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Lower Bounds for n-Term Approximations of Plane Convex Sets and Related Topics

B. S. Kashin

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Abstract—In this paper, we establish lower bounds for n-term approximations in the metric of $L^2(I^2)$ of characteristic functions of plane convex subsets of the square I^2 with respect to arbitrary orthogonal systems. It is shown that, as $n \to \infty$, these bounds cannot decrease more rapidly than $1/n$.

Key words: *n-term approximation, plane convex set, Haar system, orthonormal system, Hilbert space, Walsh system.*

Suppose that $I^d = [0,1]^d$ is the unit cube in the Euclidean space $\mathbb{R}^d$, $d = 1, 2, \dots$. By $\mathfrak{M}_d$ we denote the set of all convex subsets of the cube I^d and set

$$K_d = \{\chi_A(x),\ A \in \mathfrak{M}_d\},$$

where for $x \in I^d$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

In this paper, we establish lower bounds for best n-term approximations of the family K_2 with respect to an arbitrary orthonormal system in the metric of the Hilbert space $L^2(I^2)$. In addition, in connection with problems of approximation theory, for the family $K_2 \subset L^2(I^2)$ we introduce and study a quantitative characteristic of the "massivity" of an arbitrary subset of elements of the unit ball in Hilbert space.

1. Suppose that H is a Hilbert space, $\Phi = \{\varphi_j\}$ is an orthonormal system (o.n.s.) in H, and $f \in H$.

Definition 1. By a *best n-term approximation* of an element f with respect to the system Φ we mean the quantity

$$e_n(f, \Phi) = \inf_{P \in \Sigma_n} \|f - P\|_H, \tag{1}$$

where for $n = 1, 2, \dots$

$$\Sigma_n \equiv \left\{ \sum_{j \in \Lambda} a_j \varphi_j ; \quad a_j \in \mathbb{R}, \quad \#\Lambda \le n \right\},$$

and by $\#\Lambda$ we denote the number of elements in a (finite) set of positive integers Λ.

Further, if $\mathcal{F} \subset H$, then for $n = 1, 2, \dots$ we have

$$e_n(\mathcal{F}, \Phi) = \sup_{f \in \mathcal{F}} e_n(f, \Phi). \tag{2}$$

Best n-term approximations with respect to arbitrary systems of elements Φ in normed spaces are defined similarly (for more details, see [1, 2]). To a large extent, recent more intensive studies of n-term approximations have been related to their applications to image contraction problems. Such studies involve the quantities (2) for different classes of "sufficiently nice" functions, which are regarded as subsets of a normed function space. It is still not clear what function classes $\mathcal{F}$ yield good desscriptions of images encountered in actual practice. From this standpoint, we can regard the problem of the behavior as $n \to \infty$ of the quantities

$$e_n(K_d, \Phi), \qquad d = 1, 2, \dots, \qquad \Phi \text{ is an o.n.s.} \tag{3}$$

posed by S. V. Konyagin in 1993 as a very natural problem.

For $d = 1$, the quantities (3) may decrease exponentially; this is so if Φ is a Haar system. On the other hand, in [3] it was established that there exists an absolute constant $C > 0$ such that for any o.n.s. Φ,

$$e_n(K_1, \Phi) \ge C^{-n}, \qquad n = 1, 2, \dots .$$

It turns out that the one-dimensional case is, in a certain sense, degenerate and in the case $d > 1$ the pattern of the behavior of the quantities (3) changes. In particular, we have the following result.

Theorem 1. *There exists an absolute constant $c > 0$ such that for any o.n.s. Φ in $L^2(I^2)$ we have the following estimate for $n = 1, 2, \dots$:*

$$e_n(K_2, \Phi) \ge \frac{c}{n}.$$

To prove Theorem 1, we need the following theorem.

Theorem A [4]**.** *Suppose that for some $m \in \mathbb{N}$ the subset K of Hilbert space H contains the family Q of all vertices of a $2m$-dimensional cube*

$$Q = \left\{ \sum_{i=1}^{2m} \varepsilon_i, \psi_i, \quad \varepsilon_i = \pm 1, \quad \{\psi_i\}_{i=1}^{2m} \quad \text{is an o.n.s.} \right\}.$$

Then for any o.n.s. Φ we have

$$e_m(K, \Phi) \ge c_1 m^{1/2},$$

where $c_1 > 0$ is an absolute constant.

Set

$$\Delta_2(K_2) = \{f_1 + f_2 - 2f_3,\ f_i \in K_2,\ i = 1, 2, 3\}.$$

It is readily seen that for $n = 1, 2, \dots$ we have

$$e_{3n}(\Delta_2(K_2), \Phi) \le 4e_n(K, \Phi).$$

Therefore, to prove Theorem 1, it suffices to verify that for $n = 2, 3, \dots$ for any o.n.s. Φ we have

$$e_n(\Delta_2(K_2), \Phi) \ge c_2 n^{-1}, \qquad c_2 > 0. \tag{4}$$

Consider the closed disk B of radius $1/2$ centered at the point $(1/2, 1/2) \in I^2$, and suppose that P_{2n} is a closed regular $2n$-gon inscribed in the disk B (see Fig. 1).

In that case the difference $B \setminus P_{2n}$ consists of $2n$ circular segments $L_1, \dots, L_{2n}$. It is important that for any collection of positive integers $\Lambda \subset \{1, 2, \dots, 2n\}$ the set

$$V(\Lambda) = P_{2n} \cup \left\{ \bigcup_{i \in \Lambda} L_i \right\}$$

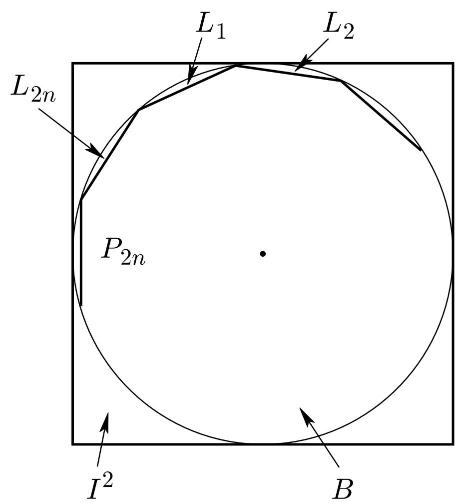

Fig. 1

is a convex set. Suppose that $f_1 = \chi_B$, $f_2 = \chi_{P_{2n}}$, $f_3 = \chi_{V(\Lambda)}$, where $\Lambda \subset \{1, \dots, 2n\}$. Then $f_1 + f_2 - 2f_3 \in \Delta_2(K_2)$ and

$$f_1(x) + f_2(x) - 2f_3(x) = \begin{cases} +1 & \text{if } x \in L_i,\ i \notin \Lambda, \\ -1 & \text{if } x \in L_i,\ i \in \Lambda, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Thus for any choice of signs $\{\varepsilon_i\}_{i=1}^{2n}$, $\varepsilon_i = \pm 1$, we have the inclusion

$$\sum_{i=1}^{2n} \varepsilon_i \chi_{L_i} \subset \Delta_2(K_2), \tag{6}$$

where the functions χ_{L_i}, $1 \le i \le 2n$, are pairwise orthogonal, have an identical norm on $L^2(I^2)$, and

$$\|\chi_{L_i}\|_{L^2(I^2)} \ge c_3 n^{-3/2}. \tag{7}$$

In view of relations (6) and (7), by applying Theorem A, we obtain the estimate

$$e_n(\Delta_2(K_2), \Phi) \ge c_4 n^{-1},$$

which is valid for any o.n.s. Φ. Inequality (4), and hence Theorem 1, is proved.

Remark. An upper bound for the form $e_n(K_2, \Phi_0) \le Cn^{-1/2}$ for the case in which Φ_0 is a multiple Haar system is a consequence of results from [5]. Also note the paper [6], in which the author studied n-term approximations of characteristic functions of plane sets with smooth boundary.

Now suppose that $\Phi = \{\varphi_j\}$ is an arbitrary frame in Hilbert space H and for $f \in H$ we have

$$\sigma_n(f, \Phi) = \inf_{\Lambda \subset \mathbb{N},\ \#\Lambda \le n} \left\| f - \sum_{j \in \Lambda} (f, \tilde{\varphi}_j)\varphi_j \right\|_H,$$

where $\{\tilde{\varphi}_j\}$ is a frame dual to Φ (for more details, see [7]). It was verified in [8] that there is an analog of Theorem A for frames of general form if $e_n(K, \Phi)$ is replaced by

$$\sigma_n(K, \Phi) = \sup_{f \in K} \sigma_n(f, \Phi).$$

Therefore, arguments given in the proof of Theorem 1, show that the following theorem is valid.

Theorem 2. *For an arbitrary frame* Φ *in* $L^2(I^2)$ *for* $n = 1, 2, \dots$ *the following inequality holds*:

$$\sigma_n(K_2, \Phi) \ge \frac{c}{n}, \qquad c = c(\Phi) > 0.$$

2. Suppose that K is a subset of the unit ball in Hilbert space H. For $n = 1, 2, \dots$, set

$$OM_n(K) = n^{-1/2} \sup_{\substack{\{\psi_j\}_{j=1}^n \text{ is an o.n.s.} \\ \{f_j\}_{j=1}^n \subset K}} \sum_{j=1}^{n} (\psi_j, f_j), \tag{8}$$

where $(\cdot\,,\,\cdot)$ is the inner product in H. The behavior of the sequence

$$\{OM_n(K)\}_{n=1}^{\infty}, \tag{9}$$

which can be called the "characteristic of orthomassivity" of the set K, contains useful information about K. It is readily seen that always

$$\sup_{f \in K} \|f\|_H \le OM_n(K) \le n^{1/2}, \qquad n = 1, 2, \dots .$$

It is also easy to note the connection of the quantities $OM_n(K)$ with the Kolmogorov n-widths of the sets K in the space H.

If $K = K_1 \subset L^2(0, 1)$, then

$$OM_n(K_1) \asymp \log n, \qquad n \to \infty.$$

This result is a consequence of the Men′shov–Rademacher theorem (see [9, pp. 333, 336]). More generally, if for $d = 2, 3, \dots$ π_d is the set of characteristic functions of d-dimensional intervals lying in I^d with sides parallel to the coordinate axes, then from results of [10, 11] we obtain the relation

$$OM_n(\pi_d) \asymp (\log n)^d, \qquad n \to \infty.$$

It turns out that for $d > 1$ the sequence $OM_n(K_d)$ increases substantially faster than in the one-dimensional case.

Theorem 3. *For* $n = 1, 2, \dots$, *we have*

$$OM_n(K_2) \ge c_5 n^{1/6}, \qquad c_5 > 0.$$

Proof. Without loss of generality, we assume that $n = m^3 = 2^{3r}$, $r = 1, 2, \dots$. Inside the annulus

$$\frac{1}{4} \le \left| x - \left(\frac{1}{2}, \frac{1}{2}\right) \right| \le \frac{1}{2}, \qquad x \in I^2,$$

consider, for $p \ge c_6 n^{2/3}$, $c_6 > 0$, the annuli Ω_ν centered at the point $(1/2, 1/2) \in I^2$ such that the boundary Ω_ν consists of the circles C_ν^{ext} and C_ν^{int} of radii r_ν^{ext} and r_ν^{int}, respectively, and

1) $C_\nu^{\text{int}} = C_{\nu+1}^{\text{ext}}$, $\nu = 1, 2, \dots, p-1$;
2) if $\mathcal{D}_\nu$ is a regular m-gon, inscribed in C_ν^{ext}, then

$$\mathcal{D}_\nu \supset C_\nu^{\text{int}}, \qquad \nu = 1, \dots, p;$$

3) $0 < c_7 n^{-2/3} \le r_\nu^{\text{ext}} - r_\nu^{\text{int}} < c_8 n^{-2/3}$, $\nu = 1, \dots, p$.

As in the proof of Theorem 1, circular sectors into which the set $\Omega_\nu\setminus\mathcal{D}_\nu$ is partitioned are denoted by L_i^ν, $i=1,\dots,m$.

Suppose that $\{w_j\}_{j=1}^m$ are the first m functions of the orthonormal Walsh system [9, p. 150]. Let us carry over these functions taking the values $+1$ or -1 onto the set $\omega_\nu=\bigcup_{i=1}^m L_i^\nu$ (for $\nu=1,2,\dots,p$) by setting

$$\psi_j^\nu(x)=\begin{cases}\alpha_\nu w_j\left(\dfrac{i-1/2}{m}\right) & \text{for } x\in L_i^\nu,\ i=1,\dots,m,\\ 0 & \text{for } x\notin\omega_\nu,\end{cases} \tag{10}$$

where $\alpha_\nu>0$ is a normalizing factor such that $\|\psi_j^\nu\|_{L^2(I^2)}=1$, $\nu=1,\dots,p$, $j=1,\dots,m$; here it is easy to verify that

$$\alpha_\nu=\{\operatorname{meas}\omega_\nu\}^{-1/2}\asymp n^{1/3},\qquad \nu=1,\dots,p. \tag{11}$$

The resulting system of functions

$$\{\psi_j^\nu,\quad \nu=1,\dots,p,\ j=1,\dots,m\}\quad\text{is an o.n.s. in } L^2(I^2). \tag{12}$$

Further, suppose that for $\nu=1,\dots,p$ and $j=1,\dots,m$,

$$Q_j^\nu=\bigcup_{i\,:\,\psi_j^\nu(x)>0\ \text{for}\ x\in L_i^\nu} L_i^\nu,\qquad E_j^\nu=\mathcal{D}_\nu\cup Q_j^\nu.$$

Then (see the proof of Theorem 1) E_j^ν is a convex set, i.e.,

$$f_j^\nu=\chi_{E_j^\nu}\in K_2,\qquad \nu=1,\dots,p,\quad j=1,\dots,m.$$

Moreover, we have (see (10), (11))

$$\sum_{(\nu,j)}\int_{I^2} f_j^\nu\psi_j^\nu\,dx=\frac12\sum_{(\nu,j)}\int_{I^2}|\psi_j^\nu|\,dx=\frac12\sum_{\nu=1}^{p}\sum_{j=1}^{m}\frac{1}{\alpha_\nu}\ge c_9n^{2/3},\qquad c_9>0,$$

i.e., $OM_n(K_2)\ge c_9n^{1/6}$, which proves the assertion. □

Let us explain how one can use the estimates of the rate of growth of the sequence (8). Suppose that for a given n the collection of elements $\{f_j\}_{j=1}^n\subset K$ almost maximizes the expression (7), i.e., there exists an o.n.s. $\{\psi_j\}_{j=1}^n$ for which

$$\frac{1}{n^{1/2}}\sum_{j=1}^{n}(\psi_j,f_j)\ge\frac12 OM_n(K).$$

Further, suppose that $\{\varphi_i\}_{i=1}^s$ is an arbitrary normed system of elements in H such that for $j=1,2,\dots,n$,

$$f_j=\sum_{i=1}^{s}c_i^j\varphi_i+\Delta_j,\qquad\text{where}\quad \|\Delta_j\|_H\le n^{-1/2},\quad j=1,2,\dots,n. \tag{13}$$

Then the following inequality holds:

$$\sum_{i=1}^{s}\left(\frac1n\sum_{j=1}^{n}(c_i^j)^2\right)^{1/2}\ge\frac12 OM_n(K)-1. \tag{14}$$

Indeed,

$$\frac{1}{2}OM_n(K) \le \frac{1}{n^{1/2}}\sum_{j=1}^{n}\sum_{i=1}^{s} c_i^j(\psi_j\,,\varphi_i) + \frac{1}{n^{1/2}}\sum_{j=1}^{n}(\psi_j\,,\Delta_j)\,;$$

hence, taking into account (13) and Bessel's inequality, we obtain

$$\frac{1}{2}OM_n(K) - 1 \le \sum_{i=1}^{s}\biggl(\frac{1}{n^{1/2}}\sum_{j=1}^{n} c_i^j\psi_j\,,\varphi_i\biggr) \le \sum_{i=1}^{s}\biggl(\frac{1}{n}\sum_{j=1}^{n}(c_i^j)^2\biggr)^{1/2},$$

as was required.

If there is additional information about the system $\{\varphi_i\}_{i=1}^s$, then in certain cases from (14) we can derive lower bounds for the values of the coefficients c_i^j, $1 \le i \le s$, for some fixed j (for more details, see [3, 12], where inequalities close to (14) were used for special cases in which $K = K_1 \subset L^2(0,1)$ and Φ is an arbitrary orthonormal system in $L^2(0,1)$).

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V. A. Steklov Mathematics Institute, Russian Academy of Sciences
E-mail: kashin@mi.ras.ru