Mathematical Notes, vol. 80, no. 2, 2006, pp. 199–203.
Translated from Matematicheskie Zametki, vol. 80, *no.* 2, 2006, *pp.* 204–208.

On a Class of Inequalities for Orthonormal Systems

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Received February 7, 2006

Abstract—In this paper, a novel approach to the proof of inequalities of Lieb–Thirring type based on the standard apparatus of the theory of orthogonal series is proposed.

Key words: *inequality of Lieb–Thirring type, orthonormal system, orthogonal series, classical Littlewood–Paley theorem, Cauchy's inequality.*

In 1976, the following assertion was proved by Lieb and Thirring in [1].

Theorem A. *For an arbitrary orthonormal system* $\Phi = \{\varphi_j\}_{j=1}^N \subset L^2(\mathbb{R}^d)$, *the following inequality holds*:

$$\left(\int_{\mathbb{R}^d} \rho_\Phi^{p/(p-1)}\,dx \right)^{2(p-1)/d} \le C_{p,d} \sum_{j=1}^N \|\nabla\varphi_j\|_2^2 \tag{1}$$

if $\max(1, d/2) < p \le 1 + d/2$, *where*

$$\rho_\Phi \equiv \sum_{j=1}^N \varphi_j^2(x) \tag{2}$$

and, as is customary, $\nabla\varphi = \bigl(\frac{\partial\varphi}{\partial x_1}, \dots, \frac{\partial\varphi}{\partial x_d}\bigr)$.

More recently, a number of inequalities of type (1) for finite orthonormal systems was established by various authors. In particular, Il′in [2] proved the following assertion.

Theorem B. *Suppose that* $\{\varphi_j\}_{j=1}^N \subset L^2(S^1)$ *is an orthonormal system of functions defined on the unit circle* S^1, *with* $\varphi_j \perp 1$, $j = 1, \dots, N$. *Then the following inequality holds*:

$$\int_{S^1} \rho_\Phi^2\,d\mu \le C \sum_{j=1}^N \|\varphi_j^{(1/2)}\|_{L^2(S^1)}^2,$$

where, for $f(z) = \sum_{k\ne 0} \hat f(k) z^k \in L^2(S^1)$,

$$f^{(1/2)} = \sum_{k\ne 0} |k|^{1/2} \hat f(k) z^k$$

is the derivative of order $1/2$ *of the function* f *and* μ *is the normalized Lebesgue measure on* S^1.

Inequalities of Lieb–Thirring type are of particular interest in view of their application to the theory of partial differential equations (for greater detail, see [3] as well as [2]). The well-known methods of proving these inequalities are based on nontrivial results from spectral theory and use information on the behavior of negative eigenvalues of operators of Schrödinger type (see, for example, [2]). In this paper, we propose an approach to inequalities of Lieb–Thirring type based on the standard apparatus of the theory of orthogonal series: on inequalities for random series and the classical Littlewood–Paley theorem. Below we demonstrate this approach by establishing a generalization of Theorem B. Moreover, the arguments given below can be applied, apparently, to the proof of other inequalities of the above-mentioned type. Besides, our approach allows us to clarify, to a certain extent, the following question: For what systems $\{\varphi_j\}$ are inequalities of Lieb–Thirring type "order-sharp"?

Suppose that

$$\Phi = \{\varphi_j\}_{j=1}^N \subset L^2(S^1), \qquad \text{where} \quad \varphi_j \perp 1, \quad 1 \le j \le N. \tag{3}$$

We define the operator

$$P_\Phi \colon l_2^N \to L^2(S^1)$$

acting by the rule:

$$P_\Phi(\{c_j\}_{j=1}^N) = \sum_{j=1}^N c_j \varphi_j.$$

Also suppose that, for $\nu = 0, 1, \dots,$

$$\pi_\nu \colon L^2(S^1) \to T_{2^\nu}$$

is the orthogonal projection onto the space of trigonometric polynomials of the form

$$T_{2^\nu} = \biggl\{ t(z) \colon t = \sum_{2^\nu \le |k| < 2^{\nu+1}} a_k z^k \biggr\},$$

and suppose that

$$\lambda_\nu(\Phi) = \|\pi_\nu \cdot P_\Phi \colon l_2^N \to T_{2^\nu}\|. \tag{4}$$

Obviously, $\lambda_\nu \le 1$, $\nu = 0, 1, \dots,$ if Φ is an orthonormal (or a suborthonormal) system. Therefore, the following inequality generalizes Theorem B.

Theorem 1. *For any system of real functions* Φ *of the form* (3),

$$\int_{S^1} \rho_\Phi^2 \, d\mu \le C \sum_{\nu=0}^\infty \gamma_\nu(\Phi) \cdot \sum_{j=1}^N \sum_{2^\nu \le |k| < 2^{\nu+1}} |\widehat{\varphi}_j(k)|^2,$$

where, for $\nu = 0, 1, 2, \dots,$

$$\gamma_\nu(\Phi) = \sum_{\beta=0}^\nu 2^\beta \cdot \lambda_\beta^2(\Phi), \tag{5}$$

the function ρ_Φ *is defined in* (2), *the numbers* $\lambda_\beta(\Phi)$ *are defined in* (4), *and* C *is an absolute constant.*

Before passing to the proof of Theorem 1, let us recall some important facts.

Suppose that $1 \le p < \infty$ and $\varphi_j \in L^p(S^1)$, $1 \le j \le N$. Then, setting

$$\mathscr{F} = \Bigl(\sum \varphi_j^2\Bigr)^{1/2},$$

we obtain (see [4, p. 52]):

$$C_1(p)\|\mathscr{F}\|_{L^p(S^1)}^p \le \int_{S^1} \biggl\| \sum_{j=1}^N r_j(t)\varphi_j \biggr\|_{L^p(S^1)}^p dt \le C_2(p)\|\mathscr{F}\|_{L^p(S^1)}^p, \tag{6}$$

where the $\{r_j(t)\}$ are Rademacher functions and $C_1(p) > 0$.

Further, we need a well-known corollary of the Littlewood–Paley theorem (see, for example, [5, p. 56]): if a real-valued function f belongs to $L^p(S^1)$, $1 < p < \infty$, $f \perp 1$, and

$$f = \sum_{\nu=0}^{\infty} \delta_\nu(f), \qquad \delta_\nu(f) = \sum_{2^\nu \le |k| < 2^{\nu+1}} \hat{f}(k) z^k,$$

then

$$C_3(p)\|f\|_{L^p(S^1)} \le \left\| \left(\sum \delta_\nu^2 \right)^{1/2} \right\|_{L^p(S^1)} \le C_4(p)\|f\|_{L^p(S^1)}, \qquad C_3(p) > 0. \tag{7}$$

We also need the following simple lemma.

Lemma. *For any system of functions* Φ *of the form* (3), *for* $\beta = 0, 1, \dots$ *the following inequality holds*:

$$A = \left\| \sum_{j=1}^{N} \delta_\beta^2(\varphi_j) \right\|_{L^\infty(S^1)} \le 2^{\beta+1} \cdot \lambda_\beta^2.$$

Indeed, suppose that $z_0 \in S^1$ is a point such that

$$\left(\sum_{j=1}^{N} \delta_\beta^2(\varphi_j)(z_0) \right)^{1/2} = A^{1/2}.$$

There exists a collection of numbers $\{c_j\}_{j=1}^N$ with $\sum_{j=1}^N c_j^2 = 1$ such that, for the trigonometric polynomial

$$\sum_{j=1}^{N} c_j \delta_\beta(\varphi_j) = P(z) = \sum_{2^\beta \le |k| < 2^{\beta+1}} a_k z^k,$$

the following inequalities hold:

$$\|P\|_{L^\infty(S^1)} \ge |P(z_0)| = A^{1/2},$$

and hence (see (4)),

$$\begin{aligned} A^{1/2} &\le \|P\|_{L^\infty(S^1)} \le \sum_{2^\beta \le |k| < 2^{\beta+1}} |a_k| \le 2^{(\beta+1)/2} \left(\sum_{2^\beta \le |k| < 2^{\beta+1}} |a_k|^2 \right)^{1/2} \\ &= 2^{(\beta+1)/2} \|P\|_{L^2(S^1)} \le 2^{(\beta+1)/2} \lambda_\beta, \end{aligned}$$

as was required (actually, we have used Nikol′skii's inequality for different metrics reducing, in this case, to Cauchy's inequality).

Proof of Theorem 1. Using (6), then (7), and introducing the notation

$$\delta_\nu(t, z) = \delta_\nu\left(\sum_{j=1}^{N} r_j(t)\varphi_j(z) \right),$$

we obtain

$$Q \equiv \int_{S^1} \rho_\Phi^2 \, d\mu \le C_5 \int_0^1 \left\| \sum r_j(t)\varphi_j \right\|_{L^4}^4 dt \le C_6 \int_0^1 \int_{S^1} \left[\sum_{\nu=0}^{\infty} \delta_\nu^2(t, z) \right]^2 d\mu \, dt.$$

Therefore,

$$\frac{Q}{C_6} \le \sum_{\nu,\beta=0}^{\infty} \int_0^1 \int_{S^1} \delta_\nu^2(t,z)\delta_\beta^2(t,z)\,d\mu\,dt. \tag{8}$$

Here

$$\delta_\nu^2(t,z) = \biggl(\sum_{j=1}^{N} r_j(t)\delta_\nu(\varphi_j)(z)\biggr)^2 = \sum_{j,q=1}^{N} r_j(t)r_q(t)\delta_\nu(\varphi_j)\delta_\nu(\varphi_q),$$

and hence, for each t,

$$\int_{S^1} \delta_\nu^2(t,z)\delta_\beta^2(t,z)\,d\mu = \sum_{j,q,h,s=1}^{N} r_j(t)r_q(t)r_h(t)r_s(t)\int_{S^1} \delta_\nu(\varphi_j)\delta_\nu(\varphi_q)\delta_\beta(\varphi_h)\delta_\beta(\varphi_s)\,d\mu.$$

Taking into account the fact that

$$\int r_j r_q r_h r_s\,dt \ne 0,$$

only if each number occurs an even number of times in the set (j,q,h,s) (and then this integral is equal to 1), we obtain

$$\begin{aligned} \mathrm{I}_{\nu,\beta} \equiv \int_0^1 \int_{S^1} \delta_\nu^2(t,z)\delta_\beta^2(t,z)\,d\mu\,dt &= \sum_{j=1}^{N}\sum_{h=1}^{N}\int_{S^1} \delta_\nu^2(\varphi_j)\delta_\beta^2(\varphi_h)\,d\mu \\ &\quad + 2\sum_{j=1}^{N}\sum_{q=1}^{N}\int_{S^1} \delta_\nu(\varphi_j)\delta_\beta(\varphi_j)\delta_\nu(\varphi_q)\delta_\beta(\varphi_q)\,d\mu - 2\sum_{j=1}^{N}\int_{S^1} \delta_\nu^2(\varphi_j)\delta_\beta^2(\varphi_j)\,d\mu \\ &\le \int_{S^1} \sum_{j=1}^{N} \delta_\nu^2(\varphi_j)\cdot\sum_{j=1}^{N}\delta_\beta^2(\varphi_j)\,d\mu + 2\int_{S^1}\biggl[\sum_{j=1}^{N}\delta_\nu(\varphi_j)\delta_\beta(\varphi_j)\biggr]^2 d\mu. \end{aligned}$$

Applying Cauchy's inequality to the sum in the last integral, we find

$$\mathrm{I}_{\nu,\beta} \le 3J_{\nu,\beta}, \qquad \text{where} \quad J_{\nu,\beta} \equiv \int_{S^1} \sum_{j=1}^{N}\delta_\nu^2(\varphi_j)\cdot\sum_{j=1}^{N}\delta_\beta^2(\varphi_j)\,d\mu.$$

Therefore (see (8)),

$$\frac{Q}{C_6} \le 3\sum_{\nu,\beta=0}^{\infty} J_{\nu,\beta} \le 6 \sum_{\nu,\beta:\ \beta\le\nu} J_{\nu,\beta}. \tag{9}$$

Using the lemma to estimate the terms in the sum (9), we obtain (see also (5))

$$\frac{Q}{C_6} \le 6\sum_{\nu=0}^{\infty}\biggl[\sum_{\beta=0}^{\nu} 2^{\beta+1}\lambda_\beta^2\biggr]\cdot\int_{S^1}\sum_{j=1}^{N}\delta_\nu^2(\varphi_j)\,d\mu = 12\sum_{\nu=0}^{\infty}\gamma_\nu(\Phi)\biggl[\sum_{j=1}^{N}\sum_{2^k\le|k|<2^{k+1}} |\widehat{\varphi}_j(k)|^2\biggr].$$

Theorem 1 is proved. □

ACKNOWLEDGMENTS

This research was supported by the Russian Foundation for Basic Research under grants no. 05-01-00062.

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