On the choice of a subsystem of convergence in a given orthonormal system

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Results have long been known in the theory of orthogonal series on the possibility of choosing in an arbitrary orthonormal system (ONS) a subsystem with certain additional "good" properties. To this type there belongs, in particular, the following theorem, which was proved in 1936 independently by Men'shov and Marcinkiewicz.

Theorem ([1], [2]). *An arbitrary ONS* $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in (0, 1)$, *contains a subsystem of convergence, that is,* $\{\varphi_{n_k}(x)\}$ *is a system of convergence for some sequence* $\{n_k\}_{k=1}^{\infty}$, $n_1 < n_2 < \dots$.

(An ONS $\{\psi_n\}$ is called a system of convergence if every series

$$\sum a_n \psi_n(x), \quad \sum a_n^2 < \infty$$

converges almost everywhere.)

We remark that the known proofs of the Marcinkiewicz-Men'shov theorem and of certain other theorems of a similar nature are non-effective in the sense that they do not give any upper estimate for the rate of increase of the numbers $\{n_k\}$.

In [3] Bennett posed the following question: is there a sequence $\{r_k\}_{k=1}^{\infty}$ of numbers such that any ONS $\{\varphi_n(x)\}$ has a subsystem of convergence $\{\varphi_{n_k}(x)\}$ for which

$$\lim_{k\to\infty} \frac{n_k}{r_k} = 0?$$

A positive answer to this question is given by the following theorem.

Theorem 1. *An arbitrary ONS* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *has a subsystem of convergence* $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ *with* $n_k \leqslant R_k$ $(k = 1, 2, \dots)$, *where*

$$R_1 = 3, \quad R_{k+1} = (R_k)! \quad (k = 1, 2, \dots). \tag{1}$$

The proof of Theorem 1 uses the following lemma.

Lemma. *For any ONS* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *and any* $\varepsilon > 0$ *there is a Lebesgue-measure preserving transformation* $\sigma(x)$ *of the interval* $(0, 1)$ *such that*

$$\sum_{n=1}^{\infty} \| \varphi_n(x) - P_n(\sigma(x)) \|_{L^2}^2 < \infty,$$

where $P_n(x)$ $(n = 1, 2, \dots)$ *is a polynomial of degree* $\leqslant (n!)^{1+\varepsilon}$ *in the Haar system.*

The upper estimate in Theorem 1 for the numbers n_k is very coarse. An essentially new technique is probably needed to obtain sufficiently sharp estimates. It is natural to ask whether in the statement of Theorem 1 the sequence (1) can be replaced by $k^{1+\varepsilon}$ $(k = 1, 2, \dots)$ for any $\varepsilon > 0$.

In conclusion we add something to our paper [4]: for $N = 1, 2, \dots$ let D_N be the set of systems $\Phi = \{\varphi_j(x)\}_{j=1}^{N}$ of functions of the form $\varphi_j(x) = \varepsilon_{ij}$ for $x \in ((i-1)/N, i/N)$, where $\varepsilon_{ij} = \pm 1$ $(i, j = 1, \dots, N)$. We introduce a measure μ_N on D_N by setting $\mu_N(\Phi) = 2^{-N^2}$ if $\Phi \in D_N$. Next, let

$$\| \Phi \| \equiv \sup_{\sum_{j=1}^{N} a_j^2 = 1} \Big\| \sum_{j=1}^{N} a_j \varphi_j(x) \Big\|_{L^2}; \quad s(\Phi) \equiv \sup_{\sum_{j=1}^{N} a_j^2 = 1} \Big\| \sup_{1 \leqslant M \leqslant N} \Big| \sum_{j=1}^{M} a_j \varphi_j(x) \Big| \Big\|_{L^2}.$$

Then $s(\Phi) \leqslant C \log N \cdot \| \Phi \|$ for any system $\Phi \in D_N$ and

$$\sup_{\Phi \in D_N} s(\Phi) \cdot \| \Phi \|^{-1} \geqslant c \log N$$

(see [4], [5] for details).

Proposition. *For any $B > 0$ there exist constants $\gamma = \gamma(B) > 1$ and $K = K(B)$ such that for* $N = 1, 2, \ldots$

$$\mu_N\{\Phi \in D_N\colon\ s(\Phi)\cdot\|\Phi\|^{-1} \geqslant B \log N\} \leqslant K\cdot\exp\{-N^\gamma\}.$$

References

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