Dokl. Akad. Nauk SSSR
Tom 222 (1975), No. 5

Soviet Math. Dokl.
Vol. 16 (1975), No. 3

ON WEYL MULTIPLIERS

UDC 517.5

B. S. KAŠIN

Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be an orthonormal system (ONS) of functions on the interval $[0, 1]$. A nondecreasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ of nonnegative numbers is called a *Weyl multiplier* for the convergence almost everywhere (a.e.) of series in this system if the condition

$$\sum_{n=1}^{\infty} c_n^2\lambda_n<\infty \qquad (1)$$

implies that the series

$$\sum_{n=1}^{\infty} c_n\varphi_n(x) \qquad (2)$$

converges a.e. on $[0, 1]$.

A Weyl multiplier $\{\lambda_n\}$ is called an *exact Weyl multiplier* if for any sequence $\beta_n = o(\lambda_n)$ there is a series (2) that diverges on a set of positive measure, although

$$\sum_{n=1}^{\infty} c_n^2\beta_n<\infty.$$

A classical theorem of Men′šov and Rademacher (1921) states that $\lambda_n = \log^2 n$ is a Weyl multiplier for convergence a.e. of series in any ONS. At the same time Men′šov constructed an ONS for which $\lambda_n = \log^2 n$ is an exact Weyl multiplier. The system constructed by Men′šov consisted of essentially unbounded functions and for a long time it was not clear what order of magnitude a Weyl multiplier for uniformly bounded series in an ONS could have.

In 1937 Men′šov (see [1]) showed that for any $K > 1$ there are ONS $\{\phi_n(x)\}$ such that

1) $|\phi_n(x)| \le K$; $n = 1, 2, \ldots$, $x \in [0, 1]$;

2) the sequence $\{\log^2 n\}$ is an exact Weyl multiplier for them.

It is interesting to consider the case $K = 1$. In 1927 Kolmogorov and Men′šov [2] constructed an ONS $\phi_n(x)$ such that

1) $|\phi_n(x)| = 1$, $n = 1, 2, \ldots$, $x \in [0, 1]$;

2) any sequence $\{\beta_n\}$ with $\beta_n = o(\log n)$ is not a Weyl multiplier for convergence a.e. of series in this system.

In 1969 Tandori [3] proved a theorem in this direction, namely, *for each* $\epsilon > 0$ *there is an ONS*

AMS (MOS) subject classifications (1970). Primary 42A18, 42A60.

$\{\varphi_n(x)\}$ c $|\varphi_n(x)| \equiv 1$ *for* $n=1, 2, \ldots, x \in [0, 1]$

and a series (2) *such that*

1) $\sum_{n=3}^{\infty} c_n^2 \log n (\log \log n)^{1-\varepsilon} < \infty$;

2) *the series* (2) *diverges a.e. after a certain rearrangement of the order of its terms.*

A lowering of the order of the Weyl multipliers for ONS $\{\phi_n(x)\}$ with $|\phi_n(x)| = 1$ would give important consequences in various problems, but the following statement is valid.

Theorem 1. *There is an ONS* $\{\phi_n(x)\}$ *with* $|\phi_n(x)| \equiv 1$, $n = 1, 2, \ldots, x \in [0, 1]$, *such that the sequence* $\lambda_n = \log^2 n$ *is an exact Weyl multiplier for it.*

Theorem 1 can be deduced using standard arguments and the following proposition.

Basic Lemma. *There exist absolute constants* $C_1 > 0$, $C_2 > 0$ *and* $M \geq 1$ *such that for any natural number* q *there is a system of measurable functions* $\{\phi_j(x)\}_{j=1}^{2^q}$, *defined on the segment* $[0, 2M+1]$, *satisfying the following conditions:*

1) $|\phi_j(x)| = 1$, $j = 1, 2, \ldots, 2^q$, $x \in [0, 2M+1]$;

2) $\int_0^{2M+1} \phi_i(x)\phi_j(x)\,dx = 0$ *for* $i \neq j$, $1 \leq i, j \leq 2^q$;

3) *for some functions* $N_1^q(x)$ *and* $N_2^q(x)$

$$\mu\left\{x : \left|\sum_{N_1^q(x)}^{N_2^q(x)} \varphi_j(x)\right| > C_1\sqrt{2^q} \cdot q\right\} > C_2.$$

We note that the proof of the Basic Lemma makes essential use of probabilistic arguments. In the proof of Theorem 1 we obtain the following intermediate result:

For each natural number n *there is a polynomial*

$$P_n(t) = \sum_{k=-n}^{n} \varepsilon_k e^{ikt}, \quad \varepsilon_k = \pm 1,$$

such that

1) $\|P_n(t)\|_C \leq A\sqrt{n}$,

2) $\|\sum_0^n \varepsilon_k e^{ikt}\|_C \geq |\sum_0^n \varepsilon_k| \geq B\sqrt{n}\log n$,

where A *and* B *are absolute positive constants.*

Using this result it is easy to prove

Theorem 2. *There is a function*

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

continuous on $[-\pi, \pi]$ *and such that*

1) *its modulus of continuity satisfies*

$$\omega(\delta, f) = O\left(\frac{1}{\log(1/\delta)}\right);$$

2) $|a_1| > |a_2| > \cdots$;

3) *the Fourier series of $f(x)$ diverges for $x = 0$.*

This strengthens a theorem of Salem (see [4]).

Steklov Institute of Mathematics
Academy of Sciences of the USSR

Received 29/JAN/75

BIBLIOGRAPHY

1. D. Men'šov, *Sur les séries de fonctions orthogonales bornées dans leur ensemble*, Mat. Sb. 3 (45) (1938), 103–120.

2. A. Kolmogorov and D. Men'šov, *Sur la convergence des séries des fonctions orthogonales*, Math. Z. 26 (1927), 432–441.

3. K. Tandori, *Eine Bemerkung zum Konvergenzproblem der Orthogonalreihen*, Acta Math. Acad. Sci. Hungar. 20 (1969), 315–322. MR 40 #3161.

4. R. Salem, *On a problem of Smithies*, Nederl. Akad. Wetensch. Proc. Ser. A 57 = Indag. Math. 16 (1954), 403–407. MR 16, 240.

Translated by J. L. B. COOPER