MAT. SBORNIK
TOM 99(141)(1976), No. 3

MATH. USSR SBORNIK
Vol. 28 (1976), No. 3

UDC 517.52

# ON A COMPLETE ORTHONORMAL SYSTEM

B. S. KAŠIN

ABSTRACT. In this paper we construct in $L_2(0, 1)$ a complete orthonormal system $\{\psi_n(x)\}_1^\infty$ for which every series $\sum_1^\infty c_n\psi_n(x)$ whose coefficients satisfy the condition $\sum_1^\infty c_n^2 = \infty$ diverges on a set of positive measure.
Bibliography: 6 titles.

In this paper we prove

THEOREM 1. *There exists a complete orthonormal system of functions*(1) $\{\psi_n(x)\}_{n=9}^\infty$ *(c.o.n.s.) in* $L_2(0, 1)$ *having the property that any series*

$$\sum_{n=9}^{\infty} c_n\psi_n(x), \tag{1}$$

*with coefficients satisfying the condition*

$$\sum_{n=9}^{\infty} c_n^2 = \infty, \tag{2}$$

*diverges on a set of positive measure.*

The question posed by P. L. Ul'janov in [6] remains open: Does there exist a c.o.n.s. $\{\psi_n(x)\}$ for which every series of the form (1) satisfying (2) diverges almost everywhere, while every series (1) not satisfying (2) converges almost everywhere?

For the formulation of some corollaries of Theorem 1, we recall some well-known concepts (for details, see [1] and [2]).

Let there be given functions $\{f_n(x)\}_1^\infty$ which are measurable on the segment [0, 1], and let there also be given a class $F$ of functions defined on [0, 1]. We call the system $\{f_n(x)\}$ a representation in the sense of convergence almost everywhere (a.e.) for the class $F$ if for every function $g(x) \in F$ there is a series $\sum_1^\infty a_n f_n(x)$ which converges to $g(x)$ a.e. on [0, 1].

Furthermore, a series $\sum_1^\infty a_n f_n(x)$ is called a null series in the sense of convergence a.e. if it converges to zero almost everywhere and $a_{n_0} \neq 0$ for some $n_0$. There is an analogous definition of a null series in the sense of convergence in measure. Talaljan [1] proved that for any c.o.n.s. there is a null series in the sense of convergence in measure.

*AMS (MOS) subject classifications* (1970). Primary 40A05, 42A56.

(1) For convenience in constructing the system $\{\psi_n(x)\}$, we have numbered the terms to begin with $n = 9$. This, of course, does not influence the essential matter.

© American Mathematical Society 1978

It was not known (see, for example, [6], §4.2) whether there exists for any c.o.n.s. a null series in the sense of convergence a.e. From Theorem 1, there follows immediately the following theorem, which gives an answer to this question.

THEOREM 2. *There exists a c.o.n.s. for which there is no null series in the sense of convergence a.e.*

Also from Theorem 1, there immediately follows

THEOREM 3. *Not every c.o.n.s. is a system of representations in the sense of convergence a.e. for the class of functions which are measurable and finite on the interval* [0, 1].

The question which Theorem 3 answers has remained open for a fairly long time (see [1], §10, Problem 1).

It follows that our system $\{\psi_n(x)\}$ is not a system of convergence a.e., and therefore some "good" functions (for example, some functions in the space $L_2(0, 1)$) are not represented by a series which is in the system and which converges a.e.

Before proving Theorem 1, we shall establish a few lemmas.

LEMMA 1. *There exists an absolute constant $b > 0$ such that for every $N > 3$ there exists an orthonormal matrix $A_N = \{a_{ji}\}_{j,i=1}^N$, with elements of the form*

$$a_{ji} = \begin{cases} 1 - \alpha_1/N & \text{for } i = j < N, \\ \alpha_2/N & \text{for } i, j < N, \ i \neq j, \\ \alpha_3/\sqrt{N} & \text{for } i = N, \ j \neq N, \\ \alpha_4/\sqrt{N} & \text{for } i \neq N, \ j = N, \\ \alpha_5/\sqrt{N} & \text{for } i = j = N, \end{cases}$$

*where the numbers $\alpha_k$ depend upon $N$ and $|\alpha_k| \leqslant b$, $k = 1, 2, 3, 4, 5$.*

PROOF. We consider the matrix $A'_N = \{\alpha'_{ji}\}_{j,i=1}^N$, where

$$a'_{ji} = \begin{cases} 1, & i = j < N, \\ \lambda, & i \neq j, \ i, j < N, \\ 1/\sqrt{N} & i = N, \ j \neq N \ \text{ or } \ j = N, \ i \neq N, \\ \beta/\sqrt{N} & i = j = N. \end{cases}$$

The numbers $\lambda$ and $\beta$ will be chosen later. Let $e_j = \{\alpha'_{j1}, \dots, \alpha'_{jN}\}$ $(j = 1, \dots, N)$ be the $j$th row of the matrix $A'_N$. Then

$$(e_k, e_r) = 2\lambda + (N-3)\lambda^2 + \frac{1}{N} \quad \text{for } k \neq r, \ k, r \neq N,$$

$$(e_k, e_N) = \frac{\beta}{N} + \frac{\lambda(N-2)}{\sqrt{N}} + \frac{1}{\sqrt{N}}.$$

We let $\lambda = -(1/(N-3))(1 + \sqrt{3/N})$ ($\lambda$ is a solution of the equation $2\lambda + (N-3)\lambda^2 + 1/N = 0$), and $\beta = -\sqrt{N}(\lambda(N-2) + 1)$. Then

$$\frac{\beta}{N} = -\left(-\frac{N-2}{N-3}\cdot\frac{1}{\sqrt{N}} - \frac{N-2}{N-3}\cdot\frac{\sqrt{3}}{N} + \frac{1}{\sqrt{N}}\right)$$
$$= -\left(-\frac{1}{\sqrt{N}(N-3)} - \frac{N-2}{N-3}\frac{\sqrt{3}}{N}\right),$$

from which it follows that $|\beta| < 6$.

Further,

$$1 < s_j \equiv \sqrt{(e_j, e_j)} = \sqrt{1 + (N-2)\lambda^2 + \frac{1}{N}} \leqslant \sqrt{1 + \frac{10}{N}} \quad \text{for } 1 \leqslant j < N,$$

$$\sqrt{1 - \frac{1}{N}} \leqslant s_N \equiv \sqrt{(e_N, e_N)} = \sqrt{\frac{N-1}{N} + \frac{\beta^2}{N}} \leqslant \sqrt{1 + \frac{36}{N}}.$$

Consequently, for $1 \leqslant j \leqslant N$ we have $|1/s_j - 1| < 100/N$.

From this last inequality it follows that the matrix $A_N$, having rows $e_j/s_j$, will satisfy all the requirements of the lemma.(2)

LEMMA 2 (see [3], Chapter V, Theorem 8.27). *If $P(x) = \sum_1^N c_k r_k(x)$ is a polynomial in the Rademacher system, then there exist absolute constants $C_1$ and $C_2$, $0 < C_1, C_2 < 1$, such that the measure*

$$\mu\left\{x : x \in [0, 1],\ |P(x)| > C_1 \sqrt{\int_0^1 P^2(x)\, dx}\right\} \geqslant C_2.$$

LEMMA 3. *Let $C \geqslant 1$. Then for each function $f(x)$ such that*

$$\int_0^1 |f(x)|\, dx \geqslant 1, \quad \int_0^1 f^2(x)\, dx \leqslant C^2,$$

*it follows that*

$$\mu\left\{x : |f(x)| \geqslant \frac{1}{4}\right\} \geqslant \frac{1}{8C^2}.$$

PROOF. We introduce the sets

$$Q = \left\{x : x \in [0, 1],\ |f(x)| \geqslant \frac{1}{4}\right\},$$

$$E = \{x : x \in [0, 1],\ |f(x)| \geqslant 2C^2\}.$$

Then $C^2 \geqslant \int_E |f(x)|^2\, dx \geqslant 2C^2 \int_E |f(x)|\, dx$, and consequently

$$\int_E |f(x)|\, dx \leqslant \frac{1}{2}; \quad \int_{[0,1]\setminus E} |f(x)|\, dx \geqslant \frac{1}{2}.$$

Hence

$$\frac{1}{2} \leqslant \int_{[0,1]\setminus E} |f(x)|\, dx \leqslant \frac{1}{4} + \mu Q \cdot 2C^2,$$

and therefore $\mu Q \geqslant 1/8C^2$. The lemma is proved.

LEMMA 4. *For each $N > 3$, define on the interval $[0, 1]$ the system of functions*(3) *$\{f_j^N(x)\}_{j=1}^N$, letting*

$$f_j^N(x) = \begin{cases} \dfrac{\sqrt{N}}{i-j} & \text{for } i \neq j,\ |i-j| < \sqrt{N}, \\ 0 & \text{for } i = j \text{ or } |i-j| \geqslant \sqrt{N}. \end{cases}$$

*for $x \in ((i-1)/(N+1), i/(N+1))$. This system of functions may be extended to the*

(2) Matrices which satisfy the conditions of Lemma 1 were earlier used by A. M. Olevskiĭ.

(3) This system resembles a system of functions considered by Men'šov (see [4], Chapter V, §3, Lemma 5.3.6).

*interval* [0, 3] *so that*

1) $\int_0^2 f_j^N(x) f_k^N(x)\,dx = \begin{cases} 0 & \text{for } j \neq k, \\ B & \text{for } j = k, \end{cases}$

*where the constant B does not depend upon N*;

2) *for* $x \in [1, 2]$ *the functions* $f_j^N(x)$ *are piecewise constant with intervals of constancy of length* $2^{-r}$ (*for some r*).

PROOF. It is easy to see that if we can extend the system $\{f_j^N(x)\}$, and if condition 1) can be satisfied, it is possible to modify it to fulfill condition 2).

By virtue of Schur's theorem (see [4], Chapter III, §2, Theorem 3.2.2), it suffices for the proof of 1) to show that (4)

$$q_N^2 \equiv \sup_{\sum_{j=1}^N a_j^2 = 1} \int_0^1 \left( \sum_{j=1}^N a_j f_j^N(x) \right)^2 dx \leq C.$$

Since the functions $f_j^N(x)$ are piecewise constant, we have

$$q_N = \frac{1}{\sqrt{N+1}} \sup_{\sum_{j=1}^N a_j^2 = \sum_{i=1}^{N+1} y_i^2 = 1} \sum_{i=1}^{N+1} y_i \sum_{j=1}^N a_j f_j^N\left(\frac{2i-1}{2(N+1)}\right)$$
$$= \frac{\sqrt{N}}{\sqrt{N+1}} \sup_{\sum_{j=1}^N a_j^2 = \sum_{i=1}^{N+1} y_i^2} \sum_{i=1}^{N+1} \sum_{j=1}^N a_j y_i \gamma_{i,j},$$

where

$$\gamma_{ij} = \frac{1}{\sqrt{N}} f_j^N\left(\frac{2i-1)}{2(N+1)}\right) = \begin{cases} 1/(i-j) & \text{for } 0 < |i-j| < \sqrt{N}, \\ 0 & \text{for } |i-j| \geq \sqrt{N} \text{ or } i = j. \end{cases}$$

Therefore

$$q_N \leq \sup_{\sum_{j=1}^\infty a_j^2 = \sum_{i=1}^\infty y_i^2 = 1} \sum_{i=1}^\infty \sum_{j=1}^\infty a_j y_i \delta_{i-j}, \tag{3}$$

where

$$\delta_\nu = \begin{cases} 1/\nu, & 0 < |\nu| < \sqrt{N}, \\ 0, & \nu = 0 \text{ or } |\nu| \geq \sqrt{N}. \end{cases}$$

It is well known that for such a sequence $\{\delta_\nu\}$ the right-hand side of (3) is bounded by a constant which is independent of $N$ (see, for example, [5], p. 223, Theorem 303).

The lemma is proved.

We note, for what follows, that for $1 \leqslant j, k < N - \sqrt{N}$

$$\int_{j/(N+1)}^1 f_j^N(x)\,dx = \int_{k/(N+1)}^1 f_k^N(x)\,dx > \frac{C \ln N}{\sqrt{N}}. \tag{4}$$

(4) In what follows, $C$ and $C'$ denote various absolute positive constants.

We pass to the proof of Theorem 1. We shall take the sequence $N_k = 2^k - 1$ ($k = 3, 4, \dots$), and for $2^k < n < 2^{k+1}$ and $x \in [0, 3]$ we let

$$\varphi_n(x) = \begin{cases} r_{s_k}(x) f_{n-2^k}^{N_k}(x) \sqrt{2B} & \text{for } x \in [0, 2], \\ r_n(x)/\sqrt{2} & \text{for } x \in [2, 3], \end{cases} \tag{5}$$

where the functions $f_j^N(x)$ are defined on the interval [0, 2] according to Lemma 4, and the sequence of Rademacher functions(5) $r_{s_k}(x)$ was chosen to be so sparse that the system of functions $\{\varphi_n(x)\}_{n=9, n \neq 2^k}^{\infty}$ was an orthogonal and normalized system on the interval [0, 3] (this is possible by virtue of condition 2) of Lemma 4). For $n = 8, 16, \dots, 2^s, \dots$ the functions $\varphi_n(x)$ are not defined.

We add to the system of functions $\{\varphi_n(x)\}_{n=9, n \neq 2^k}^{\infty}$ a set of functions $\{u_k(x)\}_{k=3}^{\infty}$ such that the system $\{\varphi_n(x)\} \cup \{u_k(x)\}$ forms a complete orthogonal and normalized system on the interval [0, 3].

Finally, we define the desired system $\{\psi_n(x)\}_{n=9}^{\infty}$. We shall construct the system $\{\psi_n(x)\}$ on the interval [0, 3], after which it will be possible to transform it to the interval [0, 1] by a similarity transformation.

Let $n = 2^k + \nu$, where $0 < \nu \leqslant 2^k$, $k = 3, 4, \dots$. We shall consider the matrix $A_{2^k} = \{\alpha_{ji}\}$, whose existence is guaranteed by Lemma 1, and for $x \in [0, 3]$ we let

$$\psi_n(x) = \sum_{r=1}^{2^k - 1} a_{\nu, r} \varphi_{2^k + r}(x) + a_{\nu, 2^k} u_k(x). \tag{6}$$

The resulting system $\{\psi_n(x)\}_{n=9}^{\infty}$ forms a c.o.n.s. on the interval [0, 3], since the matrices $A_{2^k}$ are orthonormal and it is possible to express the functions $\varphi_n(x)$ and $u_k(x)$ in terms of the functions $\psi_n(x)$. We shall show that our system satisfies all the requirements of the theorem.

We shall consider an arbitrary series of the form (1) with coefficients satisfying condition (2). We let

$$\beta_k = \sqrt{\sum_{n=2^k+1}^{2^{k+1}} c_n^2}, \quad d_k = \frac{1}{2^{k/2}} \left| \sum_{n=2^k+1}^{2^{k+1}} c_n \right|, \quad k = 3, 4, \dots .$$

By Cauchy's inequality, $\beta_k \geqslant d_k$. Let the absolute constant

$$C_0 = \frac{C_1 \sqrt{C_2}}{10b} \leqslant \frac{1}{10b}$$

(where $C_1$ and $C_2$ are the constants from Lemma 2, and $b$ is from Lemma 1). We divide all natural numbers $k > 2$ into two groups. In the first group, $S_1$, we place those numbers $k$ for which

$$C_0 \beta_k \leqslant d_k. \tag{7}$$

We place the remaining numbers in group $S_2$.

Two cases are possible:

(5) We assume that the Rademacher functions have been extended with period 1 from the interval [0, 1] to the entire real axis.

1) The series $\sum_{k\in S_1}\beta_k^2=\infty$. Then we shall show that series (1) does not converge on a set $E\subset[0,1]$ with $\mu E>0$.

2) The series $\sum_{k\in S_1}\beta_k^2<\infty$. Then by virtue of (2), the series $\sum_{k\in S_2}\beta_k^2=\infty$, and we shall show that the series (1) does not converge in measure on the interval [2, 3].

Let us consider case 1). Let the number $k\in S_1$. From properties of the matrices $A_{2^k}$ (see Lemma 1) and inequality (6), it follows at once that for $2^k<n<2^{k+1}$

$$\psi_n(x)=\varphi_n(x)+\Delta_n(x),$$

where

$$\int_0^3|\Delta_n(x)|\,dx\leqslant 3\sqrt{\int_0^3\Delta_n^2(x)\,dx}\leqslant\frac{C}{2^{k/2}}.$$

Therefore, for every measurable integral-valued function $N(x)$ such that $2^k+1\leqslant N(x)<2^{k+1}$, $x\in[0,3]$,

$$\int_0^3\left|\sum_{n=2^k+1}^{N(x)}c_n\psi_n(x)\right|dx=\int_0^3\left|\sum_{n=2^k+1}^{N(x)}c_n\varphi_n(x)+\sum_{n=2^k+1}^{N(x)}c_n\Delta_n(x)\right|dx$$
$$\geqslant\int_0^3\left|\sum_{n=2^k+1}^{N(x)}c_n\varphi_n(x)\right|dx-\sum_{n=2^k+1}^{2^{k+1}-1}|c_n|\int_0^3|\Delta_n(x)|\,dx \tag{8}$$
$$\geqslant\int_0^3\left|\sum_{n=2^k+1}^{N(x)}c_n\varphi_n(x)\right|dx-\frac{C}{2^{k/2}}\sum_{n=2^k+1}^{2^{k+1}}|c_n|\geqslant\int_0^3\left|\sum_{n=2^k+1}^{N(x)}c_n\varphi_n(x)\right|dx-C\beta_k.$$

Let $a(k)=2^k-[2^{k/2}]-2$ and

$$\widetilde N(x)=\begin{cases}i-1 & \text{for } x\in((i-1)2^{-k},i2^{-k}),\quad 2\leqslant i\leqslant a(k),\\ a(k) & \text{for } x\in((i-1)2^{-k},i2^{-k}),\quad a(k)<i\leqslant 2^k.\end{cases}$$

We estimate the number

$$q=\int_0^1\left|\sum_{n=2^k+1}^{2^k+\widetilde N(x)}c_n\varphi_n(x)\right|dx.$$

Let $\alpha_j=c_{2^k+j}$, $j=1,\dots,2^k$. It is clear that

$$\sum_{j=}^{2^k}\alpha_j^2=\beta_k^2,\qquad\frac{1}{2^{k/2}}\left|\sum_{j=1}^{2^k}\alpha_j\right|=d_k.$$

By virtue of (5),

$$q\sqrt{2B}=\int_0^1\left|\sum_{j=1}^{\widetilde N(x)}\alpha_jf_j^{N_k}(x)\right|dx, \tag{9}$$

where $N_k=2^k-1$, and the system of functions $\{f_j^{N_k}(x)\}$ is defined in Lemma 4.

From (9) it evidently follows that

$$q\sqrt{2B}\geqslant\left|\int_0^1\sum_{j=1}^{\widetilde N(x)}\alpha_jf_j^{N_k}(x)\,dx\right|.$$

By virtue of (4) and the definition of $N(x)$,

$$q\sqrt{2B} \geqslant \left| \int_0^1 \sum_{j=1}^{\widetilde{N(x)}} \alpha_j f_j^{N_k}(x)\,dx \right| = \left| \sum_{j=1}^{a(k)} \alpha_j \int_{j/2^k}^1 f_j^{N_k}(x)\,dx \right|$$
$$= \left| \left( \sum_{j=1}^{a(k)} \alpha_j \right) \int_{2^{-k}}^1 f_1^{N_k}(x)\,dx \right| \geqslant \frac{C \ln 2^k}{2^{k/2}} \left| \sum_{j=1}^{a(k)} \alpha_j \right|. \tag{10}$$

Further, since by Cauchy's inequality

$$\left| \sum_{j=a(k)+1}^{2^k} \alpha_j \right| \leqslant 2^{1+\frac{k}{4}} \sqrt{\sum_{j=1}^{2^k} \alpha_j^2} \leqslant 2\beta_k 2^{k/4},$$

for sufficiently large $k$ we obtain

$$\frac{1}{2^{k/2}} \left| \sum_{j=1}^{a(k)} \alpha_j \right| \geqslant \frac{1}{2^{k/2}} \left| \sum_{j=1}^{2^k} \alpha_j \right| - \frac{4\beta_k}{2^{k/4}} \geqslant d_k - \frac{4\beta_k}{2^{k/4}} \geqslant C\beta_k,$$

since $k \in S_1$ (see (7)). Therefore (see (9) and (10)) $q \geqslant C(\ln 2^k)\beta_k$. Recalling the definition of the number $q$, and taking (8) into account, we obtain

$$\int_0^1 \left| \sum_{n=2^k+1}^{2^k+\widetilde{N(x)}} c_n \psi_n(x) \right| dx \geqslant C(\ln 2^k)\beta_k \geqslant Ck\beta_k. \tag{11}$$

On the other hand, by virtue of the well-known Men′šov-Rademacher lemma (see [4], Chapter V, §3, Lemma 5.3.4)

$$\sqrt{\int_0^1 \left( \sum_{2^k+1}^{\widetilde{N}(x)+2^k} c_n \psi_n(x) \right)^2 dx} \leqslant C'(\ln 2^k)\beta_k \leqslant C'k\beta_k, \tag{12}$$

and it remains to apply Lemma 3 to the function

$$\frac{1}{Ck\beta_k} \left( \sum_{n=2^k+1}^{N(x)+2^k} c_n \psi_n(x) \right),$$

in order to obtain (see (11) and (12)) that

$$\mu \left\{ x \in [0, 1] : \left| \sum_{n=2^k+1}^{2^k+\widetilde{N(x)}} c_n \psi_n(x) \right| \geqslant Ck\beta_k \right\} \geqslant C. \tag{13}$$

The estimate (13) is true for every sufficiently large $k \in S_1$. Since we assumed that $\sum_{k \in S_1} \beta_k^2 = \infty$, we have

$$\varlimsup_{\substack{k \to \infty \\ k \in S_1}} k\beta_k = \infty,$$

and from (13) it follows that the series (1) diverges on a set $E$ with $\mu E > 0$, $E \subset [0, 1]$, in case 1).

Let us consider case 2).

We take a $k \in S_2$. By virtue of (6) and the orthonormality of the matrices $A_{2^k}$ we have

$$\sum_{n=2^k+1}^{2^{k+1}} c_n\psi_n(x) = \sum_{n=2^k+1}^{2^{k+1}-1} \gamma_n\varphi_n(x) + \omega_k u_k(x),$$

where

$$\beta_k^2 = \sum_{n=2^k+1}^{2^{k+1}} c_n^2 = \omega_k^2 + \sum_{n=2^k+1}^{2^{k+1}-1} \gamma_n^2.$$

Now from properties of the matrices $A_{2^k}$ it immediately follows that

$$\omega_k \leqslant \frac{b}{2^{k/2}}\left|\sum_{n=2^k+1}^{2^{k+1}} c_n\right| + \frac{2b}{2^{k/2}}\max_{2^k<n\leqslant 2^{k+1}}|c_n| \leqslant bd_k + 2^{1-\frac{k}{2}}b\beta_k,$$

$$\sum_{n=2^k+1}^{2^{k+1}-1} \gamma_n^2 = \beta_k^2 - \omega_k^2 \geqslant \beta_k^2 - (bd_k + 2^{1-\frac{k}{2}}b\beta_k)^2.$$

From these estimates, and since for $k \in S_2$

$$bd_k \leqslant \frac{C_1\sqrt{C_2}}{10}\beta_k \leqslant \frac{1}{10}\beta_k,$$

for sufficiently large $k \in S_2$ we have

$$\sum_{n=2^k+1}^{2^{k+1}-1} \gamma_n^2 \geqslant \frac{1}{2}\beta_k^2, \quad \left(\sum_{n=2^k+1}^{2^{k+1}-1} \gamma_n^2\right)\frac{C_1^2C_2}{20Q} \geqslant \omega_k^2. \tag{14}$$

We take a sequence of natural numbers $\{m_s\}_1^\infty$ such that (14) is true for $k \geqslant m_1$ ($k \in S_2$) and

$$\sum_{k=m_s}^{m_{s+1}-1} \beta_k^2 = \sum_{n=2^{m_s}+1}^{2^{m_s+1}} c_n^2 \geqslant s. \tag{15}$$

Let us consider the sum

$$\sum_{n=2^{m_s}+1}^{2^{m_s+1}} c_n\psi_n(x) = I_s'(x) + I_s''(x), \tag{16}$$

where

$$I_s'(x) = \sum_{k\in S_1\cap[m_s, m_{s+1})} \sum_{n=2^k+1}^{2^{k+1}} c_n\psi_n(x),$$

$$I_s''(x) = \sum_{k\in S_2\cap[m_s, m_{s+1})} \sum_{n=2^k+1}^{2^{k+1}} c_n\psi_n(x).$$

Since $\sum_{k\in S_1}\beta_k^2 < \infty$, we have

$$\sum_{s=1}^{\infty}\int_0^3 (I_s'(x))^2\,dx \leqslant \sum_{k\in S_1}\beta_k^2 \leqslant C < \infty. \tag{17}$$

Moreover,

$$I''_s(x) = \sum_{k\in S_s\cap[m_s,m_{s+1})} \sum_{n=2^k+1}^{2^{k+1}} c_n\psi_n(x) = \sum_{k\in S_s\cap[m_s,m_{s+1})} \left(\sum_{n=2^k+1}^{2^{k+1}-1} \gamma_n\varphi_n(x) + \omega_k u_k(x)\right).$$

Here by virtue of (14) and (15)

$$J_s \equiv \sum_{k\in S_s\cap[m_s,m_{s+1})} \left(\sum_{n=2^k+1}^{2^{k+1}-1} \gamma_n^2\right) \geqslant \left(\frac{C_1^2C_2}{20}\right)^{-1} \sum_{k\in S_s\cap[m_s,m_{s+1})} \omega_k^2, \quad J_s \geqslant \frac{s}{2}.$$

By virtue of (5), $\varphi_n(x) = r_n(x)/\sqrt{2}$ for $x \in [2, 3]$ and $n \neq 2^k$ ($k = 1, 2, \dots$), and thus

$$I''_s(x) = P_s(x) + \sum_{k\in S_s\cap[m_s,m_{s+1})} \omega_k u_k(x) \quad (x \in [2, 3]),$$

where $P_s(x)$ is a polynomial in the Rademacher system and

$$\int_2^3 P_s^2(x)\,dx = \frac{1}{2} J_s \geqslant \frac{s}{4}, \qquad \sum_{k\in S_s\cap[m_s,m_{s+1})} \omega_k^2 \leqslant J_s \frac{C_1^2C_2}{20}.$$

Since the functions $u_k(x)$ are orthogonal and $\int_0^3 u_k^2(x)\,dx = 1$, we have

$$\int_0^3 \left(\sum_{k\in S_s\cap[m_s,m_{s+1})} \omega_k u_k(x)\right)^2 dx = \sum_{k\in S_s\cap[m_s,m_{s+1})} \omega_k^2 \leqslant J_s \frac{C_1^2C_2}{20}.$$

Consequently by Čebyšev's inequality we obtain

$$\mu\left\{x \in [2, 3]: \left|\sum_{k\in S_s\cap[m_s,m_{s+1})} \omega_k u_k(x)\right| \geqslant \frac{\sqrt{J_s}\,C_1}{3}\right\} \leqslant \frac{C_2}{2}. \tag{18}$$

Furthermore, by virtue of Lemma 2

$$\mu\left\{x \in [2, 3]: |P_s(x)| \geqslant \frac{\sqrt{J_s}}{\sqrt{2}} C_1\right\} \geqslant C_2. \tag{19}$$

From (18) and (19) it follows that

$$\mu\left\{x \in [2, 3]: |I''_s(x)| \geqslant \frac{\sqrt{J_s}}{6} C_1 \geqslant \frac{\sqrt{s}}{10} C_1\right\} \geqslant \frac{C_2}{2}.$$

Hence, taking (17) into account (and see also (16)), we immediately obtain that in case 2) the series (1) does not converge in measure on the interval [2, 3]. The theorem is proved.

Remark 1. In proving the theorem, we completed a system $\{\varphi_n(x)\}$ of arbitrary form, adding to it functions $\{u_k(x)\}$. Applying the fact that the Walsh system is a system of convergence, it is easy to select functions $u_k(x)$ such that the convergence of the series (1) follows from the condition $\sum_{n=9}^{\infty} c_n^2 < \infty$. Theorem 1 with this remark gives an affirmative answer to a problem of Ul'janov (see [6], §3.4).

Remark 2. Theorem 1 remains true for a wide class of summation methods, i.e. it is possible to construct a system $\{\psi_n(x)\}$ such that the series (1) is not summable on sets of positive measure if its coefficients satisfy condition (2).

Received 20/MAY/75

## BIBLIOGRAPHY

1. A. A. Talaljan, *The representation of measurable functions by series*, Uspehi Mat. Nauk **15** (1960), no. 5 (95), 77–141 = Russian Math. Surveys **15** (1960), no. 5, 75–136. MR **23** #A2704.

2. P. L. Ul′janov, *Representation of functions by series, and the classes* $\varphi(L)$, Uspehi Mat. Nauk **27** (1972), no. 2 (164), 3–52 = Russian Math. Surveys **27** (1972), 1–54.

3. A. Zygmund, *Trigonometric series*, 2nd rev. ed., Vol. I, Cambridge Univ. Press, New York, 1959. MR **21** #6498.

4. S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, PWN, Warsaw, 1935; reprint, Chelsea, New York, 1951.

5. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, New York, 1934.

6. P. L. Ul′janov, *Solved and unsolved problems in the theory of trigonometric and orthogonal series*, Uspehi Mat. Nauk **19** (1964), no. 1 (115), 3–69 = Russian Math. Surveys **19** (1964), no. 1, 1–62. MR **28** #4294.

Translated by C. V. and G. A. COMISKY