Mathematical Notes, Vol. 56, Nos. 5-6, 1994

LOWER BOUND FOR THE MAXIMUM OF A STOCHASTIC PROCESS

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Below we consider stochastic processes of the form

$$\sum_{j=1}^{n} \xi_j(t)\varphi_j(x),$$

where $\{\xi_j\}$ is a collection of independent random variables defined on a probability space $(T, \mathcal{T}, \tau)$, and $\{\varphi_j\}_{j=1}^n$ is a collection of functions from $L^2(X, \Sigma, \mu)$, where (X, Σ, μ) is another probability space.

Many problems of analysis and probability theory reduce to needing estimates of the expectation

$$\mathsf{E}\left(\left\|\sum_{j=1}^{n} \xi_j(t)\varphi_j(x)\right\|_{L^\infty(\mu)}\right). \tag{1}$$

A well-known example of such a result is the estimate

$$\mathsf{E}\left(\left\|\sum_{j=1}^{n} r_j(t)e^{2\pi i j x}\right\|_{L^\infty}\right) \le C(n\log n)^{1/2}, \quad n = 2, 3, \dots, \tag{2}$$

($r_j(t)$ is the Rademacher function), first explicitly formulated by Salem and Zygmund [1] and which has been applied many times. That (2) is sharp is established in [1], i.e., it is shown that

$$\mathsf{E}\left(\left\|\sum_{j=1}^{n} r_j(t)e^{2\pi i j x}\right\|_{L^\infty}\right) \ge c(n\log n)^{1/2}; \quad c > 0, \quad n = 1, 2, \dots. \tag{3}$$

Later (cf. [2]) analogous questions were studied in the theory of Gaussian processes in finding necessary and sufficient conditions for the continuity for a.a. t of the trajectories of the process

$$\sum_{j=1}^{\infty} \gamma_j(t)\cdot a_j \cdot e^{2\pi i j x},$$

where $a_j \in \mathbb{R}$, $\{\gamma_j(t)\}_{j=1}^\infty$ is a sequence of independent Gaussian random variables with $\mathsf{E}(\gamma_j) = 0$, $\mathsf{E}(\gamma_j^2) = 1$, $j = 1, 2, \dots$.

The behavior of (1) when $\{\varphi_j(x)\}$, $x \in V$ is a system of characters of a locally compact Abelian group G and V is a compact neighborhood of zero in G was studied in papers of Marcus, Pisier, Talagrand, etc. These studies are described in detail in [3] and [4]. The approach to lower bounds used in [3] and [4] is based on the comparison of (1) with

$$\mathsf{E}\left(\left\|\sum_{j=1}^{n} \gamma_j(t)\varphi_j(x)\right\|_{L^\infty(\mu)}\right) \tag{4}$$

followed by the use of results of the type of the familiar Slepian lemma for estimating (4). The execution of this scheme is not trivial even in quite simple situations.

The first author's participation in the preparation of this paper was supported by the Russian Foundation for Basic Research (Grant No. 93-01-00240).

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 0001-4346/94/5656-1306$12.50

In the present note we take a different approach to finding lower bounds for (1), using sharper versions of the central limit theorem (with estimates of the error term) for sums of independent vectors in R^2. This approach is in a certain sense a return to the method of Salem and Zygmund [1] (although their arguments are not directly applicable to the case we consider). The method's possibilities are not confined to the theorem proved below but can be used in studying more general processes.

THEOREM. For each M one can find a constant $c_M > 0$ such that for any system of functions $\{\varphi_j\}_{j=1}^n \subset L^2(X, \mu)$ with

$$\|\varphi_j\|_{L^2(\mu)} = 1, \quad \|\varphi_j\|_{L^3(\mu)} \leq M, \qquad 1 \leq j \leq n,$$

such that

$$\left\|\sum_{j=1}^n a_j\varphi_j\right\|_{L^2(\mu)} \leq M\left(\sum_{j=1}^n a_j^2\right)^{1/2}, \qquad \{a_j\}_{j=1}^n \in R^n,$$

and for any system of independent random variables $\{\xi_j\}_{j=1}^n$ with

$$\mathsf{E}(\xi_j) = 0, \quad \mathsf{E}(\xi_j^2) = 1, \quad \{\mathsf{E}(|\xi_j|^3)\}^{1/3} \leq M, \qquad 1 \leq j \leq n,$$

one has

$$\mathsf{E}\left(\left\|\sum_{j=1}^n \xi_j\varphi_j(x)\right\|_{L^\infty(\mu)}\right) \geq c_M(n\log n)^{1/2}.$$

To prove the theorem we use as the lower bound of (1)

$$\mathsf{E}\left(\max_{1\leq\nu\leq n}\left|\sum_{j=1}^n \xi_j\varphi_j(x_\nu)\right|\right), \tag{5}$$

where $\{x_\nu\}_{\nu=1}^n \subset X$ is a collection of points such that for $\nu = 1, 2, \ldots, n$

$$\frac{1}{n}\sum_{j=1}^n \varphi_j^2(x_\nu) > c_1(M), \quad \frac{1}{n}\sum_{j=1}^n |\varphi_j(x_\nu)|^3 \leq C_2(M)$$

and

$$\frac{1}{n^2}\sum_{\nu,\nu'=1}^n\left(\sum_{j=1}^n \varphi_j(x_\nu)\varphi_j(x_{\nu'})\right)^2 \leq C_3(M)\cdot n \tag{6}$$

(the existence of a collection $\{x_\nu\}$ with the property (6) follows directly from the hypotheses of the theorem). Then, setting, for $\rho > 0$,

$$E_\nu^\rho = \left\{t \in T\colon \sum_{j=1}^n \xi_j(t)\varphi_j(x_\nu) > \rho(n\log n)^{1/2}\right\}$$

and using sharp versions of the central limit theorem for sums of one-dimensional and two-dimensional independent vectors, we verify that for small ρ, $0 < \rho \leq \rho(M)$,

$$C_M\cdot\|\chi_\rho\|_{L^1} \geq \|\chi_\rho\|_{L^2}, \quad \chi_\rho = \sum_{\nu=1}^n \chi_{E_\nu^\rho}(t), \tag{7}$$

where we use the standard notation

$$\chi_E(t) = \begin{cases} 1, & \text{if } t \in E, \\ 0, & \text{if } t \notin E. \end{cases}$$

The estimate $\mathcal{T}\left(\bigcup_{\nu=1}^n E_\nu^\rho\right) \geq c_4(M) > 0$, which completes the proof of the theorem, follows directly from (7).

We note that the comparison of the norms of L^1 and L^2 (cf. (7)) is also a characteristic of the method of [1] and was recently applied by S. Konyagin [5] in estimating the minimum on the unit circle of the modulus of a random trigonometric polynomial.

The process of choosing the points $\{x_\nu\}$ satisfying (6) establishes a certain connection of the approach we use with theorems on the restriction of an operator to a coordinate subspace considered in [6]—[9]. We take the opportunity to cite two results which supplement [6]—[9]. We consider linear operators A: $l_2^n \to l_2^n$ of rank q, $1 \le q \le n$, n = 1, 2, ... We denote by $\| A \|_{2\to 2}$ the norm of the operator A. If $\{e_j\}_{j=1}^n$ is the standard basis in l_2^n and $\sigma \subset \{1, \ldots, n\}$, then we denote by R_σ the orthoprojector to the subspace

$$L_\sigma = \left\{ x \in l_2^n : x = \sum_{j\in\sigma} x_j e_j \right\}.$$

PROPOSITION 1. There exists an absolute constant C such that for any n = 1, 2, ..., $1/n \le \delta \le 1$, and an arbitrary operator A of rank $\le$ q in l_2^n one can find sets $\sigma' \subset \{1, \ldots, n\}$, $\sigma'' \subset \{1, \ldots, n\}$ with $|\sigma'| \ge \delta n$, $|\sigma''| \ge \delta n$ such that

$$\|R_{\sigma'} A R_{\sigma''}\|_{2\to 2} \le C\left(\delta + \left(\frac{\delta q}{n}\right)^{1/2}\right)\|A\|.$$

PROPOSITION 2. There exists an absolute constant C_1 such that for any n = 1, 2, ..., $1/n \le \delta \le 1$, and any operator A of rank $\le$ q in l_2^n with

$$\langle Ae_j, e_j\rangle = 0, \qquad 1 \le j \le n,$$

one can find a set $\sigma \subset \{1, \ldots, n\}$ with $|\sigma| \ge \delta n$ such that

$$\|R_\sigma A R_\sigma\|_{2\to 2} \le C_1\left(\delta + \left(\frac{\delta q}{n}\right)^{1/2}\right)\|A\|.$$

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