Mathematical Notes, vol. 72, no. 2, 2002, pp. 281–284.
*Translated from Matematicheskie Zametki, vol. 72, no. 2, 2002, pp. 312–315.*


# A Note on the Approximation Properties of Frames of General Form

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Received March 13, 2002

Key words: *frame system, separable Hilbert space, Riesz basis, canonical expansion in the frame system.*

The approximation properties of redundant systems have recently been studied intensively in approximation theory. In particular, approximations using frame systems are studied in connection with the development of the theory of wavelets and its applications.

**Definition 1** [1]. By a *frame* (or *frame system*) we mean a collection of nonzero vectors $\Phi = \{\varphi_j\}_{j=1}^{\infty}$ in a separable Hilbert space $H$ such that

$$A\|v\|^2 \le \sum_{j=1}^{\infty}(v,\varphi_j)^2 \le B\|v\|^2 \qquad \forall v \in H, \tag{1}$$

where $0 < A \le B < \infty$, and $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$ are, respectively, the norm and inner product on $H$.

The constants $A$ and $B$ are called, respectively, the *lower* and *upper bounds* of the frame $\Phi$ and their ratio

$$\kappa = \kappa(\Phi) = \frac{B}{A}$$

is called the *condition of the frame*. Finally, a frame is said to be *tight* if $\kappa(\Phi) = 1$, i.e., $A = B$. Here let us note the paper of Kozlov [2] in which tight frames were studied (in other terms).

Obviously, any complete orthonormal system (o.n.s.) in $H$, as well as an arbitrary Riesz basis, is a frame (see the definition, for example, in [3, p. 17]). Frame systems are, in general, not minimal; however, some properties of orthogonal expansions remain valid for them. In particular, we use the inequality (see [1])

$$\biggl\|\sum_{j=1}^{\infty} c_j\varphi_j\biggr\| \le B^{1/2}\biggl(\sum_{j=1}^{\infty} c_j\biggr)^{1/2}, \tag{2}$$

where $\{c_j\}$ is an arbitrary sequence from $\ell^2$.

For each frame $\Phi = \{\varphi_j\}$, the dual frame $\widetilde{\Phi} = \{\widetilde{\varphi}_j\}$ is defined (see, for example, [4]), and each element $f \in H$ can be expressed as the following series converging in norm:

$$f = \sum_{j=1}^{\infty}(f,\widetilde{\varphi}_j)\varphi_j. \tag{3}$$

If $\kappa(\Phi) = 1$, i.e., the frame $\Phi$ is tight, then

$$\widetilde{\varphi}_j = \frac{1}{A}\varphi_j, \qquad j = 1, 2, \dots .$$

In the general case, for a frame $\Phi$ with bounds $A$ and $B$ the dual frame $\widetilde{\Phi}$ has bounds $B^{-1}$, $A^{-1}$ $(B^{-1} \le A^{-1})$.

The series (3) is called the *canonical expansion* of an element $f \in H$ in the frame system $\Phi$. When frames are used as an apparatus in approximation problems, the question arises about estimates for

$$\sigma_m(f, \Phi) = \inf_{\Lambda, \sharp\Lambda \le m} \left\| f - \sum_{j \in \Lambda} (f, \widetilde{\varphi}_j)\varphi_j \right\| \tag{4}$$

(here $\Lambda \subset \mathbb{N}$ is a subset of the set of positive integers, and $\sharp$ is the number of elements in the set),

$$\sigma_m(K, \Phi) = \sup_{f \in K} \sigma_m(f, \Phi), \qquad m = 1, 2, \dots \tag{5}$$

(here $K$ is a given subset in $H$; usually, $K$ is a class of smooth functions in a functional Hilbert space).

For the case in which $\Phi$ is a complete o.n.s. in $H$, the quantity (4) coincides with the best $m$-term approximation of the element $f$ in the system $\Phi$, i.e., with the quantity

$$e_m(f, \Phi) = \inf_{\{c_j\}_{j \in \Lambda}, \Lambda \subset \mathbb{N}, \sharp\Lambda \le m} \left\| f - \sum_{j \in \Lambda} c_j \varphi_j \right\|. \tag{6}$$

However, for general frames, estimates of the quantities (6) may turn out to be of little value. It is easy to give an example of a frame $\Phi_0 = \{\varphi_j^0\}$ for which the set $\{\lambda\varphi_j^0\}_{\lambda \in \mathbb{R}, j \in \mathbb{N}}$ is dense in $H$, i.e., $e_1(f, \Phi_0) = 0$ for any element $f \in H$.

The goal of this paper is to show that the geometric approach proposed in [5] in order to obtain best $m$-term approximations can also be applied to study the quantities (5) for frames of general form.

Let us introduce some notation: for $x = \{x_k\}_{k=1}^N \subset \mathbb{R}^N$, we set

$$\|x\|_2 = \|x\|_{l_2^N} = \left( \sum_{k=1}^N x_k^2 \right)^{1/2}, \qquad \|x\|_\infty = \|x\|_{l_\infty^N} = \max_{1 \le i \le N} |x_i|, \qquad B_\infty^N = \{x \in \mathbb{R}^N : \|x\|_\infty \le 1\}$$

is an $N$-dimensional cube and $W_\infty^N$ is the set of vertices of this cube. We also consider the $N$-dimensional cubes embedded in Hilbert space $H$, i.e., sets of form

$$B_N = \left\{ f : f = \sum_{k=1}^N a_k \psi_k,\ |a_k| \le 1,\ k = 1, \dots, N \right\}, \qquad \{\psi_k\}_{k=1}^N \text{ is an o.n.s.,} \tag{7}$$

as well as the set of vertices of the cube (7):

$$W_N = \left\{ f : f = \sum_{k=1}^N \varepsilon_k \psi_k,\ \varepsilon_k = \pm 1 \right\}, \qquad \{\psi_k\}_{k=1}^N \text{ is an o.n.s..} \tag{8}$$

Finally, we denote by $E_m$, $m = 1, 2, \dots$, the set of all subsets of the set of positive integers with number of elements $\le m$.

**Theorem 1.** *Suppose that $\Phi = \{\varphi_j\}_{j=1}^\infty \subset H$ is a frame with condition $\kappa = \kappa(\Phi)$, and suppose that $\delta > 0$. If $W_N$ is the set of vertices of an $N$-dimensional cube (see (8)), then for $m \le c(\delta)N$ the following inequality holds*:

$$\sigma_m(W_N, \Phi) \ge (1 - \kappa^{1/2}\delta) \cdot N^{1/2},$$

*where $c(\delta) \in (0, 1)$ is a positive constant depending only on $\delta$.*

The following result was obtained in [5].

**Proposition 1.** *Suppose that in* $\mathbb{R}^N$, $N=1,2,\dots$, *we are given a sequence of vectors* $\{e_i\}_{i=1}^{\infty}$, *with*

$$\sum_{i=1}^{\infty}\|e_i\|_2^2 \le 1, \qquad \max_{1\le i<\infty}\|e_i\|_2=\rho.$$

*Suppose that we are also given a number* $\delta>0$. *If* $m\rho^2\le c(\delta)$, *then there exists a vertex* $w\in W_\infty^N$ *for which*

$$\sup_{\Lambda\in E_m}\biggl(\sum_{i\in\Lambda}(w,e_i)^2\biggr)^{1/2}\le\delta$$

(*here* $c(\delta)$, $0<c(\delta)<1$, *is a constant depending only on* $\delta$).

**Remark.** In [5], the case $\delta=1/2$ was treated; for other $\delta$, the assertion can be verified in a similar way. In addition, it was assumed in [5] that $\sum\|e_i\|_2^2=1$; the difference between this assumption and that given in the statement of Proposition 1 is unessential.

**Proof of Theorem 1.** Suppose that $\widetilde{\Phi}=\{\widetilde{\varphi}_j\}_{j=1}^{\infty}$ is the frame dual to $\Phi$, $L$ is the linear hull of the functions $\psi_k$, $k=1,\dots,N$ (see (8)), and

$$\tilde e_j=\pi_L(\widetilde{\varphi}_j), \qquad j=1,2,\dots, \tag{9}$$

where $\pi_L\colon H\to L$ is the operator of orthogonal projection onto $L$. In view of the fact that $\{\psi_k\}_{k=1}^{N}$ is an orthonormal basis in $L$, we have

$$\sum_{j=1}^{\infty}\|\tilde e_j\|^2=\sum_{k=1}^{N}\sum_{j=1}^{\infty}(\widetilde{\varphi}_j,\psi_k)^2. \tag{10}$$

Since $\widetilde{\Phi}$ is a frame with bounds $B^{-1}$, $A^{-1}$, where $A$, $B$ are bounds of the frame $\Phi$, it follows that for $k=1,2,\dots,N$ we have the estimate

$$B^{-1}=B^{-1}\|\psi_k\|^2\le\sum_{j=1}^{\infty}(\widetilde{\varphi}_j,\psi_k)^2\le A^{-1}\|\psi_k\|^2=A^{-1}. \tag{11}$$

From (10) and (11) we obtain

$$B^{-1}N\le\sum_{j=1}^{\infty}\|\tilde e_j\|^2\le A^{-1}N,$$

i.e.,

$$\frac{1}{\kappa(\Phi)}\le\sum_{j=1}^{\infty}\|N^{-1/2}A^{1/2}\tilde e_j\|^2\le 1. \tag{12}$$

In addition,

$$\|\tilde e\|\le\|\widetilde{\varphi}_j\|\le A^{-1/2}, \qquad j=1,2,\dots$$

(the last inequality follows from the definition of frames), and hence

$$\max_j\|N^{-1/2}A^{1/2}\tilde e_j\|\equiv\rho\le N^{-1/2}. \tag{13}$$

Further, for arbitrary numbers $m\in\mathbb{N}$ and element $z\in L$, using (2), we obtain

$$\sigma_m(z,\Phi)=\inf_{\Lambda\in E_m}\biggl\|z-\sum_{j\in\Lambda}(z,\widetilde{\varphi}_j)\varphi_j\biggr\|=\inf_{\Lambda\in E_m}\biggl\|z-\sum_{j\in\Lambda}(z,\tilde e_j)\varphi_j\biggr\|$$
$$\ge\inf_{\Lambda\in E_m}\biggl(\|z\|-\biggl\|\sum_{j\in\Lambda}(z,\tilde e_j)\varphi_j\biggr\|\biggr)\ge\|z\|-B^{1/2}\sup_{\Lambda\in E_m}\biggl(\sum_{j\in\Lambda}(z,\tilde e_j)^2\biggr)^{1/2}.$$

Therefore, for any point $w \in W_N$, the following inequality holds:

$$\sigma_m(w, \Phi) \geq N^{1/2}\left(1 - \left(\frac{B}{A}\right)^{1/2} \sup_{\Lambda \in E_m} \left(\sum_{j \in \Lambda}\left(w, \left(\frac{A}{N}\right)^{1/2} \tilde{e}_j\right)^2\right)^{1/2}\right).$$

Combining this assertion with (13), we prove the existence of a point $w_0 \in W_N$ for which for $m \leq c(\delta)N$ we have the estimate

$$\sigma_m(w_0, \Phi) \geq N^{1/2}\left(1 - \left(\frac{B}{A}\right)^{1/2} \delta\right) = (1 - \delta\kappa^{1/2})N^{1/2}.$$

Theorem 1 is proved. □

If $\Phi$ is a Riesz basis, then for $f \in H$ we have the inequality

$$\sigma_m(f, \Phi) \leq \kappa^{1/2} e_m(f, \Phi).$$

Therefore, Theorem 1 yields the following result.

**Corollary 1.** *Suppose that $\Phi = \{\varphi_j\}$ is a Riesz basis in $H$ and $W_N$ is the set of vertices of an $N$-dimensional cube (see (8)). Then for $m \leq c(\kappa)N$ we have the inequality*

$$\max_{w \in W_N} e_m(w, \Phi) \geq 0.9 \cdot N^{1/2}.$$

Applications based on Theorem 1 and Corollary 1 are obtained in the same way as similar results in [5]. In particular, suppose that $H = L^2(0,1)$ and $H^{r,\alpha}$ is the following class of smooth functions:

$$H^{r,\alpha} = \left\{f \in L^2(0,1) \colon \|f\|_C + \|f^{(r)}\|_C \leq 1,\ \frac{|f^{(r)}(x) - f^{(r)}(y)|}{|x-y|^\alpha} \leq 1\right\},$$
$$r = 0, 1, \dots, \qquad \alpha \in (0, 1].$$

We have the following result.

**Corollary 2.** *For $r = 0, 1, \dots$ and $\alpha \in (0,1]$, in the case of an arbitrary frame $\Phi$ with condition $\kappa$ the following inequalities hold:*

$$\sup_{f \in H^{r,\alpha}} \sigma_m(f, \Phi) \geq c(r, \alpha, \kappa) \cdot m^{-(r+\alpha)} > 0, \qquad m = 1, 2, \dots .$$


## ACKNOWLEDGMENTS

The research of the first author was supported by the Russian Foundation for Basic Research under grant no. 02-01-00315.


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