EAST JOURNAL ON APPROXIMATIONS
Volume 1, Number 3 (1995), 373-377

LOWER ESTIMATES FOR THE SUPREMUM OF SOME RANDOM PROCESSES, II

B. KASHIN AND L. TZAFRIRI *

Steklov Mathematical Institute, Vavilov str. 42
Moscow, RUSSIA;
Department of Mathematics, Hebrew University of Jerusalem,
Jerusalem, ISRAEL

In this paper we give lower estimates for the supremum of random polynomials (with arbitrary coefficients) with respect to general orthogonal systems.

In this paper we give some lower estimates for the supremum of random processes of the type

$$\sum_{i=1}^{n} a_i \xi_i(t) \varphi_i(x) \ , \tag{1}$$

where $\{a_i\}_{i=1}^n$ are real coefficients, $\{\xi_i\}_{i=1}^n$ is a system of independent random variables on a probability space $(T, \mathcal{T}, \tau)$ normalized in $L_2(T, \mathcal{T}, \tau)$, and $\{\varphi_i\}_{i=1}^n$ is a system of norm one functions in an $L_2(X, \Sigma, \mu)$ space with (X, Σ, μ) being another probability space.

This note continues the investigation done in [2], where the simpler case $a_i = 1$, $1 \leq i \leq n$ was considered. The method of proof of the theorem below is the same as in [2] and is based on the application of a sharper version of the central limit theorem for sequences of independent vectors in $\mathbb{R}^2$ (see [1] and [4]). Our theorem generalizes results from Salem and Zygmund [5] (see Theorems 4.5.1 and 5.4.1 there), where only the case $\varphi_i = cos(ix)$ was considered. Let us remark that the method used in [5] can be also applied, after corresponding modifications, for studying the process (1) involving more general orthogonal systems.

*This work was performed while both authors visited the Max-Planck- Institute in Bonn, Germany.

As an application of the main estimate, we study random d-dimensional trigonometric (and more general) polynomials, and give some lower L_∞-estimates which could be useful in harmonic analysis and approximation theory.

Theorem. *For every $M < \infty$ there exist constants $C_j = C_j(M) > 0$, $j = 1, 2, 3$, and $q = q(M) > 0$ such that, whenever $\{\varphi_i\}_{i=1}^n$ is a system of functions in an $L_2(\mu)$ -space satisfying*

$$(1^\circ) \qquad \|\varphi_i\|_{L_2(\mu)} = 1 \quad \text{and} \quad \|\varphi_i\|_{L_3(\mu)} \le M, \quad \text{for all} \quad 1 \le i \le n,$$

$$(2^\circ) \qquad \|\sum_{i=1}^n a_i\varphi_i\|_{L_2(\mu)} \le M(\sum_{i=1}^n |a_i|^2)^{1/2}, \quad \text{for all} \quad \{a_i\}_{i=1}^n,$$

and $\{\xi_i\}_{i=1}^n$ are independent random variables over a probability space $(T, \mathcal{T}, \tau)$ with

$$(3^\circ) \qquad \mathbb{E}(\xi_i) = 0,\ \mathbb{E}|\xi_i|^2 = 1 \quad \text{and} \quad (\mathbb{E}|\xi_i|^3)^{1/3} \le M, \quad \text{for all} \quad 1 \le i \le n,$$

then, for any choice of the coefficients $\{a_i\}_{i=1}^n$, we have

$$\text{(i)} \quad \tau\{t \in T, \|\sum_{i=1}^n a_i\,\xi_i(t)\varphi_i(x)\|_\infty \le C_1(\sum_{i=1}^n |a_i|^2)^{1/2}(1+\log R)^{1/2}\} \le C_2R^{-q} \ ,$$

where

$$R = \frac{(\sum_{i=1}^n |a_i|^2)^2}{\sum_{i=1}^n |a_i|^4} \ ,$$

and hence

$$\text{(ii)} \qquad \mathbb{E}\|\sum_{i=1}^n a_i\,\xi_i\,\varphi_i\|_{L_\infty(\mu)} \ge C_3(\sum_{i=1}^n |a_i|^2)^{1/2}(1+\log\frac{(\sum_{i=1}^n |a_i|^2)^2}{\sum_{i=1}^n |a_i|^4})^{1/2}.$$

Remark. Using the so-called contraction principle (see Theorem 4.9 in [3]), one can p rove the inequality (ii) under the weaker assumption

$$(3') \qquad \mathbb{E}\,(\xi_i) = 0, \quad \mathbb{E}|\xi_i|^2 = 1 \quad \text{and} \quad \mathbb{E}|\xi_i| \ge 1/M, \quad \text{for} \quad 1 \le i \le n,$$

instead of (3) above. However, we are not aware of other reduction methods that would allow us to prove (i) under similar weaker assumptions.

We do not reproduce here the proof since, as we have mentioned already, it is quite similar to that from [2]. In this proof we estimate from below not only the L_∞-norm of the realization of the process (1):

$$f_t(x) = \sum_{i=1}^n a_i\xi_i(t)\varphi_i(x),$$

but also the generally smaller norm

$$\|f\|_{(m,\infty)} = \int \ldots \int max(|f(x_1)|, \ldots |f(x_m)|) d\mu(x_1) \ldots d\mu(x_m) \, .$$

Here the parameter m , taken to be equal to n in [2], is now connected with the size of the coefficients $\{a_i\}_{i=1}^n$ (like in all other considerations of the proof). Precisely,

$$m \asymp C(1 + \log R)^{1/2} R^{\gamma} \, ,$$

for some constants $C = C(M)$ and $0 \leq \gamma = \gamma(M) \leq 1$.

As in the previous paper [2], the theorem above remains true if we replace $p = 3$ with any other value of p greater than 2.

The main theorem can be used, for instance, to study random processes of the type

$$F_{N,\alpha,d}(t,x) = \sum_{\substack{n = (n_1,\ldots,n_d) \in \mathbb{Z}_+^d \\ 1 \leq |n_1 \ldots n_d| \leq N}} \frac{\xi_n(t)\varphi_n(x)}{(n_1 \ldots n_d)^{\alpha}} \, ,$$

for $0 < \alpha < \frac{1}{2}$ and any dimension $d \geq 1$, where the random variables $\{\xi_n\}$ and the system $\{\varphi_n\}$, (both indexed by $n \in \mathbb{Z}_+^d$) satisfy the conditions of the theorem for some choice of M. In this case we derive the existence of constants $K_1 = K_1(M,\alpha)$, $K_2 = K_2(M,\alpha)$ and $q = q(M,\alpha) > 0$ such that

$$(4) \qquad \tau\{t \in T, \|F_{N,\alpha,d}(t,x)\|_{L_{\infty(\mu,x)}} \leq K_1 N^{\frac{1}{2}-\alpha}(1+\log N)^{\frac{d}{2}}\} \leq K_2 N^{-q} \, ,$$

for all N.

This estimate is obtained using the fact that the cardinality of the set

$$\Gamma_N := \{n = (n_1,\ldots,n_d) \in \mathbb{Z}_+^d \ : \ 1 \leq |n_1 \ldots n_d| \leq N\}$$

satisfies

$$|\Gamma_N| \asymp N(1+\log N)^{d-1}.$$

Hence, the computation of the expression

$$\left(\sum_{n \in \Gamma_N} \frac{1}{(n_1 \ldots n_d)^{2\alpha}} \right)^{1/2} , \qquad 0 < \alpha \leq \frac{1}{2},$$

shows that it is of order of magnitude

$$\left(\frac{1}{N^{2\alpha}} N(1+\log N)^{d-1}\right)^{1/2} = N^{\frac{1}{2}-\alpha}(1+\log N)^{\frac{d-1}{2}}.$$

Then (4) follows immediately from the main theorem since R is of a polynomial growth with respect to N.

This estimate is sharp in the case $\{\varphi_n\}$ is the d-dimensional trigonometric system and $\{\xi_n\}$ are the usual Rademacher functions, since in this case

$$\mathbb{E}\|F_{N,\alpha,d}(t,x)\|_{L_\infty(\mu,x)} \le K_3 N^{\frac{1}{2}-\alpha}(1+\log N)^{\frac{d}{2}} ,$$

with some constant K_3.

In the case $\alpha = \frac{1}{2}$, the direct application of the main theorem does not yield a sharp estimate since R is now of logarithmic order. In order to overcome this difficulty, we consi der the random process

$$F'_{N,\frac{1}{2},d}(t,x) = \sum_{\substack{n=(n_1,\dots,n_d)\in\mathbb{Z}_+^d \\ \sqrt{N}<|n_1\dots n_d|\le N}} \frac{\xi_n(t)\varphi_n(x)}{(n_1\dots n_d)^{\frac{1}{2}}}$$

instead of the original one. The theorem now gives

$$\mathbb{E}\|F'_{N,\frac{1}{2},d}(t,x)\|_{L_\infty(\mu,x)} \ge c(1+\log N)^{\frac{d}{2}} ,$$

with some constant $c > 0$. Since $\{\xi_n\}$ are independent random variables of mean zero, it follows further that

$$\mathbb{E}\|F_{N,\frac{1}{2},d}(t,x)\|_{L_\infty(\mu,x)} \ge \mathbb{E}\|F'_{N,\frac{1}{2},d}(t,x)\|_{L_\infty(\mu,x)} \ge c(1+\log N)^{\frac{d}{2}}.$$

This estimate is sharp in the classical case discussed above.

In a similar manner, we can study also random processes of the form

$$Q_{N,\beta,d}(t,x) = \sum_{\substack{n=(n_1,\dots,n_d)\in\mathbb{Z}_+^d \\ 1<|n_1\dots n_d|\le N}} \frac{\xi_n(t)\varphi_n(x)}{(n_1\dots n_d)^{\frac{1}{2}}(1+\log^\beta(n_1\dots n_d))} ,$$

where $\frac{d}{2} < \beta < \frac{d+1}{2}$ and the systems $\{\xi_n\}$ and $\{\varphi_n\}$ are as before.

For $\beta > \frac{d}{2}$, a direct calculation shows that

$$\mathbb{E}\|Q_{N,\beta,d}(t,x)\|_{L_2(\mu,x)} \le K_4 ,$$

for some $K_4 = K_4(d) < \infty$ and all N. On the other hand, applying the main theorem again to the auxiliary process

$$Q'_{N,\beta,d}(t,x) = \sum_{\substack{n\in\mathbb{Z}_+^d \\ \sqrt{N}<|n_1\dots n_d|\le N}} \frac{\xi_n(t)\varphi_n(x)}{(n_1\dots n_d)^{\frac{1}{2}}(1+\log^\beta(n_1\dots n_d))}$$

we obtain, as before, that

$$\mathbb{E}\|Q_{N,\beta,d}(t,x)\|_{L_\infty(\mu,x)} \geq c(1+\log N)^{\frac{d+1}{2}-\beta} \ ,$$

which is of interest only if $\beta < \frac{d+1}{2}$. Again, this estimate is sharp in the case of trigonometric functions and that of Rademacher functions. This fact can be verified as in [5] p. 284.

Remark. As we have seen in the above examples, it is sometimes useful to replace the assertion (ii) of the theorem by

$$(ii') \ \ \mathbb{E}\|\sum_{i=1}^{n} a_i\,\xi_i\,\varphi_i\|_{L_\infty(\mu)} \geq C_3 \max_{\Lambda\subset\{1,\ldots,n\}} (\sum_{i\in\Lambda}|a_i|^2)^{1/2}(1+\log\frac{(\sum_{i\in\Lambda}|a_i|^2)^2}{\sum_{i\in\Lambda}|a_i|^4})^{1/2}.$$

References

[1] R. N. BHATTACHARYA AND R. RANGA RAO, Normal Approximation and Asymptotic Expansions, J. Wiley, 1976.

[2] B. KASHIN AND L. TZAFRIRI, Lower estimates for the supremum of some random processes, *East J. Approx.* **1**, 1(1995), 125–139.

[3] M. MARCUS AND G. PISIER, Random Fourier series with applications to harmonic analysis, Princeton Univ. Press, 1981.

[4] V. ROTAR', Nonuniform estimate of the rate of convergence in multidimensional central limit theorem, *Theory of Probab. and Appl.* **15** (1970), 647–665.

[5] R. SALEM AND A. ZYGMUND, Some properties of trigonometric series whose terms have random signs, *Acta Math.* **91**(1954), 245–301.

Received August 14, 1995.