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Moscow Institute of Engineering Physics (MFTI)
E-mail address: bulinski@math.mipt.ru

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On the Possibility of Generalizing "Correction" Theorems

B. S. Kashin

Key words: Men'shov correction theorem, multiple orthogonal series, majorants of partial sums, Fourier series.

In [1] the author established an analog of the well-known Men'shov "correction" theorem for arbitrary discrete complete orthonormal systems. In [2] in the course of investigation of another problem of orthogonal series theory, properties of a majorant of partial sums of an orthogonal series with respect to a certain family of subsets were considered. To be more precise, let I be an arbitrary set of indices with a finite number $\#I$ of elements, and let Ω be a family of its subsets. Further, let $\{\varphi_\alpha\}_{\alpha\in I}$ be an orthonormal system of functions defined on a measure space (X, Σ, μ), $\mu(X) = 1$ and "numbered" by elements of I. For an orthogonal expansion

$$f(x) \sim \sum c_\alpha \varphi_\alpha(x),$$

let the "Ω-majorant of partial sums" be defined by the relation

$$S^*_\Omega f(x) = \sup_{\Lambda\in\Omega} \Big|\sum_{\alpha\in\Lambda} c_\alpha \varphi_\alpha(x)\Big|. \tag{1}$$

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In the "usual" case, we have $I = I_N \equiv \{1, 2, \dots, N\}$ and

$$\Omega = \{I_k : 1 \le k \le N\}.$$

For d-fold multiple orthogonal series, the study of the majorant of rectangular partial sums is reduced to (1) with $I = (I_N)^d$ and

$$\Omega = \{I_{k_1} \times \dots \times I_{k_d} : 1 \le k_s \le N,\ s = 1, 2, \dots, d\}.$$

In this paper, which supplements [1], we show that, under certain conditions on the family Ω and the system $\{\varphi_\alpha\}$, an arbitrary bounded function f can be corrected to a function with a bounded Ω-majorant. In what follows, we only consider finite orthonormal systems $\Phi_N = \{\varphi_k^N(x)\}_{k=1}^N$, $N = 1, 2, \dots$, satisfying the following three conditions:

(i) $\|\varphi_k^N(x)\|_{L^\infty(X,\mu)} \le M_1$, $1 \le k \le N$, $N = 1, 2, \dots$;

(ii) for $N = 1, 2, \dots$, there exists a set of points $\{x_j\}_{j=1}^N \subset X$ possessing the property that the matrix $\{N^{-1/2}\varphi_k^N(x_j)\}_{k,j=1}^N$ is orthonormal;

(iii) for $N = 1, 2, \dots$, there exists a set of points $\{z_j\}_{j=1}^Q$ with $Q \le M_2 N$ such that any polynomial

$$P = \sum_{k=1}^N a_k \varphi_k^N(x) \tag{2}$$

satisfies the inequality

$$\|P\|_{L^\infty(X,\mu)} \le M_3 \max_{1 \le j \le Q} |P(z_j)|$$

(here and in what follows, M_i, $i = 1, 2, \dots$, are some absolute constants).

Note that condition (iii), the property of *quasimatrixness* (or *quasidiscreteness*) of the systems Φ_N, which was introduced in [3], naturally arises in a number of problems of analysis. Besides, note that conditions (i)–(iii) are obviously true if Φ_N, $N = 1, 2, \dots$, are uniformly bounded discrete systems (i.e., $X = (0, 1)$ and $\varphi_k^N(x) = \text{const}$ for $x \in ((i-1)/N, 1/N)$, $1 \le i \le N$).

In what follows, we denote by $P_N(x, \{y_k\}_{k=1}^N)$ the (unique, in view of (ii)) polynomial of the form (2) such that

$$P(x_k) = y_k, \qquad 1 \le k \le N.$$

Let Ω_N, $N = 1, 2, \dots$, be some families of subsets of the initial segments I_N of the set of positive integers. We shall introduce a restriction on the "complexity" of the sequence Ω_N. Namely, let us assume that

($*$) for a some $\rho < 1$ and for $N = 1, 2, \dots$, there exist families Δ_s, $\varnothing \in \Delta_s$, $s = 1, \dots, s_0$, with $\#\Delta_s \le M_4 \exp\exp s^\rho$ such that each set $\omega \in \Omega_N$ is representable in the form

$$\omega = \bigcup_{s=1}^{s_0} E_s, \qquad E_s \in \Delta_s, \quad \#E_s \le \frac{N}{2^s}, \quad E_s \cap E_{s'} = \varnothing \quad \text{for } s \ne s'.$$

Remark 1. Conditions on the families of sets Ω_N close to ($*$) play an important role in the theory of communication complexity (see, for instance, [4]).

Below a *correction of a polynomial of the form* (2) is understood as its replacement by a polynomial $\widetilde{P}$ that differs from P only on a small portion of points x_k, $1 \le k \le N$. Clearly, for discrete systems such a correction amounts to changing the function on a set of small measure.

The method of [1] yields the following result.

Theorem. *Suppose that for all $N = 1, 2, \dots$, $\Phi_N = \{\varphi_k^N(x)\}_{k=1}^N$ is an orthonormal system satisfying conditions (i)–(iii) and Ω_N is a family of subsets of I_N with property $(*)$. Then for each $\varepsilon > 0$ there exists a constant C_ε depending only on ε and on ρ and M_i, $1 \le i \le 4$, such that for all $N = 1, 2, \dots$ and for any set $\{y_j\}_{j=1}^N$ with $|y_j| \le 1$, $1 \le j \le N$, there exists a set $\{\widetilde{y}_j\}_{j=1}^N$ satisfying*

$$\#\{j : y_j \ne \widetilde{y}_j\} \le \varepsilon N \qquad \text{and} \qquad \|S_\Omega^* P(x, \{\widetilde{y}_j\})\|_{L^\infty} \le C_\varepsilon.$$

Remark 2. The technique of the proof of this theorem also allows us to consider majorants whose definition admits families of subsets Ω depending on x (i.e., $\Omega = \Omega(x)$).

As was mentioned in [2], the estimates of the majorants S_Ω^* can be applied to the study of majorants of rectangular partial sums of a multiple orthogonal series. In particular, let

$$F_{d,N} = \{\bar{\jmath} = (j_1, \dots, j_d) \in \mathbb{Z}^d : -N \le j_s \le N,\ s = 1, \dots, d\},$$

and

$$\Gamma_{d,N} = \left\{ z_{\bar{\jmath}} = \left(\frac{2\pi j_1}{2N+1}, \dots, \frac{2\pi j_d}{2N+1} \right),\ \bar{\jmath} = (j_1, \dots, j_d) \in F_{d,N} \right\}$$

for $d = 1, 2, \dots$ and $N = 1, 2, \dots$. Denote by $T(x, \{y_{\bar{\jmath}}\}_{\bar{\jmath} \in F_{d,N}})$ a trigonometric polynomial t of degree $\le N$ in each of the d variables such that $t(z_{\bar{\jmath}}) = y_{\bar{\jmath}}$ for $z_{\bar{\jmath}} \in \Gamma_{d,N}$. The theorem readily implies the following statement:

Corollary. *For $\varepsilon > 0$ and $d = 2, 3, \dots$ there exists a constant $C_{\varepsilon,d}$ such that for each set $\{y_{\bar{\jmath}}\}$ with $|y_{\bar{\jmath}}| \le 1$, $\bar{\jmath} \in F_{d,N}$, there exists a set $\{\widetilde{y}_{\bar{\jmath}}\}$, $\bar{\jmath} \in F_{d,N}$, for which*

$$\#\{\bar{\jmath} \in F_{d,N} : y_{\bar{\jmath}} \ne \widetilde{y}_{\bar{\jmath}}\} \le \varepsilon (2N+1)^d$$

and

$$\max_{\bar{r}} \left\| S_{\bar{r}}(T(x, \{\widetilde{y}_{\bar{\jmath}}\})) \right\|_\infty \le C_{\varepsilon,d}, \tag{3}$$

where

$$S_{\bar{r}}(f) = \sum_{\bar{k} : -r_s \le k_s \le r_s} \widehat{f}(k) e^{i\bar{k}x}$$

is a rectangular partial sum of the trigonometric Fourier series for a function $f(x)$ of d variables and the maximum in (3) is taken over all the sets $r = \{r_s\} \in \mathbb{Z}^d$ with $0 \le r_s \le N$ for $s = 1, \dots, d$.

Remark 3. The statement of the corollary for $d = 1$ was proved in [5].

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V. A. Steklov Mathematics Institute, Russian Academy of Sciences
E-mail address: kashin@ipsun.ras.ru

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