ON WEYL MULTIPLIERS

B. S. KAŠIN

Let \{\phi_n(x)\}_{n=1}^\infty be an orthonormal system (ONS) of functions on the interval \([0, 1]\). A nondecreasing sequence \{\lambda_n\}_{n=1}^\infty of nonnegative numbers is called a Weyl multiplier for the convergence almost everywhere (a.e.) of series in this system if the condition

\[ \sum_{n=1}^\infty c_n\lambda_n < \infty \]  

implies that the series

\[ \sum_{n=1}^\infty c_n\phi_n(x) \]  

converges a.e. on \([0, 1]\). A Weyl multiplier \{\lambda_n\} is called an exact Weyl multiplier if for any sequence \(\beta_n = o(\lambda_n)\) there is a series (2) that diverges on a set of positive measure, although

\[ \sum_{n=1}^\infty c_n\beta_n < \infty. \]

A classical theorem of Men'sov and Rademacher (1921) states that \(\lambda_n = \log^2 n\) is a Weyl multiplier for convergence a.e. of series in any ONS. At the same time Men'sov constructed an ONS for which \(\lambda_n = \log^2 n\) is an exact Weyl multiplier. The system constructed by Men'sov consisted of essentially unbounded functions and for a long time it was not clear what order of magnitude a Weyl multiplier for uniformly bounded series in an ONS could have.

In 1937 Men'sov (see [1]) showed that for any \(K > 1\) there are ONS \{\phi_n(x)\} such that

1) \(|\phi_n(x)| \leq K; n = 1, 2, \ldots, x \in [0, 1];\)
2) the sequence \(|\log^2 n|\) is an exact Weyl multiplier for them.

It is interesting to consider the case \(K = 1\). In 1927 Kolmogorov and Men'sov [2] constructed an ONS \(\phi_n(x)\) such that

1) \(|\phi_n(x)| = 1; n = 1, 2, \ldots, x \in [0, 1];\)
2) any sequence \(|\beta_n|\) with \(\beta_n = o(\log n)\) is not a Weyl multiplier for convergence a.e. of series in this system.

In 1969 Tandori [3] proved a theorem in this direction, namely, for each \(\epsilon > 0\) there is an ONS

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\{q_n(x)\} \subset \{|q_n(x)|=1\} \ for \ n=1, 2, \ldots, x \in [0, 1]

and a series (2) such that

1) \( \sum_{n=1}^{\infty} c_n \log n (\log \log n)^{-1} < \infty; \)

2) the series (2) diverges a.e. after a certain rearrangement of the order of its terms.

A lowering of the order of the Weyl multipliers for ONS \( \{\phi_n(x)\} \) with \( |\phi_n(x)| = 1 \) would give important consequences in various problems, but the following statement is valid.

**Theorem 1.** There is an ONS \( \{\phi_n(x)\} \) with \( |\phi_n(x)| = 1, n = 1, 2, \ldots, x \in [0, 1] \) such that the sequence \( \lambda_n = \log^2 n \) is an exact Weyl multiplier for it.

Theorem 1 can be deduced using standard arguments and the following proposition.

**Basic Lemma.** There exist absolute constants \( C_1 > 0, C_2 > 0 \) and \( M > 1 \) such that for any natural number \( q \) there is a system of measurable functions \( \{\phi_j(x)\}_{j=1}^q \), defined on the segment \( [0, 2M + 1] \), satisfying the following conditions:

1) \( |\phi_j(x)| = 1, j = 1, 2, \ldots, 2^q, x \in [0, 2M + 1]; \)

2) \( \int_0^{2M+1} \phi_i(x) \phi_j(x) \, dx = 0 \) for \( i \neq j, 1 \leq i, j \leq 2^q; \)

3) for some functions \( N_1(x) \) and \( N_2(x) \)

\[
\mu \left\{ x : \left| \sum_{j \leq n} q_j(x) \right| > C_1 \sqrt{2^q \cdot q} \right\} > C_2.
\]

We note that the proof of the Basic Lemma makes essential use of probabilistic arguments. In the proof of Theorem 1 we obtain the following intermediate result:

For each natural number \( n \) there is a polynomial

\[ P_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}, \quad c_k = \pm 1, \]

such that

1) \( \|P_n(t)\|_c \leq A \sqrt{n}, \)

2) \( \|\sum_{k} c_k e^{ikt}\|_c \geq |\sum_{k} c_k| \geq B \sqrt{n} \log n, \)

where \( A \) and \( B \) are absolute positive constants.

Using this result it is easy to prove

**Theorem 2.** There is a function

\[ f(x) = \sum_{k=1}^{m} a_k \cos kx \]

continuous on \([-\pi, \pi]\) and such that

1) its modulus of continuity satisfies

\[ \omega(\delta, f) = O \left( \frac{1}{\log(1/\delta)} \right); \]
2) $|a_1| > |a_2| > \cdots$

3) the Fourier series of $f(x)$ diverges for $x = 0$.

This strengthens a theorem of Salem (see [4]).

BIBLIOGRAPHY

1. D. Men’sov, Sur les séries de fonctions orthogonales bornées dans leur ensemble, Mat. Sb. 3 (45) (1938), 103–120.


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