The integrals in (7) have a singularity at the point $p = 0$. Let us divide the region of integration with respect to p and k into the singular region ($p < h^{\varepsilon}$, $\varepsilon > 1/2$) and the nonsingular region ($p > h^{\varepsilon}$). By introducing the corresponding partition of unity, we express the original integral as the sum of integrals over the singular and nonsingular regions. The integral over the singular region is an $O(h)$ summand as $h \to 0$. In the nonsingular region, an application of the stationary phase method leads to the same result as for a flat bottom with the only difference that now H is a function of $(x_0 + q/kz_0)$. The other terms can be studied in a similar way; note that in studying $\mathcal{I}^{\perp}$, one must separately consider a neighborhood of the set $k^2 - (c^2 - 1)p^2 = 0$ exactly as we considered a neighborhood of the point $p = 0$ above. Since, for a flat bottom, the theorem is valid for any H, we obtain an identity independent of H. By thoroughly analyzing this identity and by carrying out the appropriate estimates, we complete the proof of the theorem. □

The theorem allows us to express the solution of the Cauchy problem (1)–(3) with the initial conditions (3a) as the sum of two types of solutions: the first term (corresponding to longitudinal and transverse waves) is of the same form as in [1] with coefficients A_1^-, B_1^-, A_2^-, and B_2^- determined by (5), and the second summand (corresponding to surface waves) is determined by the formulas of [4] via Maslov's canonical operator.

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Estimate of Approximate Characteristics for Classes of Functions with Bounded Mixed Derivative

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This note is a supplement to our recent paper [1], where we estimated [1, §2] the volume of the set of Fourier coefficients of polynomials from the unit L_1-ball in the subspace $T(\Lambda)$ spanned by an arbitrary finite set of harmonics $e^{i(k,x)}$, $k \in \Lambda \subset \mathbb{Z}^d$, and, as a consequence, obtained lower bounds for the entropy numbers $\varepsilon_m(W^r_{\infty,0}, L_1)$ and the Kolmogorov widths $d_m(W^r_{\infty,\alpha}, L_p)$ of classes $W^r_{\infty,\alpha}$ of functions in d variables ($d > 1$) with bounded mixed derivative. In the present paper, we use the constructions from [1] and the approach suggested by É. S. Belinskii in [2] (unfortunately, this paper was unknown to us when we were writing the paper [1]) to complete the determination of the orders of $\varepsilon_m(W^r_{\infty,\alpha}, L_p)$ and $d_m(W^r_{\infty,\alpha}, L_p)$ for all $p \in [1, \infty)$ and $r > 0$.

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We first present the results concerning ε-entropy. For all $1 < q, p < \infty$, the orders of $\varepsilon_m(W^r_{q,\alpha}, L_p)$ were obtained in [3, 4] (where the history of the problem is also reviewed). Temlyakov [4] proved that

$$\varepsilon_m(W^r_{q,\alpha}, L_1) \gg m^{-r}(\log m)^{r(d-1)} \tag{1}$$

for all $1 \le q < \infty$ and $r > 0$. Belinskii [2] established the estimate

$$\varepsilon_m(W^r_{\infty,\alpha}, L_1) \gg m^{-r}(\log m)^{r(d-1)}, \tag{2}$$

which sharpens (1) for $r > 1/2$: The authors proved the estimate (2) for $\alpha = 0$ and for all $r > 0$ in [1]. Here we prove the following assertion.

Theorem 1. *For all $\alpha \in \mathbb{R}^d$ and $r > 0$ we have*

$$\varepsilon_m(W^r_{\infty,\alpha}, L_1) \gg m^{-r}(\log m)^{r(d-1)}. \tag{2'}$$

Proof. We apply the same constructions and notation as those in the proof of Theorem 2.2 in [1] (see also [1, p. 70]). In that paper, for $m = 1, 2, \dots$ and for the smallest $n = n(m)$ such that $|D_n| \ge m$ (note that $2^n \cdot n^{d-1} \asymp m$ as $m \to \infty$) we constructed polynomials $t_j \in T(D_n)$, $j = 1, \dots, 2^m$, with the following properties:

$$E^{\perp}_{D_n}(t_j)_\infty \le 1, \qquad j = 1, \dots, 2^m, \quad \|t_i - t_j\|_2 \ge \frac{1}{2}\varepsilon_0 > 0, \quad i \ne j.$$

By choosing $t_j^{\perp} \in T(D_n)^{\perp}$ so that $\|t_j - t_j^{\perp}\|_\infty \le 2$ and by setting

$$\varphi_j = \frac{1}{2}(t_j - t_j^{\perp}), \quad f_j = \varphi_j * F_r(x, \alpha), \qquad j = 1, \dots, 2^m,$$

we obtain a collection of functions $f_j \in W^r_{\infty,\alpha}$ such that

$$\|f_i - f_j\|_2 \ge \frac{1}{2}\|t_i - t_j\|_2 \cdot 2^{-rn} \ge \frac{1}{4}\varepsilon_0 \cdot 2^{-rn}, \qquad i \ne j.$$

Consequently ($r > 0$),

$$\varepsilon_m(W^r_{\infty,\alpha}, L_2) \gg m^{-r}(\log m)^{r(d-1)}. \tag{3}$$

Following [2], we derive the assertion of Theorem 1 from the estimate (3) by combining the interpolation inequality [5, p. 189 of the Russian translation]

$$\varepsilon_{2m-1}(W^r_{\infty,\alpha}, L_2) \le 2\varepsilon_m(W^r_{\infty,\alpha}, L_1)^{\frac{p-2}{2(p-1)}}\varepsilon_m(W^r_{\infty,\alpha}, L_p)^{\frac{p}{2(p-1)}},$$

$2 < p < \infty$, for entropy numbers with inequality (3) and the well-known estimate [6]

$$\varepsilon_m(W^r_{\infty,\alpha}, L_p) \le \varepsilon_m(W^r_{p,\alpha}, L_p) \ll m^{-r}(\log m)^{r(d-1)}. \quad \square$$

Let us proceed to the estimates of Kolmogorov widths. The orders of the widths $d_m(W^r_{q,\alpha}, L_p)$ for $1 < q, p < \infty$ and sufficiently large r are known [7]. The authors [1] established the following lower bound for all $p \in [1, \infty)$ and $r > 0$:

$$d_m(W^r_{\infty,\alpha}, L_p) \gg m^{-r}(\log m)^{r(d-1)}. \tag{4}$$

The corresponding upper bound for $1 < p < \infty$ follows from the well-known estimate [8, p. 69]

$$d_m(W^r_{p,\alpha}, L_p) \ll m^{-r}(\log m)^{r(d-1)}. \tag{5}$$

In this paper we prove the following assertion.

Theorem 2. *For all $\alpha \in \mathbb{R}^d$ and $r > 0$ we have*

$$d_m(W^r_{\infty,\alpha}, L_1) \gg m^{-r}(\log m)^{r(d-1)}. \tag{6}$$

Proof. It suffices to apply a general result due to G. Lorentz which permits one to derive (6) from (2′). A similar approach was used earlier in [2].

To use this approach here, let us state an assertion that directly follows from a result of G. Lorentz [9, Theorem 6, p. 915] in the form of a lemma.

Lemma 1. *Let A be an arbitrary compact set in a separable Banach space X; suppose that for some real numbers $r > 0$ and a we have*

$$\varepsilon_m(A, X) \asymp m^{-r}(\log m)^a$$

as $m \to \infty$. Then for the Kolmogorov width we have the relation

$$d_m(A, X) \gg m^{-r}(\log m)^a.$$

The assertion of Theorem 2 now readily follows from Theorem 1 and Lemma 1. □

Remark 1. Estimate (6) in Theorem 2 cannot be sharpened, as can be seen from the evident inequality

$$d_m(W^r_{\infty,\alpha}, L_1) \ll d_m(W^r_{2,\alpha}, L_2)$$

and from the well-known [8] estimate

$$d_m(W^r_{2,\alpha}, L_2) \ll m^{-r}(\log m)^{r(d-1)}. \tag{7}$$

Remark 2. In the proof of Theorem 2, Lemma 1 can be replaced by Remark 1 and the following assertion, which is a consequence of a result due to B. Carl [10].

Lemma 2. *Let A be an arbitrary compact set in a separable Banach space X; suppose that for some real numbers $r > 0$ and a we have*

$$d_m(A, X) \ll m^{-r}(\log m)^a \qquad \text{and} \qquad \varepsilon_m(A, X) \gg m^{-r}(\log m)^a.$$

Then

$$d_m(A, X) \asymp \varepsilon_m(A, X) \asymp m^{-r}(\log m)^a.$$

In closing, we note the analogy between Lemmas 1 and 2 and between their proofs.

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