The authors wish to express their gratitude to Professor A. M. Chebotarev for many stimulating discussions.

This research was partially supported by the State Committee of the Russian Federation for Higher Education under grant No. 96-1.7-100.

References

MOSCOW INSTITUTE OF ENGINEERING PHYSICS (MFTI)
E-mail address: bulinski@math.mipt.ru

Translated by M. A. Shishkova

On the Possibility of Generalizing “Correction” Theorems

B. S. Kashin

KEY WORDS: Men’shov correction theorem, multiple orthogonal series, majorants of partial sums, Fourier series.

In [1] the author established an analog of the well-known Men’shov “correction” theorem for arbitrary discrete complete orthonormal systems. In [2] in the course of investigation of another problem of orthogonal series theory, properties of a majorant of partial sums of an orthogonal series with respect to a certain family of subsets were considered. To be more precise, let I be an arbitrary set of indices with a finite number |I| of elements, and let Ω be a family of its subsets. Further, let \{φ_α\}_α∈I be an orthonormal system of functions defined on a measure space \((X, Σ, μ)\), \(μ(X) = 1\) and “numbered” by elements of I. For an orthogonal expansion

\[ f(x) \sim \sum c_αφ_α(x), \]

let the “Ω-majorant of partial sums” be defined by the relation

\[ S^*_Ωf(x) = \sup_{\Lambda \in Ω} \left| \sum_{α \in Λ} c_αφ_α(x) \right|. \]
In the "usual" case, we have $I = I_N \equiv \{1, 2, \ldots, N\}$ and
\[ \Omega = \{I_k : 1 \leq k \leq N\}. \]

For $d$-fold multiple orthogonal series, the study of the majorant of rectangular partial sums is reduced to (1) with $I = (I_N)^d$ and
\[ \Omega = \{I_{k_1} \times \cdots \times I_{k_d} : 1 \leq k_s \leq N, \ s = 1, 2, \ldots, d\}. \]

In this paper, which supplements [1], we show that, under certain conditions on the family $\Omega$ and the system $\{\varphi_k\}$, an arbitrary bounded function $f$ can be corrected to a function with a bounded $\Omega$-majorant. In what follows, we only consider finite orthonormal systems $\Phi_N = \{\varphi^N_k(x)\}_{k=1}^N$, $N = 1, 2, \ldots$, satisfying the following three conditions:
\begin{enumerate}
  \item $\|\varphi_k^N(x)\|_{L^\infty(X, \mu)} \leq M_1, \ 1 \leq k \leq N, \ N = 1, 2, \ldots$;
  \item for $N = 1, 2, \ldots$, there exists a set of points $\{x_j\}_{j=1}^N \subset X$ possessing the property that the matrix $\{N^{-1/2}\varphi_k^N(x_j)\}_{k,j=1}^N$ is orthonormal;
  \item for $N = 1, 2, \ldots$, there exists a set of points $\{x_j\}_{j=1}^Q$ with $Q \leq M_2N$ such that any polynomial $P = \sum_{k=1}^N a_k\varphi_k^N(x)$ satisfies the inequality
    \[ \|P\|_{L^\infty(X, \mu)} \leq M_3 \max_{1 \leq j \leq Q} |P(x_j)| \]
    (here and in what follows, $M_i, \ i = 1, 2, \ldots$, are some absolute constants).
\end{enumerate}

Note that condition (iii), the property of quasinatrixness (or quasidiscreteness) of the systems $\Phi_N$, which was introduced in [3], naturally arises in a number of problems of analysis. Besides, note that conditions (i)–(iii) are obviously true if $\varphi_k^N(x) = \text{const}$ for $x \in ((i-1)/N, 1/N), \ 1 \leq i \leq N$.

In what follows, we denote by $P_N(x, \{y_k\}_{k=1}^N)$ the (unique, in view of (ii)) polynomial of the form (2) such that
\[ P(x_k) = y_k, \quad 1 \leq k \leq N. \]

Let $\Omega_N, \ N = 1, 2, \ldots$, be some families of subsets of the initial segments $I_N$ of the set of positive integers. We shall introduce a restriction on the "complexity" of the sequence $\Omega_N$. Namely, let us assume that
\begin{enumerate}
  \item for a some $\rho < 1$ and for $N = 1, 2, \ldots$, there exist families $\Delta_s, \ 0 \in \Delta_s, \ s = 1, \ldots, s_0$, with
    \[ \#\Delta_s \leq M_4 \exp \rho^s \] such that each set $\omega \in \Omega_N$ is representable in the form
    \[ \omega = \bigcup_{s=1}^{s_0} E_s, \quad E_s \in \Delta_s, \quad \#E_s \leq \frac{N}{2^s}, \quad E_s \cap E_{s'} = \emptyset \quad \text{for} \ s \neq s'. \]
\end{enumerate}

Remark 1. Conditions on the families of sets $\Omega_N$ close to (*) play an important role in the theory of communication complexity (see, for instance, [4]).

Below a correction of a polynomial of the form (2) is understood as its replacement by a polynomial $\tilde{P}$ that differs from $P$ only on a small portion of points $x_k, \ 1 \leq k \leq N$. Clearly, for discrete systems such a correction amounts to changing the function on a set of small measure.

The method of [1] yields the following result.
Theorem. Suppose that for all $N = 1, 2, \ldots$, $\Phi_N = \{\varphi^N_k(z)\}_{k=1}^N$ is an orthonormal system satisfying conditions (i)–(iii) and $\Omega_N$ is a family of subsets of $I_N$ with property (*). Then for each $\varepsilon > 0$ there exists a constant $C_\varepsilon$ depending only on $\varepsilon$ and on $\rho$ and $M_i$, $1 \leq i \leq N$, such that for all $N = 1, 2, \ldots$ and for any set $\{y_j\}_{j=1}^N$ with $|y_j| \leq 1$, $1 \leq j \leq N$, there exists a set $\{\tilde{y}_j\}_{j=1}^N$ satisfying

$$\#\{j : y_j \neq \tilde{y}_j\} \leq \varepsilon N \quad \text{and} \quad \|S^*_N P(x, \{\tilde{y}_j\})\|_{L^\infty} \leq C_\varepsilon.$$ 

Remark 2. The technique of the proof of this theorem also allows us to consider majorants whose definition admits families of subsets $\Omega$ depending on $x$ (i.e., $\Omega = \Omega(x)$).

As was mentioned in [2], the estimates of the majorants $S^*_N$ can be applied to the study of majorants of rectangular partial sums of a multiple orthogonal series. In particular, let

$$F_{d,N} = \{\bar{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d : -N \leq j_s \leq N, \ s = 1, \ldots, d\},$$

and

$$\Gamma_{d,N} = \left\{z_j = \left(\frac{2\pi j_1}{2N+1}, \ldots, \frac{2\pi j_d}{2N+1}\right), \bar{j} = (j_1, \ldots, j_d) \in F_{d,N}\right\}$$

for $d = 1, 2, \ldots$ and $N = 1, 2, \ldots$. Denote by $T(x, \{y_j\}_{j \in F_{d,N}})$ a trigonometric polynomial $t$ of degree $\leq N$ in each of the $d$ variables such that $t(z_j) = y_j$ for $z_j \in \Gamma_{d,N}$. The theorem readily implies the following statement:

Corollary. For $\varepsilon > 0$ and $d = 2, 3, \ldots$ there exists a constant $C_{\varepsilon,d}$ such that for each set $\{y_j\}$ with $|y_j| \leq 1$, $\bar{j} \in F_{d,N}$, there exists a set $\{\tilde{y}_j\}$, $\bar{j} \in F_{d,N}$, for which

$$\#\{\bar{j} \in F_{d,N} : y_j \neq \tilde{y}_j\} \leq \varepsilon(2N + 1)^d$$

and

$$\max_F\|S_F(T(x, \{\tilde{y}_j\}))\|_{L^\infty} \leq C_{\varepsilon,d}, \quad (3)$$

where

$$S_F(f) = \sum_{\bar{k} : -r_s \leq k_s \leq r_s} \hat{f}(k) e^{i\bar{k}x}$$

is a rectangular partial sum of the trigonometric Fourier series for a function $f(x)$ of $d$ variables and the maximum in (3) is taken over all the sets $r = \{r_s\} \in \mathbb{Z}^d$ with $0 \leq r_s \leq N$ for $s = 1, \ldots, d$.

Remark 3. The statement of the corollary for $d = 1$ was proved in [5].

The author is indebted to A. A. Razborov for useful discussions.

This research was supported by the Russian Foundation for Basic Research under grants Nos. 96-01-00094 and No. 96-15-96102 and by the INTAS Foundation under grant No. 93-1376.

References


Translated by V. N. Dubrovsky