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SHORT COMMUNICATIONS

Remark on Estimates of Orthomassivity

B. S. Kashin*

Steklov Mathematical Institute, Russian Academy of Sciences

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In [1], the author introduced a characteristic of the complexity of a subset K of the unit ball $B(H)$ in Hilbert space H, called orthomassivity. *Orthomassivity* is defined by the sequence of numbers $(n = 1, 2, \dots)$

$$OM_n(K) = n^{-1/2} \sup \sum_{j=1}^{n} (f_j, \varphi_j), \qquad \{\varphi_j\}_{j=1}^{n} \text{ is an o.n.s.}, \quad \{f_j\}_{j=1}^{n} \subset K, \tag{1}$$

where $(\,\cdot\,,\,\cdot\,)$ is the inner product in H and o.n.s. denotes an orthonormal system in H.

We can easily see that always

$$\sup_{f \in K} \|f\|_H \le OM_n(K) \le n^{1/2}, \qquad n = 1, 2, \dots .$$

The behavior of the quantities (1) is related to the behavior of the Kolmogorov widths $d_n(K, H)$ of the set K regarded as a subset in H. In particular, it is easy to verify that, for $n = 1, 2, \dots$,

$$OM_n(K) \ge n^{-1/2} \sum_{j=1}^{n} d_{j-1}(K, H). \tag{2}$$

However, estimate (2) is not always order-sharp. For example, for the case in which

$$K = \{X_t, 0 < t < 1\} \subset L^2(0, 1), \tag{3}$$

where

$$X_t(x) = \begin{cases} 1 & \text{if } 0 \le x \le t, \\ 0 & \text{if } t < x \le 1, \end{cases}$$

the order relation

$$OM_n(K) \asymp \log n, \qquad n \to \infty, \tag{4}$$

was noted in [1], while (2) yields only the trivial estimate $OM_n(K) \ge 1$.

The estimate of the quantities (1) is related to other problems in the theory of orthogonal series and combinatorics. So relation (4) actually is equivalent to the combination of the classical Men'shov–Rademacher and Men'shov theorems on convergence and divergences almost everywhere of orthogonal series (see [2, Chap. 9]). Thus, these classical theorems can be regarded as assertions on the complexity of the family of characteristic functions of the intervals (3).

In the statement given below which generalizes, in effect, the proof of the Men'shov–Rademacher theorem, we establish an upper bound for orthomassivity; this bound leads to sharp results in a number of cases. In what follows, by $\#A$ we denote the number of elements in a finite set A.

*E-mail: kashin@mi.ras.ru

Theorem. *Suppose that the set $F \subset B(H)$ is such that there exist finite sets*

$$V_0, V_1, \dots, V_R \subset H$$

satisfying

1) $0 \in V_q \subset 2^{-q/2}B(H)$, $0 \le q \le R$;
2) $\#V_q \le h_q \cdot 2^q$, $0 \le q \le R$;
3) *any element $f \in F$ can be expressed as*

$$f = \sum_{q=0}^{R}\sum_{\mu=1}^{l_q} v_\mu^{(q)},$$

where $v_\mu^{(q)} \in V_q$, $1 \le \mu \le l_q$.

Then, uniformly on $n = 1, 2, \dots$,

$$OM_n(F) \le \left(R \cdot \sum_{q=0}^{R} l_q h_q\right)^{1/2}.$$

Proof. Using property 3) of the set F and applying Cauchy's inequality twice to an arbitrary pair $f \in F$, $\varphi \in H$, we obtain

$$(f,\varphi)^2 = \left[\sum_{q=0}^{R}\left(\sum_{\mu=1}^{l_q} v_\mu^{(q)}, \varphi\right)\right]^2 \le R\sum_{q=0}^{R}\left(\sum_{\mu=1}^{l_q} v_\mu^{(q)}, \varphi\right)^2 \le R\sum_{q=0}^{R} l_q \cdot \sum_{\mu=1}^{l_q}(v_\mu^{(q)}, \varphi)^2$$
$$\le R\sum_{q=0}^{R} l_q \sum_{v\in V_q}(v,\varphi)^2. \tag{5}$$

If now $\{\varphi_j\}_{i=1}^n$ is an arbitrary orthonormal system and $\{f_j\}_{i=1}^n \subset F$, then again using Cauchy's inequality, estimate (5), Bessel's inequality, and properties 1), 2) of the family F, we obtain

$$\left[n^{-1/2}\sum_{j=1}^{n}(f_j,\varphi_j)\right]^2 \le \sum_{j=1}^{n}(f_j,\varphi_j)^2 \le \sum_{j=1}^{n} R \cdot \sum_{q=0}^{R} l_q \sum_{v\in V_q}(v,\varphi_j)^2 = R\sum_{q=0}^{R} l_q \sum_{v\in V_q}\sum_{j=1}^{n}(v,\varphi_j)^2$$
$$\le R\sum_{q=0}^{R} l_q \sum_{v\in V_q}\|v\|_H^2 \le R\sum_{q=0}^{R} l_q(\#V_q)\cdot 2^{-q} \le R\sum_{q=0}^{R} l_q h_q,$$

which proves the assertion. □

Suppose that, for $d = 2, 3, \dots$,

$$\pi_d = \{X_P; P = (0,t_1)\times \dots (0,t_d),\ 0 < t_i < 1,\ 1 \le i \le d\}$$

is the set of characteristic functions of d-dimensional intervals, regarded as a subset in $L^2(0,1)^d$.

Corollary. *For $d = 2, 3, \dots$,*

$$OM_n(\pi_d) \asymp (\log n)^d, \qquad n \to \infty.$$

Proof. For simplicity, we restrict ourselves to the case $d = 2$.

a) *Upper bound.* Suppose that, for $r = 1, 2, \dots$,

$$\pi_2(r) = \left\{X_P : P = \left(0, \frac{a}{2^r}\right) \times \left(0, \frac{b}{2^r}\right),\ a, b \in \{1, \dots, 2^r - 1\}\right\}.$$

For a given n, we choose a natural number r so that

$$2^{r-1} < n \le 2^r. \tag{6}$$

It is easily verified that, for any function $f \in \pi_2$, there exists an $\tilde{f} \in \pi_2(r)$ with

$$\|f - \tilde{f}\|_{L^2} \le cn^{-1/2}.$$

this yields

$$OM_n(\pi_2) \le OM_n(\pi_2(r)) + C, \tag{7}$$

where C is an absolute constant.

Suppose that the interval P is of the form

$$P = \left(0, \frac{a}{2^r}\right) \times \left(0, \frac{b}{2^r}\right), \qquad a, b \in \{1, \dots, 2^r - 1\}.$$

The following relation holds up to a finite set of points:

$$\left(0, \frac{a}{2^r}\right) = \bigsqcup_{s \in \Lambda} \Delta_s,$$

where the sign $\bigsqcup$ denotes, as is customary, the union of disjoint sets, Δ_s is a binary interval of length 2^{-s} (i.e., the interval of the form

$$\left(\frac{c-1}{2^s}, \frac{c}{2^s}\right), \qquad c \in \{1, \dots, 2^s\}),$$

and Λ is a subset of the collection $\{1, \dots, r\}$. Similarly,

$$\left(0, \frac{b}{2^r}\right) = \bigsqcup_{m \in \Lambda'} \Delta'_m$$

and

$$X_P = \sum_{s \in \Lambda, m \in \Lambda'} X_{\Delta_s \times \Delta'_m}. \tag{8}$$

It is easy to verify that, for $q = 1, 2, \dots, 2r$, the representation (8) contains $\le 2q$ characteristic functions of two-dimensional binary intervals of area 2^{-q}. In all, there are $\le 2q \cdot 2^q$ binary intervals of area 2^{-q} lying in $(0,1)^2$. Using the theorem for the case in which $F = \pi_2(r)$, $R = 2r$, and V_q, $q = 0, 1, \dots, R$, is the family of two-dimensional binary intervals of area 2^{-q}, we obtain

$$OM_n(\pi_2(r)) \le \left(R \cdot \sum_{q=1}^{R} q^2\right)^{1/2} \le 4R^2 \le 16r^2 \le 16(\log_2 n + 1)^2.$$

The last inequality, together with (7), implies the upper bound.

b) *Lower bound.* The lower bound is obtained if, for $n = p^2$, $p = 2, 3, \dots$, the orthonormal system $\{\varphi_j\}_{j=1}^n$ is taken as the double system

$$\{\psi_{j_1}(x)\psi_{j_2}(y),\ 1 \le j_1, j_2 \le p\},$$

where $\{\psi_j(x)\}_{j=1}^p$ are functions from the Men'shov orthonormal system (see, for example, [2, Theorem 9.6]) for which

$$\int_o^{t_j} \psi_j(x)\, dx > c(\log p)p^{-1/2}, \qquad t_j \in (0,1), \quad 1 \le j \le \frac{p}{2}.$$

Then

$$\frac{1}{\sqrt{n}} \sum_{1 \le j_1, j_2 \le p/2} (\psi_{j_1}(x)\psi_{j_2}(y), X_{[0,t_{j_1}]\times[0,t_{j_2}]}(x,y))$$
$$\ge \frac{1}{p} \sum_{1 \le j_1, j_2 \le p/2} c^2(\log p)^2/p \ge c'(\log n)^2, \qquad c' > 0,$$

as required. □

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