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Lower Bounds for n-Term Approximations

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Let X be a real normed space, let $\Phi \subset X$ be a subset in X (a dictionary), and let $f \in X$.

Definition. The *n-term approximation* of an element f with respect to the dictionary Φ is defined as the expression

$$e_n(f, \Phi, X) \equiv \inf_{P \in \Sigma_n} \|f - P\|_X, \tag{1}$$

where for $n = 1, 2, \dots$

$$\Sigma_n \equiv \left\{ \sum_{j=1}^{n} a_j x_j, \ a_j \in \mathbb{R}, \ x_j \in \Phi \right\}.$$

Further, if K is a subset of X, then

$$e_n(K, \Phi, X) \equiv \sup_{f \in K} e_n(f, \Phi, X). \tag{2}$$

Estimates of the expressions (2) for different K, Φ, and X are of importance in both theory and practice (for more details, see [1, 2]).

In the present paper, we restrict our consideration to the case in which $X = L^2(\Omega)$ and Φ is a complete orthonormal system (o.n.s.) in X. In this case the expression (1) was introduced by S. B. Stechkin [3]. As is easy to see, it is equal to

$$e_n(f, \Phi, L^2(\Omega)) = \left\{ \sum_{k \ge n+1} [c_k^*(f)]^2 \right\}^{1/2}, \tag{3}$$

where $\{c_k^*(f)\}$ is a nonincreasing rearrangement of the sequence of absolute values of the Fourier coefficients for the function f with respect to the system Φ.

In the author's work [4], a geometric scheme for obtaining the lower bounds for the variables (2) was proposed for the case in which Φ is an orthonormal system. More precisely, it was shown in [4] that the embedding in K of the set of vertices of the $2n$-dimensional cube, i.e., of the set Q of the form

$$Q = \left\{ \sum_{i=1}^{2n} \varepsilon_i \psi_i, \ \varepsilon_i = \pm 1, \ L^2(\Omega) \supset \{\psi_i\}_{i=1}^{2n} \text{ is an o.n.s.} \right\}, \tag{4}$$

implies the inequality

$$e_n(K, \Phi, L^2(\Omega)) \ge c n^{1/2}, \qquad c > 0. \tag{5}$$

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To use this result in applications, it suffices to solve the problem of inscribing the largest possible cube in a given function class K (i.e., of finding, for a given n, a sufficiently large number λ and a set Q of the form (4) such that $\lambda \cdot Q \subset K$). This problem can be solved easily for the classical function classes K. As a result, this allows one, in several cases, to find the lower estimates for n-term approximations that are sharp in order. Various generalizations and analogs of the estimate (5) were established in [5, 6].

In 1993 S. V. Konyagin posed the problem of estimating the expressions (2) in the case where $X = L^2(I^d)$, Φ is an o.n.s. in X, and K is the set of characteristic functions of convex subsets of the unit cube $I^d \subset \mathbb{R}^d$. Already for $d = 1$ the problem remained unsolved. For $d = 1$ this problem is, in fact, reduced to finding bounds for (2) for the "one-parametric family"

$$K = \mathbb{X} \equiv \{\chi_t\}_{t\in[0,1]}, \qquad \chi_t(x) = \begin{cases} 0 & \text{if } 0 \le x < t, \\ 1 & \text{if } t \le x \le 1. \end{cases} \tag{6}$$

Konyagin draw the author's attention to the problem of obtaining lower bounds for n-term approximations of the family (6), by pointing out that it is possible to obtain upper bounds for these expressions. More precisely, if $\Phi = H$ is a Haar system, then

$$e_n(\mathbb{X}, H, L^2(0,1)) \le C\, 2^{-n/2}. \tag{7}$$

To verify (7), it suffices to use the standard estimate for the error of the L^2-approximation of the functions χ_t $(0 \le t \le 1)$ by partial sums of the Fourier–Haar series (e.g., see [7, p. 75]) and to take into account the fact that each block of the Fourier–Haar series of the function χ_t contains only one nonzero coefficient.

Since the "set of the family $\mathbb{X}$ is extremely small," it is impossible to use the above geometric scheme for finding the lower bounds for n-term approximations of this family. It turns out that, instead of this scheme, the technique of the theory of general orthogonal series can be used.

Theorem 1. *There exists an absolute positive constant C such that for an arbitrary orthonormal system $\Phi \subset L^2(0,1)$ the inequality*

$$e_n(\mathbb{X}, \Phi, L^2(0,1)) \ge C^{-n}$$

holds for $n = 1, 2, \dots$

Remark 1. The problem of finding the exact value of the constant C in Theorem 1 remains open. However, it follows from the proof that this constant is "not too large."

As is shown below, for uniformly bounded o.n.s. Φ, the lower bound for $e_n(\mathbb{X}, \Phi, L^2(0,1))$ can be improved significantly.

Theorem 2. *If Φ is a uniformly bounded complete o.n.s: $\Phi = \{\varphi_j\}_{j=1}^{\infty} \subset L^2(0,1)$,*

$$\|\varphi_j\|_{L^\infty(0,1)} \le M, \qquad j = 1, 2, \dots,$$

then for $n = 1, 2, \dots$ we have

$$e_n(\mathbb{X}, \Phi, L^2(0,1)) \ge \frac{C_M}{\sqrt{n}} > 0. \tag{8}$$

Remark 2. The accuracy of the estimate (8) can be verified by using the special example of trigonometric systems: if $\Phi = T$ is a trigonometric system, then $e_n(\mathbb{X}, T, L^2) \le Cn^{-1/2}$.

In the proofs of Theorems 1 and 2 the results of the author's paper [8] play an essential role. In particular, the proof of Theorem 1 is based on the following inequality (in fact, this is a special case of Theorem 1 in [8]).

Lemma. *Suppose that* $N = 1, 2, \dots$ *and* $\{\varphi_j\}_{j\in\Lambda}$ *is an arbitrary normed system of functions in* $L^2(0,1)$. *Suppose also that the representation*

$$\chi_{k/N} = \sum_{j\in\Lambda} a_{k,j}\varphi_j + \Delta_k, \qquad \|\Delta_k\|_{L^1} \le \frac{1}{N}$$

holds for $k = 1, 2, \dots, N$. *Then*

$$\sum_{j\in\Lambda}\left(\frac{1}{N}\sum_{k=1}^{N} a_{k,j}^2\right)^{1/2} \ge B\ln N,$$

where $B > 0$ *is an absolute constant.*

The proof of Theorem 2 is based on an argument close to that used in establishing the estimate (9) in the paper [8]. In conclusion, we note that the first lower bounds for the coefficients of the expansion of functions from the family $\mathbb{X}$ (see (6)) in the series with respect to general uniformly bounded o.n.s. were obtained by S. V. Bochkarev [9].

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REFERENCES

1. R. A. DeVore, *Acta Numer.*, **7** (1998), 51–150.
2. V. N. Temlyakov, *Nonlinear Methods of Approximation*, IMI Preprint 2001:09, University of South Carolina, 2001.
3. S. B. Stechkin, *Dokl. Akad. Nauk SSSR* [*Soviet Math. Dokl.*], **102** (1955), no. 1, 37–40.
4. B. S. Kashin, *Trudy Mat. Inst. Steklov* [*Proc. Steklov Inst. Math.*], **172** (1985), 187–191.
5. B. S. Kashin and V. N. Temlyakov, *Mat. Zametki* [*Math. Notes*], **56** (1994), no. 5, 57–86.
6. D. L. Donoho, *Ann. Statist.*, **25** (1997), no. 5, 1870–1911.
7. B. S. Kashin and A. A. Saakyan, *Orthogonal Series* [in Russian], 2nd ed., AFTs, Moscow, 1999.
8. B. S. Kashin, *Sibirsk. Mat. Zh.* [*Siberian Math. J.*], **18** (1977), no. 1, 122–131.
9. S. V. Bochkarev, *Mat. Zametki* [*Math. Notes*], **15** (1974), no. 3, 363–370.

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