complete orthonormal system $\mathbb{X}$ for which

$$\sigma_n(\mathbb{I},\mathbb{X}) \le 2^{-n/4+2}, \qquad n = 1,2,\dots \tag{5}$$

(see, e.g., [2] where an exponential lower bound for $\sigma_n(\mathbb{I},\Phi)$ and an arbitrary orthonormal system Φ was also established) shows that, for the system Φ without the constraint (1), the quantities $\sigma_n(\mathbb{I},\Phi)$ can decrease exponentially.

On the other hand, it was shown in [2] (see also [3]) that, for the orthonormal system Φ with property (1) the following estimate holds for $p=\infty$:

$$\sigma_n(\mathbb{I},\Phi) \ge Cn^{-1/2}, \quad n = 1,2,\dots, \qquad C>0. \tag{6}$$

Comparing estimates (5) and (6), we naturally come to the question of a possible order of decrease of the quantities $\sigma_n(\mathbb{I},\Phi)$ for orthonormal systems or frames satisfying condition (1) for $2<p<\infty$.

Theorem 1. *For an arbitrary tight frame Φ satisfying condition* (1)*, the following estimate is valid*:

$$\sigma_n(\mathbb{I},\Phi) \ge c_{p,D}\cdot n^{-p/(2(p-2))}, \qquad n=1,2,\dots$$

(here and elsewhere, by $c_{p,D}$ and C_p we denote positive quantities depending on the parameters indicated in the notation).

Theorem 2. *There exists a tight frame Φ satisfying condition* (1) *such that*

$$\sigma_n(\mathbb{I},\Phi) \le C_p\cdot n^{-p/(2(p-1))}, \qquad n=1,2,\dots .$$

Remark 1. For the frame Φ constructed in Theorem 2, the quantity D in (1) depends on p.

Proof of Theorem 1. The proof is very simple. Let n be given, and let N be a natural parameter whose value will be given below. Also let

$$f_0 = I_\omega \subset \mathbb{I}, \qquad \omega = \left(0, \frac{1}{N}\right).$$

Obviously,

$$\sigma_n(f_0,\Phi) \ge \inf_{\Lambda\subset\mathbb{N},\,\#\Lambda\le n} \left\| f_0 - \sum_{k\in\Lambda}\langle f_0,\varphi_k\rangle\varphi_k \right\|_{L^2(0,1/N)}. \tag{7}$$

Using the definition of a tight frame for any $\Lambda\subset\mathbb{N}$, $\#\Lambda\le n$, we can write

$$\begin{aligned}\left\| f_0 - \sum_{k\in\Lambda}\langle f_0,\varphi_k\rangle\varphi_k \right\|_{L^2(0,1/N)} &\ge N^{-1/2} - \left(\sum_{k\in\Lambda}|\langle f_0,\varphi_k\rangle|\right)\sup_{k=1,2,\dots}\|\varphi_k\|_{L^2(0,1/N)}\\ &\ge N^{-1/2} - n^{1/2}\left(\sum_{k\in\Lambda}\langle f_0,\varphi_k\rangle^2\right)^{1/2}\sup_{k=1,2,\dots}\|\varphi_k\|_{L^2(0,1/N)}\\ &\ge N^{-1/2} - n^{1/2}N^{-1/2}\sup_{k=1,2,\dots}\|\varphi_k\|_{L^2(0,1/N)}.\end{aligned} \tag{8}$$

For $k=1,2,\dots$, in view of (1) and Hölder's inequality with exponents $p/2$ and $p/(p-2)$, we have

$$\begin{aligned}\int_0^{1/N}\varphi_k^2\,dx &= \int_0^{1/N} 1(|\varphi_k|^p)^{2/p}\,dx\\ &\le \left(\int_0^{1/N}|\varphi_k|^p\,dx\right)^{2/p}\left(\int_0^{1/N}1\,dx\right)^{(p-2)/p} \le D^2\cdot N^{-(p-2)/p}.\end{aligned}$$

Therefore,

$$\sup_{k=1,2,\dots} \|\varphi_k\|_{L^2(0,1/N)} \le D \cdot N^{-(p-2)/(2p)}. \tag{9}$$

Set $N = \left[(4nD^2)^{p/(p-2)}\right] + 1$. Then it follows from (7)–(9) that

$$\sigma_n(f_0, \Phi) \ge \frac{1}{2} N^{-1/2} \ge c_{p,D} \cdot n^{-p/(2(p-2))},$$

as required. □

Proof of Theorem 2. Let us construct a complete (in $L^2(0,1)$) orthonormal system $\Psi = \{\psi_m\}_{m=1}^{\infty}$ with

$$\|\psi_m\|_{L^p(0,1)} \le D_p, \qquad m = 1, 2, \dots, \tag{10}$$

such that, for any interval $\omega \subset (0,1/2)$ and $n = 1, 2, \dots$, there exists a set $\Lambda = \Lambda(n, \omega) \subset \mathbb{N}$ with

$$\left(\sum_{m \notin \Lambda} \langle I_\omega, \psi_m \rangle^2\right)^{1/2} \le C_p \cdot n^{-p/(2(p-1))}, \qquad \#\Lambda \le n. \tag{11}$$

Then, for a tight frame $\widetilde{\Phi}$ in the space $L^2(0,1/2)$ with elements $\psi_m(x)$, $x \in (0,1/2)$, $m = 1, 2, \dots$, we have

$$\sigma_n(\mathbb{I}_{(0,1/2)}, \widetilde{\Phi}) \le C_p \cdot n^{-p/(2(p-1))}, \qquad \text{where} \quad \mathbb{I}_{(0,1/2)} = \left\{ I_\omega \subset \mathbb{I},\, \omega \subset \left(0, \frac{1}{2}\right) \right\}.$$

Indeed, it follows from (11) and Bessel's inequality for series in the system Ψ that, for $\omega \subset (0,1/2)$, the following esimate holds:

$$\left\| I_\omega - \sum_{m \in \Lambda} \langle I_\omega, \psi_m \rangle \psi_m \right\|_{L^2(0,1/2)} \le C_p \cdot n^{-p/(2(p-1))}, \qquad n = 1, 2, \dots .$$

The transformation

$$T \colon L^2\left(0, \frac{1}{2}\right) \to L^2(0,1), \qquad \text{where} \quad (Tf)(z) = \frac{1}{\sqrt{2}} f\left(\frac{z}{2}\right)$$

takes the frame $\widetilde{\Phi}$ to a tight frame Φ satisfying all the assumptions of Theorem 2.

We shall construct the required system Ψ by transforming the Haar system (see [4, p. 70])

$$\mathbb{X} = \{\chi_0^0, \chi_k^i,\, k = 0, 1, \dots,\, i = 1, \dots, 2^k\}.$$

For $k = 2, 3, \dots$, consider the orthogonal $2^k \times 2^k$ matrices $A_k = \{\alpha_{is}^k\}$ with

$$\alpha_{is}^k = \begin{cases} 1 - (2^k - 1)^{-1} & \text{if } 1 \le i = s < 2^k, \\ 0 & \text{if } i = s = 2^k, \\ -(2^k - 1)^{-1} & \text{if } i \ne s,\, i, s < 2^k, \\ (2^k - 1)^{-1/2} & \text{if } i \ne s,\, i = 2^k \text{ or } s = 2^k. \end{cases}$$

The matrices A_k have been used in the theory of orthogonal series, beginning with Olevskii's papers (also see [4, p. 441]).

Let us define the new complete (in $L^2(0,1)$) orthonormal system $\{\widetilde{\chi}_k^i\}$ by setting

$$\widetilde{\chi}_k^i = \chi_k^i, \quad k = 0, 1, \qquad \text{and} \qquad \widetilde{\chi}_k^i = \sum_{s=1}^{2^k} \alpha_{is}^k \chi_k^s, \quad k = 2, 3, \dots, \quad i = 1, \dots, 2^k.$$

It is easy to see that, for $k = 2, 3, \dots,$

$$\begin{cases} \text{a)} \quad w_k(x) \equiv \widetilde{\chi}_k^{2^k}(x) = r_{k+1}(x)\dfrac{2^{k/2}}{2^k-1}\, I_{(0,1-2^{-k})}(x), \\ \text{where } r_{k+1}(x) \text{ is the } (k+1)\text{th Rademacher function (see [4, p. 22]);} \\ \text{b) for } i = 1, \dots, 2^k - 1 \\ \widetilde{\chi}_k^i(x) = \begin{cases} \chi_k^i(x) - r_{k+1}\dfrac{2^{k/2}}{(2^k-1)} & \text{if } 0 < x < 1 - 2^{-k}, \\ (2^k-1)^{-1/2}\chi_k^{2^k} & \text{if } 1 - 2^{-k} < x < 1. \end{cases} \end{cases} \tag{12}$$

It follows from (12) that, for $2 < p \le \infty$,

$$\begin{cases} 1) \quad \text{uniformly in } i = 1, 2, \dots, 2^k - 1 \text{ as } k \to \infty \ \|\widetilde{\chi}_k^i\|_p \cdot 2^{-k(1/2-1/p)} \to 1; \\ 2) \quad \|\widetilde{\chi}_k^i\|_{L^p(1-2^{-k},1)} \le 2 \cdot 2^{-k/p}, \qquad i = 1, 2, \dots, 2^k - 1. \end{cases} \tag{13}$$

Let

$$\beta_k = 2^{[k(1-2/p)]+1}, \qquad k = 2, 3, \dots, \tag{14}$$

and let U_{β_k} be a Walsh orthogonal matrix of order β_k whose entries satisfy the equality

$$|(U_{\beta_k})_{rs}| = \beta_k^{-1/2}, \qquad 1 \le r, s \le \beta_k.$$

For brevity, let $u_k = U_{\beta_k}$, $k = 2, 3, \dots$. We shall also define the sequence of natural numbers s_k, $k = 2, 3, \dots,$ setting

$$s_2 = 1, \qquad s_k = s_{k-1} + \beta_{k-1} - 1, \quad k = 3, 4, \dots.$$

We now construct the required complete (in $L^2(0,1)$) orthonormal system Ψ whose elements are numbered by three indices k, i, ν, setting

$$\begin{cases} \psi_k^{i,1} = \chi_k^i & \text{for } k = 0, 1,\ 1 \le i \le 2^k,\ \nu = 1, \\ \psi_k^{i,\nu} = (u_k)_{\nu,1} \cdot \widetilde{\chi}_k^i + \sum_{\mu=2}^{\beta_k} (u_k)_{\nu,\mu} \cdot w_{s_k+\mu-1} & \\ & \text{for } k = 2, 3, \dots,\ 1 \le i \le 2^k,\ \nu = 1, \dots, \beta_k. \end{cases} \tag{15}$$

In view of relations (12)–(14) and Khintchine's inequality for the Rademacher system (see [4, p. 34]), we have

$$\sup_{k,i,\nu} \|\psi_k^{i,\nu}\|_p \le D_p, \tag{16}$$

where sup in (16) is taken over all admissible values of the parameters k, i, ν.

Let us estimate the coefficients of the decomposition of the function I_ω, $\omega \subset (0, 1/2)$, with respect to the system Ψ. First, note that, for each $k = 0, 1, \dots,$ there exists at most two Haar functions of "rank" k for which $\langle \chi_k^i, I_\omega \rangle \ne 0$. Obviously, such Haar functions satisfy the estimate $|\langle I_\omega, \chi_k^i \rangle| \le 2^{-k/2}$, and hence (see the definition of the functions $\widetilde{\chi}_k^i$)

$$|\langle I_\omega, \widetilde{\chi}_k^i \rangle| \le 2^{-k/2}$$

for the same values of i. For the other values of i, the inner product $\langle I_\omega, \widetilde{\chi}_k^i \rangle$ is estimated as

$$|\langle I_\omega, \widetilde{\chi}_k^i \rangle| \le 2 \cdot 2^{-(3k)/2}.$$

Therefore, (see (15)), for $k = 2, 3, \dots,$ among the inner products $\langle I_\omega, \psi_k^{i,\nu} \rangle$, there are at most $2\beta_k$ numbers estimated as follows:

$$|\langle I_\omega, \psi_k^{i,\nu} \rangle| \le 2^{-k/2}\beta_k^{-1/2} + 2^{-s_k+1}\beta_k^{-1/2} \le C_p \cdot 2^{-k/2}\beta_k^{-1/2}. \tag{17}$$

All the other numbers $\langle I_\omega, \psi_k^{i,\nu}\rangle$ admit the estimate

$$|\langle I_\omega, \psi_k^{i,\nu}\rangle| \le 2^{-(3k)/2+1}\beta_k^{-1/2} + 2^{-s_k+1}\beta_k^{-1/2} \le C_p \cdot 2^{-(3k)/2}\beta_k^{-1/2} \tag{18}$$

(in (17), (18), we have taken into account the fast growth of the sequences s_k, $k = 2, 3, \dots$). For a given n, let us define the index set Λ, $\#\Lambda \le n$, for which

$$\biggl(\sum_{(k,i,\nu)\notin\Lambda} \langle I_\omega, \psi_k^{i,\nu}\rangle^2\biggr)^{1/2} \le C_p \cdot n^{-p/(2(p-1))}.$$

We shall include in Λ at most $n/2$ triplets (k, i, ν) corresponding to all "large" coefficients $\langle I_\omega, \psi_k^{i,\nu}\rangle$ that are estimated by inequality (17) with $k \le k_0$. The number k_0 is defined by the relation

$$k_0 = \max\biggl\{\nu : \sum_{k=2}^{\nu} 2\beta_k \le \frac{n}{2}\biggr\}. \tag{19}$$

Using (19) and (14), we obtain

$$k_0 \ge \frac{p}{p-2}(\log_2 n - C_p'). \tag{20}$$

The remaining part Λ', $\#\Lambda' \ge n/2$, of the set Λ consists of all the remaining triplets (k, i, ν) with $k \le k_1$. The numbers k_1 satisfy the estimate $2^{k_1}\beta_{k_1} \ge C_p'' n$, whence we obtain

$$k_1 \ge \frac{p}{2(p-1)}(\log_2 n - C_p'''). \tag{21}$$

Using relations (20), (21), and estimates (17), (18), for the coefficients $\langle I_\omega, \psi_k^{i,\nu}\rangle$ we obtain

$$\begin{aligned} S^2 &\equiv \sum_{(k,i,\nu)\notin\Lambda} |\langle I_\omega, \psi_k^{i,\nu}\rangle|^2 \\ &\le C_p\biggl(\sum_{k>k_0} \beta_k \frac{2^{-k}}{\beta_k} + \sum_{k>k_1} 2^k\beta_k \frac{2^{-3k}}{\beta_k}\biggr) \le C_p'(2^{-k_0} + 2^{-2k_1}) \le C_p'' \cdot n^{-p/(p-1)}. \end{aligned}$$

Thus, $S \le C_p \cdot n^{-p/(2(p-1))}$ and the required relation (11) iss verified. Theorem 2 is proved. □

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (grant no. 11-01-00476-a).

REFERENCES

1. I. Ya. Novikov, V. Yu. Protasov, and M. A. Skopina, *Wavelet Theory* (Fizmatlit, Moscow, 2005) [in Russian].
2. B. S. Kashin, "On lower estimates for n-term approximation in Hilbert spaces," in *Approximation Theory* (DARBA, Sofia, 2002), pp. 241–257.
3. B. S. Kashin and T. Yu. Kulikova, "On the validity for frames of a result concerning orthogonal systems," Mat. Zametki **77** (2), 311–312 (2005) [Math. Notes **77** (2), 280–282 (2005)].
4. B. C. Kashin and A. A. Saakyan, *Orthogonal Series* (Izd. AFTs, Moscow, 1999) [in Russian].