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ON UNCONDITIONAL CONVERGENCE IN THE SPACE L_1

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Abstract. The paper contains a proof of the following

Theorem. *Suppose* $\Sigma_1^\infty f_k(x)$ *converges unconditionally in* $L_1[0, 1]$. *Then for any* $\epsilon > 0$ *there exists a set* $E_\epsilon \subset [0, 1]$, $\mu E_\epsilon > 1 - \epsilon$, *such that* $\Sigma_1^\infty f_k(x)$ *converges unconditionally in* $L_q(E_\epsilon)$ *for every* $q < 2$.

This result is obtained as a corollary of a more general theorem.

Bibliography: 2 items.

Let X be a real Banach space. Then the series

$$\sum_{k=1}^{\infty} x_k \quad (x_k \in X,\ k = 1, 2, \ldots) \tag{1}$$

is said to be *unconditionally convergent in* X if any series derived from it by a rearrangement of terms converges in the norm of X. The definition of unconditional convergence in measure of the functional series

$$\sum_{i=1}^{\infty} f_i(x), \tag{2}$$

where the $f_i(x)$ are measurable functions on a set E, is completely analogous.

A well-known theorem of Orlicz (see [2], subdivision [168]) states that the series (1) converges unconditionally in X if and only if every series of the form

$$\sum_{k=1}^{\infty} \varepsilon_k x_k,$$

where $\epsilon_k = \pm 1$, converges in the norm of X.

In connection with the investigations of general functional series, the following problem was posed by E. M. Nikišin in a seminar on the theory of functions at Moscow University.

Suppose $\{f_i\}_1^\infty$ is a sequence of measurable functions on $[0, 1]$ and the series (2) converges unconditionally in Lebesgue measure on $[0, 1]$. Does there exist for any

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$\epsilon > 0$ a set $E_\epsilon \subset [0, 1]$ such that $mE_\epsilon > 1 - \epsilon$ and the series (2) converges unconditionally in $L_2(E_\epsilon)$?

Nikišin encountered this problem as a limiting case in a certain sense of a general theorem of his (see [1], Theorem 7).

The present paper contains a partial solution of this problem, viz. a proof of the following fact.

Theorem 1. *Suppose the series* (2) *converges unconditionally in the space* $L_1[0, 1]$. *Then for any* $\epsilon > 0$ *there exists a set* $E_\epsilon \subset [0, 1]$, $mE_\epsilon > 1 - \epsilon$, *such that the series* (2) *converges unconditionally in the space* $L_q(E_\epsilon)$ *for every* $q < 2$.

It follows from the results of Nikišin that unconditional convergence in measure of the series (2) implies its unconditional convergence in the (no longer Banach) space L_p for any $p < 1$.

Theorem 1 is a corollary of Theorem 2 proved below, which is of independent interest.

Let $\gamma = \{\gamma_i\}_1^n$ denote an arbitrary vector of the n-dimensional space $\mathbf{R}^n$.

Suppose given a system of functions $\{e_i\}_1^n$ with $e_i \in L_1[0, 1]$. For any $p \in [1, \infty]$ we put

$$B_p \equiv B_p(\{e_i\}_{i=1}^n) = \sup_{\|\gamma\|_{l_p}=1} \left\| \sum_{i=1}^n \gamma_i e_i \right\|_{L_1}. \tag{3}$$

It is well known that

$$B_\infty = \sup_{\gamma_i=\pm 1} \left\| \sum_{i=1}^n \gamma_i e_i \right\|_{L_1}.$$

Since the norm in the space l_p of $\gamma = \{\gamma_i\}_1^n$, where $\gamma_i = \pm 1$, is equal to $n^{1/p}$, we have

$$B_\infty = \sup_{\gamma_i=\pm 1} \left\| \sum_{i=1}^n \gamma_i e_i \right\|_{L_1} \leqslant \sup_{\|\gamma\|_{l_p}=n^{1/p}} \left\| \sum_{i=1}^n \gamma_i e_i \right\|_{L_1} \leqslant B_p n^{1/p},$$

which implies

$$B_p \geqslant \frac{B_\infty}{n^{1/p}}. \tag{4}$$

The reverse of inequality (4) is valid in the following sense.

Theorem 2. *Let* $\{e_i\}_1^n$ *be a system of* n *functions* $e_i \in L_1[0, 1]$. *Then for any* $p > 2$ *and* $\epsilon > 0$ *there exist constants* $c_{\epsilon,p}$ *and a subsystem* $\{e_{i_\nu}\}_{\nu=1}^s$ *such that* $s > n(1 - \epsilon)$ *and*

$$B_p(\{e_{i_\nu}\}) \leqslant \frac{B_\infty(\{e_i\}_{i=1}^n)}{n^{1/p}} \cdot c_{\varepsilon,p}. \tag{5}$$

We will first prove some lemmas.

Lemma 1. *For any* $p > 2$ *there exists a number* d *such that the following inequality* *is valid for any natural number* $n \geq d$ *and system* $\{e_i(x)\}_1^n$ *of functions from* $L_1[0, 1]$ *satisfying the condition* $\|e_i\|_{L_1} \geq 1$ $(1 \leq i \leq n)$:

$$\sup_{\gamma_i = \pm 1} \left\| \sum_{i=1}^n \gamma_i e_i \right\|_{L_1} > 2n^{1/p}.$$

Proof. Let $\{r_i(t)\}_1^\infty$ be Rademacher's system (see [2], subdivision [225]). Then

$$\sup_{\gamma_i = \pm 1} \left\| \sum_{i=1}^n \gamma_i e_i \right\|_{L_1} = \sup_{t \in [0,1]} \left\| \sum_{i=1}^n r_i(t) e_i(x) \right\|_{L_1}$$
$$\geqslant \int_0^1 \int_0^1 \left| \sum_{i=1}^n r_i(t) e_i(x) \right| dxdt = \int_0^1 \int_0^1 \left| \sum_{i=1}^n r_i(t) e_i(x) \right| dtdx.$$

Applying Khintchine's inequality (see [2], subdivision [456]), we get

$$\int_0^1 \int_0^1 \left| \sum_{i=1}^n r_i(t) e_i(x) \right| dtdx \geqslant \frac{1}{8} \int_0^1 \sqrt{\sum_{i=1}^n e_i^2(x)}\, dx$$
$$\geqslant \frac{1}{8} \int_0^1 \frac{\sum_{i=1}^n |e_i(x)|}{\sqrt{n}} dx \geqslant \frac{1}{8} \frac{n}{\sqrt{n}} > 2n^{1/p},$$

if $n \geq d$, where $d = d(p)$ is a sufficiently large number. Lemma 1 is proved.

The assertion of Lemma 1 is well known; we will use it in precisely this, which is not the strongest, form.

We further require some notation and definitions. We introduce in $\mathbf{R}^n$ the system of vectors $U_\infty = \{u\} = \{$all vectors, except the zero vector, whose coordinates are equal to $0, \pm 1\}$.

The system of vectors U_∞ is normalized in l_p for each $p \geq 1$; the resultant system of vectors is denoted by U_p. By the *support* of a vector $u = \{u_i\}_1^n \in U_p$ is meant the set of numbers i such that $u_i \neq 0$. We denote the support of a vector $u \in U_p$ by $N(u)$, and the number of elements in $N(u)$ by $|N(u)|$. Clearly, if $|N(u)| = k$, then

$$u_i = 0, \quad \text{if} \quad i \notin N(u); \quad |u_i| = \frac{1}{k^{1/p}}, \quad \text{if} \quad i \in N(u). \tag{6}$$

Lemma 2. *Suppose given a system* $\{e_i\}_1^n$ *of functions* $e_i \in L_1[0, 1]$. *Then for any numbers* $p > 2$ *and* $\epsilon > 0$ *there exist a number* $c_{\epsilon,p}$ *and a set* $\{e_{j_k}\}_{k=1}^s \subset \{e_i\}_1^n$ *such that* $s > n(1 - \epsilon)$ *and for any vector* $\gamma = \{\gamma_i\}_1^n \in U_p$

$$\left\| \sum_{k=1}^s \gamma_{j_k} e_{j_k} \right\|_{L_1[0,1]} \leqslant \frac{c_{\varepsilon,p} B_\infty(\{e_i\}_{i=1}^n)}{n^{1/p}}.$$

Proof. We will make use of the following property of the system U_p.

Suppose $\gamma \in U_p$. Suppose, further, $\gamma_1, \ldots, \gamma_t$ are vectors from U_p,

$$s_i = |N(\gamma_i)|, \quad i = 1, \ldots, t,$$

and $N(\gamma_i) \cap N(\gamma_j) = \varnothing$ if $i, j \in [1, t]$, $i \neq j$. Then it is easily verified that for any system $\{\epsilon_i\}_1^t$ of numbers $\epsilon_i = \pm 1$ the vectors

$$\frac{\sum_{i=1}^{t} \varepsilon_i \gamma_i s_i^{1/p}}{(s_1 + s_2 + \ldots + s_t)^{1/p}} \in U_p \tag{6'}$$

and their supports contain $s_1 + \cdots + s_t$ elements.

Suppose $\gamma = \{\gamma_i\}_1^n \in U_p$ and $N(\gamma)$ contains $k \geq \epsilon n/4d$ elements (d is the number in Lemma 1). We have

$$\left\| \sum_{i=1}^{n} \gamma_i e_i \right\|_{L_1} \leqslant B_\infty \|\gamma\|_{l_\infty}. \tag{7}$$

But in view of (6) either $\gamma_i = 0$ or $|\gamma_i| = k^{-1/p}$, and hence

$$\|\gamma\|_{l_\infty} = \frac{1}{k^{1/p}} < \frac{(4d)^{1/p}}{(\varepsilon n)^{1/p}}. \tag{8}$$

Combining (7) and (8), we get that

$$\left\| \sum_{i=1}^{n} \gamma_i e_i \right\|_{L_1} \leqslant \frac{B_\infty (4d)^{1/p}}{(\varepsilon n)^{1/p}} \leqslant \frac{B_\infty}{n^{1/p}} \cdot \left(\frac{4d}{\varepsilon}\right)^{1/p}. \tag{9}$$

Let us prove that the lemma is valid for $c_{\epsilon, p} = (4d/\epsilon)^{1/p}$. Inequality (9) can be formulated as follows.

Suppose $\gamma = \{\gamma_i\}_{i=1}^n \in U_p$ and

$$\left\| \sum_{i=1}^{n} \gamma_i e_i \right\|_{L_1} > M \equiv \frac{B_\infty}{n^{1/p}} \cdot \left(\frac{4d}{\varepsilon}\right)^{1/p}.$$

Then

$$|N(\bar{\gamma})| < \frac{n\varepsilon}{4d}. \tag{10}$$

We must prove that there exists a set of numbers $A = \{i_j\}_1^m$ such that $m \leq n\epsilon$ and for any $\gamma_0 = \{\gamma_i\}_1^n \in U_p$ satisfying the condition $N(\gamma_0) \cap A = \varnothing$ we have

$$\left\| \sum_{i=1}^{n} \gamma_i e_i \right\|_{L_1} \leqslant M. \tag{11}$$

Let us construct a system $B = \{\gamma_i\}_1^k$ of vectors $\gamma_i \in U_p$ and a system $\{z_i\}_1^k$ of functions $z_i \in L_1$ as follows.

Suppose there exists a vector $\beta = \{\beta_i\}_1^n \in U_p$ such that $\|\Sigma_{i=1}^n \beta_i e_i\|_{L_1} > M$. (If such a vector does not exist, the lemma is valid for $A = \varnothing$.) In view of (10) it is clear that $N(\beta)$ contains less than $n\epsilon/4d$ elements. We put $\gamma_1 = \beta$ and $z_1 = \Sigma_1^n \beta_i e_i$.

Suppose the vectors $\gamma_1, \ldots, \gamma_\nu$ and the elements $z_1, \ldots, z_\nu$ have been constructed. Then, if there exists a vector $\beta' = \{\beta_i'\}_1^n \in U_p$ such that $\|\Sigma_1^n \beta_i' e_i\| > M$ and $N(\beta') \cup$

$(\bigcup_1^\nu N(\gamma_i)) = \varnothing$, we put

$$\gamma_{\nu+1} = \beta', \quad z_{\nu+1} = \sum_{i=1}^{n} \beta_i' e_i. \tag{12}$$

(If such a vector β' does not exist, the construction is complete.)

According to the construction of the system $B = \{\gamma_i\}_1^k$, when $\gamma_0 = \{\gamma_i\}_1^n \in U_p$ and $N(\gamma_0) \cap (\bigcup_1^k N(\gamma_i)) = \varnothing$ we have

$$\left\| \sum_{j=1}^{n} \gamma_j e_j \right\| \leqslant M,$$

i.e. inequality (11) holds if $A = \bigcup_1^k N(\gamma_i)$. It remains to prove that the number of elements in A is not greater than $n\epsilon$, i.e.

$$\sum_{i=1}^{k} |N(\gamma_i)| < n\varepsilon. \tag{13}$$

The inequality $1 \le |N(\gamma_i)| < n\epsilon/4d$ holds for any $i \in [1, k]$ in view of (10) and the construction of the system $\{\gamma_i\}$. Suppose $2^r \le n\epsilon/4d < 2^{r+1}$. We divide the interval $[1, n\epsilon/4d]$ into the intervals $[2^{i-1}, 2^i - 1]$, $1 \le i \le r$, and $[2^r, n\epsilon/4d]$.

Let $\alpha_i(B)$ denote the group of vectors $\{\gamma_j\} \subset B$ such that $2^{i-1} \le |N(\gamma_j)| \le 2^i - 1$ $(1 \le i \le r)$, and let $\alpha_{r+1}(B)$ denote the group of vectors $\{\gamma_j\} \subset B$ such that $2^r \le |N(\gamma_j)| < n\epsilon/4d$.

We will successively alter the system $B = \{\gamma_i\}_1^k$, obtaining systems $B^1, B^2, \dots$, etc. Each of the systems B^l (their construction is described below) consists of vectors belonging to U_p and having the following properties:

$$\begin{aligned}
&1)\ \sum_{\gamma\in B} |N(\gamma)| = \sum_{\gamma \in B^l} |N(\gamma)|;\\
&2)\ \text{if}\quad \gamma_1 \in B^l,\ \gamma_2 \in B^l,\ \gamma_1 \neq \gamma_2,\ \text{then } N(\gamma_1) \cap N(\gamma_2) = \varnothing;\\
&3)\ \text{if}\quad \gamma = \{\gamma_i\}_{i=1}^n \in B^l,\ \text{then} \left\| \sum_{i=1}^{n} \gamma_i e_i \right\| > M.
\end{aligned} \tag{14}$$

It therefore suffices for the proof of (13) (by virtue of property 1) of (14)) to prove it for some system B^l.

We will successively consider the groups α_i, $i = 1, \dots, r+1$.

Suppose the group $\alpha_1(B)$ contains less then d elements (d is the number in Lemma 1). Then system B is not altered and we proceed to the group $\alpha_2(B)$.

Suppose the group $\alpha_1(B)$ contains at least d elements: $\gamma_{i_1}, \dots, \gamma_{i_t}$, $t \ge d$, and let $S_\mu = |N(\gamma_{i_\mu})|$ $(1 \le \mu \le t)$. Then, according to (6′), the vectors

$$\frac{\varepsilon_1 \gamma_{i_1} s_1^{1/p} + \dots + \varepsilon_t \gamma_{i_t} s_t^{1/p}}{(s_1 + \dots + s_t)^{1/p}}$$

belong to U_p under any choice of the system $\{\epsilon_r\}_1^t$, $\epsilon_r = \pm 1$, $1 \le r \le t$, and hence the elements

$$\frac{\varepsilon_1 z_{i_1} s_1^{1/p} + \dots + \varepsilon_t z_{i_t} s_t^{1/p}}{(s_1 + \dots + s_t)^{1/p}} \qquad (15)$$

of the space $L_1[0, 1]$ have the form $\Sigma_1^n \gamma_i e_i$, $\{\gamma_i\} = \gamma \in U_p$, under any choice of the system $\{\epsilon_r\}_1^t$.

By construction, we have $\|z_{i_j}\| > M$ for $j = 1, \dots, t$ (see (12)). Lemma 1 therefore implies the existence of a collection $\{\epsilon_r\}_1^t$ such that

$$\left\| \frac{\sum_{r=1}^{t} \varepsilon_r z_{i_r} s_r^{1/p}}{\left(\sum_{r=1}^{t} s_r\right)^{1/p}} \right\| > \frac{2Mt^{1/p} (\min_{1 \leqslant r \leqslant t} s_r^{1/p})}{\left(\sum_{r=1}^{t} s_r\right)^{1/p}} > M.$$

(We have made use of the fact that all of the vectors γ_{i_r} belong to the group $\alpha_1(B)$ and hence that $s_i/s_j \leq 2$ for $i, j \in [1, t]$.)

We replace in system B all of the vectors of the group $\alpha_1(B)$ by the vector

$$\frac{\sum_{r=1}^{t} \varepsilon_r \gamma_{i_r} s_r^{1/p}}{\left(\sum_{r=1}^{t} s_r\right)^{1/p}} = \mathbf{v}, \quad N(\mathbf{v}) = \bigcup_{r=1}^{t} N(\gamma_{i_r}), \quad \mathbf{v} \in U_p. \qquad (16)$$

The resultant system is denoted by B^1. Conditions (14) are satisfied by virtue of (16). It is important that $\alpha_1(B^1)$ contain less than d elements.

We proceed to the group $\alpha_2(B^1)$. As with the group $\alpha_1(B)$, if the group $\alpha_2(B^1)$ contains less than d elements, we alter nothing in system B^1, while if $\alpha_2(B^1)$ contains at least d elements, we replace in system B^1 all of the vectors of α_2 by a linear combination of them such that (14) is satisfied, and then proceed to the group $\alpha_3(B^2)$.

At the $(r + 1)$th step we obtain a system B^{r+1} such that each group α_i contains less than d elements, i.e.

$$\sum_{\gamma \in B^{r+1}} |N(\gamma)| = \sum_{i=1}^{r+1} \sum_{\gamma \in \alpha_i} |N(\gamma)| < d \sum_{i=1}^{r+1} 2^i \leqslant d \cdot 2^{r+2} < 4d \cdot \frac{n\varepsilon}{4d} = n\varepsilon,$$

which proves (13) and, consequently, the lemma.

Lemma 3. *Let S_p^n denote the convex hull of the system U_p of vectors in $\mathbf{R}^n$ $(p \geq 1)$, and let $\mathbf{z}_0$ belong to the boundary of S_p^n. Then for any $q > p$ there exists a constant $c_{pq} > 0$, not depending on n, such that*

$$\|\mathbf{z}_0\|_{l_q} \geqslant \frac{c_{pq}}{n^{1/p-1/q}}.\text{(1)}$$

Proof. It suffices to prove that for any $q > p$ there exists a $c'_{pq} > 0$ such that for any vector $\mathbf{z}_0 \in \mathbf{R}^n$ there exists a decomposition $\mathbf{z}_0 = \Sigma_1^n \lambda_i \gamma_i$, where

(1) If $\mathbf{z}_0$ belongs to the boundary of S_p^n, by reasoning as in the proof of Lemma 3 we get that $|\mathbf{z}_0|_{l_p} > (\ln n)^{-\alpha} \cdot c_p$ (for $n > 2$), where α depends only on p (e.g. $\alpha = ½$ if $p = 2$).

1) $\lambda_i > 0$ for any i,

2) $\sum_{i=1}^{n} \lambda_i \leqslant c'_{pq} \cdot n^{1/p-1/q} \cdot \|\mathbf{z}_0\|_{l_q}$, and (17)

3) $\gamma_i \in U_p$ for any $i \in [1, n]$.

For suppose (17) is satisfied, while z_0 belongs to the boundary of S_p^n and

$$\|\mathbf{z}_0\|_{l_q} < \frac{1}{c'_{pq}} \cdot \frac{1}{n^{1/p-1/q}}$$

(i.e. $1/c'_{pq}$ has been taken as c_{pq}). Then $z_0 = \Sigma_1^n \lambda_i \gamma_i$, where by virtue of (17)

$$\text{a) } \lambda_i > 0, \quad \text{b) } \sum_{i=1}^{n} \lambda_i < 1, \quad \text{c) } \gamma_i \in U_p.$$

But this contradicts the fact that z_0 belongs to the boundary of S_p^n.

Let us prove (17). Let $z = \{z_i\}_1^n$. Since the set U_p is the same in any orthant of $\mathbf{R}^n$, it can be assumed that $z_i > 0$, $1 \le i \le n$, and $z_1 \ge z_2 \ge \cdots \ge z_n$. Then $z = z_n \alpha_n + (z_{n-1} - z_n)\alpha_{n-1} + \cdots + (z_1 - z_2)\alpha_1$, where

$$\alpha_n = (\underbrace{1, \ldots, 1}_{n}), \quad \alpha_{n-1} = \{\underbrace{1, \ldots, 1}_{n-1}, 0\}, \ldots, \quad \alpha_1 = (1, \underbrace{0, \ldots, 0}_{n-1}).$$

The vectors $\gamma_i = \alpha_i / i^{1/p}$ clearly belong to U_p for $1 \le i \le n$. Thus

$$\mathbf{z} = z_n \cdot n^{1/p}\gamma_n + (z_{n-1} - z_n)(n-1)^{1/p}\gamma_{n-1} + \ldots + (z_1 - z_2) \cdot 1^{1/p}\gamma_1.$$

Let us prove that conditions (17) are satisfied for this decomposition. Parts 1) and 3) are obviously satisfied. Let us prove that part 2) is also satisfied, i.e. that

$$\Omega = \sum_{i=1}^{n} \lambda_i = z_n n^{1/p} + (z_{n-1} - z_n)(n-1)^{1/p} + \ldots + (z_1 - z_2)\, 1^{1/p} \leqslant c'_{p,q} n^{1/p-1/q} \cdot \|\mathbf{z}\|_{l_q}.$$

Transforming, we get

$$\Omega = \sum_{i=1}^{n} z_i \left(i^{1/p} - (i-1)^{1/p}\right),$$

i.e.

$$\Omega \leqslant c_p \cdot \sum_{i=1}^{n} \frac{z_i}{i^{1-\frac{1}{p}}}. \tag{18}$$

Let $q' = 1/(1 - 1/q)$ and $p' = 1/(1 - 1/p)$. We apply Hölder's inequality with exponents q and q' to the sum of (18):

$$\Omega \leqslant c_p \cdot \left(\sum_{i=1}^{n} z_i^q\right)^{1/q} \cdot \left(\sum_{i=1}^{n} \frac{1}{i^{(1/p')q'}}\right)^{1/q'}.$$

Since it follows from the inequality $p < q$ that $p' > q'$, we have

$$\sum_{i=1}^{n} \frac{1}{i^{q'/p'}} \leqslant c''_{p,q} \cdot n^{1-\frac{q'}{p'}} \quad \text{and} \quad \Omega \leqslant c_p \cdot c'_{p,q} \cdot \|\mathbf{z}\|_{l_q} \cdot n^{\left(1-\frac{q'}{p'}\right)\frac{1}{q'}},$$

But

$$n^{\left(1-\frac{q'}{p'}\right)\frac{1}{q'}}=n^{\frac{1}{q'}-\frac{1}{p'}}=n^{\frac{1}{p}-\frac{1}{q}}.$$

Thus property (17) and, with it, Lemma 3 are proved.

Proof of Theorem 2. Suppose $p > 2$ and suppose given a system $\{e_i\}_1^n$ of functions $e_i \in L_1[0, 1]$. We take a p' such that $2 < p' < p$. According to Lemma 2, for any $\epsilon > 0$ it is possible to choose from the system $\{e_i\}_1^n$ a subsystem $\{e_{i_k}\}_{k=1}^s$ such that $s > n(1-\epsilon)$ and the following inequality is valid for any vector $\gamma = \{\gamma_j\}_1^n \in U_{p'}$:

$$\left\|\sum_{k=1}^{s} \gamma_{i_k} e_{i_k}\right\|_{L_1} \leqslant c_{\varepsilon,p'} \cdot \frac{B_\infty(\{e_i\}_{i=1}^n)}{n^{1/p'}}. \tag{19}$$

Then, if $x = \{x_j\}_1^n \in S_p^n$, we have $x = \sum_1^m \alpha_i \gamma_i$, where 1) $\gamma_i \in U_{p'}$ and 2) $\sum_1^m |\alpha_i| \le 1$ (this follows from the properties of convex hulls), and hence

$$\left\|\sum_{k=1}^{s} x_{i_k} \cdot e_{i_k}\right\|_{L_1} \leqslant \sup_{\gamma\in U_{p'}} \left\|\sum_{k=1}^{s} \gamma_{i_k} e_{i_k}\right\|_{L_1} \leqslant c_{\varepsilon,p'} \cdot \frac{B_\infty(\{e_i\}_{i=1}^n)}{n^{1/p'}}.$$

The latter inequality is satisfied by virtue of (19). But, according to Lemma 3, $S_{p'}^n$ contains a ball of the space $l_p(\mathbf{R}^n)$ of radius $c_{p,p'}/n^{1/p'-1/p}$, so that for any $y = \{y_j\}_1^n$, $\|y\|_{l_p} = 1$, we have

$$\left\|\sum_{k=1}^{s} y_{i_k} e_{i_k}\right\|_{L_1} \leqslant \frac{c_{\varepsilon,p'} B_\infty(\{e_i\}_{i=1}^n)}{c_{p',p} n^{1/p'}} \cdot n^{1/p'-1/p}$$

and

$$B_p(\{e_{i_k}\}) = \sup_{\|y\|_{l_p}=1} \left\|\sum_{k=1}^{s} y_{i_k} e_{i_k}\right\| \leqslant c'_{\varepsilon,p} \frac{B_\infty(\{e_j\}_{j=1}^n)}{n^{1/p}},$$

which proves (5). Theorem 2 is proved.

Remark. Theorem 2 admits a generalization. We will say that a Banach space E satisfies the *α_p-condition* ($p > 1$) if for any $q > p$ there exists a number d such that for any natural number $n > d$ and system $\{e_i\}_1^n$ of elements $e_i \in E$ with $\|e_i\|_E \geq 1$ ($i = 1, \ldots, n$) we have

$$\sup_{\gamma_i=\pm 1} \left\|\sum_{i=1}^{n} \gamma_i e_i\right\|_E \geqslant n^{1/q}.$$

Suppose given a system $\{e_i\}_1^n$ of elements $e_i \in E$. Analogously to the preceding, we put

$$B_q(\{e_i\}_{i=1}^n) = \sup_{\|\{\gamma_i\}_{i=1}^n\|_{l_q}=1} \left\|\sum_{i=1}^{n} \gamma_i e_i\right\|_E.$$

We now formulate the generalization of Theorem 2.

Theorem 2'. *Suppose a Banach space E satisfies the α_p-condition for some $p > 1$ and suppose $\{e_i\}_1^n$ is a system of elements of E. Then for any $q > p$ and $\epsilon > 0$ there exist a constant $c_{\epsilon,p}$ and a subsystem $\{e_{i_\nu}\}_{\nu=1}^s$ such that $s > n(1-\epsilon)$ and*

$$B_q(\{e_{i_\nu}\}) \leqslant \frac{B_\infty(\{e_i\}_{i=1}^n)}{n^{1/q}} c_{\varepsilon,q}.$$

The proof of Theorem 2′ is completely analogous to the proof of Theorem 2. The assertion of Lemma 1 of the present paper consists in the fact that the space $L_1[0, 1]$ satisfies the α_2-condition. The specific properties of $L_1[0, 1]$ are not used elsewhere in the proof of Theorem 2.

We note, in addition, that one can prove, in the same way as Lemma 1, that any space $L_p[0, 1]$ $(1 \leq p < \infty)$ satisfies the α_2-condition.

Let us now prove Theorem 1. Theorem 1 is easily deduced from the following assertion, which is a weakened formulation of it.

Suppose given a series

$$\sum_{i=1}^{\infty} f_i(x). \tag{20}$$

that converges unconditionally in $L_1[0, 1]$. Then for any $p < 2$ and $\epsilon > 0$ there exists a set $E_{\epsilon,p} \subset [0, 1]$ with measure $mE_{\epsilon,p} \geq 1 - \epsilon$ such that the series (20) converges unconditionally in $L_p(E_{\epsilon,p})$.

With the use of a simple result proved by Nikišin (see [1], §3, Lemma 3), this assertion can be obtained from the following lemma.

Lemma 4. *Suppose given a system $\{g_i\}_1^m$ of functions $g_i \in L_1[0, 1]$, and let*

$$A_\infty = \sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{m} \varepsilon_i g_i \right\|_{L_1}.$$

Then for any $p < 2$ and $\epsilon > 0$ there exist a set $E_{\epsilon,p} \subset [0, 1]$ with $mE_{\epsilon,p} > 1 - \epsilon$, and a number $c_{\epsilon,p}$, such that

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{m} \varepsilon_i g_i \right\|_{L_p(E_{\varepsilon,p})} \leqslant A_\infty c_{\varepsilon,p}.$$

In fact, suppose that Lemma 4 is valid. Since the series (20) converges unconditionally in $L_1[0, 1]$, it follows by virtue of the theorem of Orlicz formulated at the beginning of the article that

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{\infty} \varepsilon_i f_i \right\|_{L_1} = k < \infty.$$

Taking fixed numbers $\epsilon > 0$ and $p < 2$ and applying Lemma 4 successively to the systems of functions $\{f_i\}_1^m$, $m = 1, 2, \ldots$, we construct a sequence of sets $E_{\epsilon,p}^m$ such that $m(E_{\epsilon,p}^m) > 1 - \epsilon$ and

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{m} \varepsilon_i f_i \right\|_{L_p(E_{\varepsilon,p}^m)} \leqslant c_{\varepsilon,p} \cdot k.$$

Let $\chi^m(x) \equiv \chi(E^m_{\epsilon,p})$ be the characteristic function of the set $E^m_{\epsilon,p}$. From the sequence of functions $\chi^m(x)$ it is possible to distinguish a subsequence $\chi^{m_k}(x)$ that is weakly convergent in $L_2[0, 1]$ to a nonnegative function $\chi(x)$. The lemma of Nikišin guarantees that there exists an $\epsilon_0 > 0$ such that

$$m\{x : x \in [0, 1];\ \chi(x) \geqslant \varepsilon_0\} \geqslant 1 - \varepsilon.$$

Let $E_{\epsilon,p}$ denote the set of those $x \in [0, 1]$ for which $\chi(x) > \epsilon_0$. It is easily seen that

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{\infty} \varepsilon_i f_i \right\|_{L_p(E_{\varepsilon,p})} \leqslant \frac{c_{\varepsilon,p}\ k}{\varepsilon_0},$$

and this is equivalent to the unconditional convergence of (20) in $L_p(E_{\epsilon,p})$.

Let us now prove Lemma 4 with the use of Theorem 2.

It suffices to carry out the proof of the lemma for the case when the g_i are piecewise constant functions that are constant on intervals of length $1/s$. Suppose $g_i(k/s) = a_{ik}$, $1 \le i \le m$, $1 \le k \le s$. Then

$$A_\infty = \sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{m} \varepsilon_i g_i \right\|_{L_1} = \frac{1}{s} \sup_{\varepsilon_i = \pm 1} \sum_{k=1}^{s} \left| \sum_{i=1}^{m} \varepsilon_i a_{ik} \right| = \frac{1}{s} \sup_{\substack{\varepsilon_i = \pm 1 \\ \alpha_k = \pm 1}} \sum_{k=1}^{s} \sum_{i=1}^{m} \alpha_k \varepsilon_i a_{ik}.$$

Analogously,

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{m} \varepsilon_i g_i \right\|_{L_p} = \frac{1}{s^{1/p}} \sup_{\substack{\varepsilon_i = \pm 1 \\ \|\{\alpha_k\}\|_{l_{p'}} = 1}} \sum_{k=1}^{s} \sum_{i=1}^{m} \alpha_k \varepsilon_i a_{ik} \quad (1/p + 1/p' = 1).$$

What we must prove can now be reformulated as follows.

Suppose given a bilinear form

$$\sum_{k=1}^{s} \sum_{i=1}^{m} \varepsilon_i \alpha_k a_{ik}$$

with matrix $A = \{a_{ik}\}_{i=1}^{m}\,{}_{k=1}^{s}$. It must be proved that for any $\epsilon > 0$ there exists a $c_{\epsilon,p}$ such that in A it is possible to retain s' columns with indices $k_1, \ldots, k_{s'}$ for which $s' > s(1 - \epsilon)$ and

$$\frac{1}{s^{1/p}} \sup_{\substack{\varepsilon_i = \pm 1 \\ \|\{\alpha_k\}\|_{l_{p'}} = 1}} \sum_{j=1}^{s'} \sum_{i=1}^{m} \varepsilon_i \alpha_{k_j} a_{i,k_j} \leqslant \frac{c_{\varepsilon,p}}{s} \sup_{\substack{\varepsilon_i = \pm 1 \\ \alpha_k = \pm 1}} \sum_{k=1}^{s} \sum_{i=1}^{m} \varepsilon_i \alpha_k a_{ik},$$

i.e.

$$\sup_{\substack{\varepsilon_i = \pm 1 \\ \|\{\alpha_k\}\|_{l_{p'}} = 1}} \sum_{j=1}^{s'} \sum_{i=1}^{m} \varepsilon_i \alpha_{k_j} a_{i,k_j} \leqslant \frac{c_{\varepsilon,p}}{s^{1/p'}} \sup_{\substack{\alpha_k = \pm 1 \\ \varepsilon_i = \pm 1}} \sum_{k=1}^{s} \sum_{i=1}^{m} \varepsilon_i \alpha_k a_{ik}. \tag{21}$$

We note that $p' > 2$ since $p < 2$.

We now apply Theorem 2 in the following case: the system $\{e_r\}_1^s$ consists of piecewise constant functions e_r that are constant on intervals of length $1/m$ and are given by the columns

$$\begin{array}{|c|c|c|c|c|}\hline e_1 & \ldots & e_r & \ldots & e_s \\ \hline a_{11} & \cdot & a_{1r} & \cdot & a_{1s} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & a_{mr} & \cdot & a_{ms} \\ \hline \end{array},$$

where the a_{ij} are the elements of A. Then

$$B_\infty\ (\{e_r\}_{r=1}^{s}) = \sup_{\alpha_k=\pm 1} \left\| \sum_{k=1}^{s} \alpha_k e_k \right\|_{L_1}$$

$$= \frac{1}{m} \sup_{\alpha_k=\pm 1} \sum_{i=1}^{m} \left| \sum_{k=1}^{s} a_{i,k}\alpha_k \right| = \frac{1}{m} \sup_{\substack{\alpha_k=\pm 1\\ \varepsilon_i=\pm 1}} \sum_{k=1}^{s} \sum_{i=1}^{m} \varepsilon_i \alpha_k a_{i,k}.$$

Analogously,

$$B_{p'} = \frac{1}{m} \sup_{\substack{\varepsilon_i=\pm 1\\ \|\{\alpha_k\}\|_{l_{p'}}=1}} \sum_{k=1}^{s} \sum_{i=1}^{m} \varepsilon_i \alpha_k a_{i,k}.$$

According to Theorem 2, it is possible to retain $s' > s(1-\epsilon)$ columns of A for which

$$\sup_{\substack{\varepsilon_i=\pm 1\\ \|\{\alpha_k\}\|_{l_{p'}}=1}} \sum_{j=1}^{s'} \sum_{i=1}^{m} \varepsilon_i \alpha_{k_j} a_{i,k_j} \leqslant \frac{c_{\varepsilon,p'}}{s^{1/p'}} \sup_{\substack{\varepsilon_i=\pm 1\\ \alpha_k=\pm 1}} \sum_{k=1}^{s} \sum_{i=1}^{m} \varepsilon_i \alpha_k a_{ik},$$

which coincides with (21). The lemma is proved.

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