RANDOM SETS OF UNIFORM CONVERGENCE

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In [1], P. L. Ul'yanov posed the question of characterization of sets σ ⊂ Z for which the functions \( \{e^{2\pi inx}\}_{n \in \sigma} \) form a basis for the subspace of \( C(0, 1) \) which they generate. In other words one must determine for which sets \( \sigma \) the quantity

\[
U(\sigma) = \sup_{s, \{a_n\}} \left\| \sum_{n \in \sigma \cap [-s, s]} a_n e^{2\pi inx} \right\|_{\infty}
\]

is finite, where the supremum is taken with respect to all \( s \in \mathbb{N} \) and all finite sequences \( \{a_n\}_{n \in \sigma} \neq 0 \).

We shall call a set \( \sigma \) for which \( U(\sigma) \) is finite a set of uniform convergence (U. C. set).

The article [1] contains two basic questions: one of them is to determine whether \( \sigma = \{k^2\}_{k \in \mathbb{N}} \) is a set of uniform convergence, and the second, of more general character, is to determine how dense a U. C. set can be.

The first problem is completely solved by K. I. Oskolkov [2] (cf. also [3]), who showed that for any polynomial \( P(x) \) with integer coefficients \( \sigma = \{P(n)\}_{n \in \mathbb{N}} \) is not a U. C. set.

The problem of possible density of sets of uniform convergence remains open. It can be reduced to the question of finding for given \( N \) a subset \( \sigma \subset \{-N, \ldots, N\} \) with maximal number of elements for which \( U(\sigma) \) is bounded by a constant independent of \( N \). This problem can also be considered for other orthonormal systems, in particular for a Walsh system \( \{W_i\}_{i=1}^{\infty} \).

The simplest examples of sets of uniform convergence are Sidon sets, for which it is known that their density is quite small. More precisely, for Sidon sets \( \sigma \in \mathbb{Z} \), \( | \sigma \cap [-N, N] | \leq C \log N \), where \( C < \infty \) is a constant, \( N = 1, 2, \ldots \); sets of larger density of order \( (\log N)^2 \) were constructed in [4], of order \( (\log N)^k \), \( k \in \mathbb{N} \) in [5]. U. C. sets of greater density are not yet known.

The goal of the present note is to consider random subsets \( \sigma \subset \{-N, \ldots, N\} \) of the trigonometric system \( \{e^{2\pi inx}\}_{n=-N}^{N} \) or \( \sigma \subset \{1, \ldots, N\} \) for the Walsh system \( \{W_i\}_{i=1}^{2^r=N} \). In both cases estimates of the Lebesgue constant (i.e., the norm of the operators of partial sums) guarantee that \( U(\sigma) \leq C \log N \) for all \( \sigma \) and \( N \) and some constant \( C \).

It is shown in [6] (cf. also [7, p. 283]) that for any uniformly bounded orthonormal system \( \{\varphi_n\}_{n=1}^{N} \) a random set \( \sigma \subset \{1, \ldots, N\} \) with number of elements \( | \sigma | \leq (1/6) \log N \) is a Sidon set with Sidon constant independent of \( N \). Consequently, for a random set \( \sigma \) with \( | \sigma | \leq (1/6) \log N \) we have \( U(\sigma) \leq C \) with constant \( C \) independent of \( N \).

The basic result of this paper is that for a random subset \( \sigma \subset \{1, \ldots, N\} \) with number of elements \( \gg \log N \), \( U(\sigma) \to \infty \) as \( N \to \infty \) and in addition if the number of elements of the random set \( \sigma \) satisfies \( | \sigma | \geq N^\varepsilon \) for some \( \varepsilon > 0 \), then with high probability \( U(\sigma) \) has maximal order, i.e., \( \log N \).

First we consider the case of a Walsh system. For natural numbers \( q \) and \( N \) such that \( 1 \leq q \leq N \), we denote by \( S_{N^q} \) the family of all sets \( \sigma \subset \{1, \ldots, N\} \) with \( | \sigma | = q \) and by \( \nu \) the normalized counting measure on \( S_{N^q} \). For \( \sigma \subset S_{N^q} \) let

\[
U(\sigma) = \sup \left\{ \left\| \sum_{j=1}^{s} a_j W_j \right\|_{\infty} : \left\| \sum_{j=1}^{N} a_j W_j \right\|_{\infty} = 1, \supp \{a_j\} \subset \sigma \right\}.
\]
THEOREM 1. There exists an absolute constant $c > 0$ such that if $N = 2^r$ for some natural number $r$ and $1 < q \leq N/2$, then

$$\nu\{\sigma \in S_N^r : U(\sigma) \leq c \log \left( 2 + \frac{q}{\log N} \right) \} < \frac{1}{N^2}.$$  

Proof. When $q$ is not very large compared with $\log N$ one can find a constant $c > 0$ such that

$$c \log \left( 2 + \frac{q}{\log N} \right) < 1,$$

and hence the measure considered in the theorem is zero since $U(\sigma) \geq 1$ for any set $\sigma$. Hence we assume below that $N$ is sufficiently large and $q \geq 20 \log N$.

Let $\delta = q/N$ and we note that $0 < \delta \leq 1/2$. Further, let $\{\xi_i\}_{i=1}^N$ be a collection of independent random variables defined on a probability space $(\Omega, \Sigma, \mu)$ and assuming values 0 or 1 with mean $\delta$. For $\omega \in \Omega$ let

$$\sigma(\omega) = \{i; 1 \leq i \leq N, \xi_i(\omega) = 1\}.$$  

We show below that for some constant $c > 0$

$$\mu\{\omega \in \Omega; U(\sigma(\omega)) < c \log \left( 2 + \frac{q}{\log N} \right) \} \leq \frac{5}{N^3}. \quad (*)$$

Thus we prove Theorem 1 since

$$\mu\{\omega \in \Omega; |\sigma(\omega)| = q = \delta N \} \geq \frac{B}{\sqrt{N}}$$

for some constant $B > 0$ that is independent of $N$ and $(\frac{\mu}{N^3}) / (\frac{\mu}{N^2}) < \frac{1}{N^2}$ for $N$ sufficiently large.

To prove (*) we need some auxiliary lemmas.

**Lemma 1.** Let $b = (b_1, b_2, \ldots, b_2^r)$ be a sequence such that for $s = 1, 2, \ldots, r - 1$ and some $\beta_0 > 0$ the set

$$\Delta_s = \left\{ k; \frac{1}{2^{s+1}} < |b_k| \leq \frac{1}{2^s} \right\}$$

has cardinality $|\Delta_s| \geq \beta_0 2^s$. Then for any sequence $a = (a_1, a_2, \ldots, a_{2^r})$ with $\|a\|_1 \leq \lambda$, for some $1 \leq \lambda \leq (r - 2)\beta_0 / 8$ one has

$$\|a - b\|_2 \geq 2^{-8\lambda/\beta_0 \sqrt{\beta_0}}.$$  

Proof. We fix $a = (a_1, a_2, \ldots, a_{2^r})$ with $\|a\|_1 \leq \lambda$ and for each $s$ let $\lambda_s = \Sigma_{k \in \Delta_s} |a_k|$. Since

$$\lambda \geq \Sigma_{s=1}^{(8\lambda/\beta_0)+1} \lambda_s \geq \frac{8\lambda}{\beta_0} \min_{1 \leq s \leq (8\lambda/\beta_0)+1} \lambda_s,$$

we can find a natural number $1 \leq s_0 \leq 8\lambda/\beta_0 + 1$ such that $\lambda_{s_0} \leq \beta_0 / 8$. Let

$$\Delta'_{s_0} = \left\{ k \in \Delta_{s_0}; |a_k| \geq \frac{1}{2^{s_0+2}} \right\}$$

and we note that

$$\beta_0 \geq \Sigma_{k \in \Delta'_{s_0}} |a_k| \geq \frac{|\Delta'_{s_0}|}{2^{s_0+2}},$$

i.e.,

$$|\Delta'_{s_0}| \leq \frac{\beta_0 2^{s_0+2}}{8} = \frac{\beta_0 2^{s_0}}{2}.$$  

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Consequently

\[ ||a - b||^2 \geq \sum_{k \in \Delta_{\alpha_0}} |a_k - b_k|^2 \geq \sum_{k \in \Delta_{\alpha_0} \Delta_{\alpha_0}} (|b_k| - |a_k|)^2 \geq \frac{|\Delta_{\alpha_0} \setminus \Delta_{\alpha_0}'|}{2^{2\alpha_0 + 4}} \geq \frac{\beta_0 2^{\alpha_0}}{2^{2\alpha_0 + 5}} = \frac{\beta_0}{32} \cdot \frac{1}{2^{2\alpha_0}}, \]

and we get

\[ ||a - b||_2 \geq \sqrt{\frac{\beta_0}{32} \cdot \frac{1}{2^{2\alpha_0}} \geq \frac{\beta_0}{8} \cdot \frac{1}{2^{2\alpha_0}}}. \]

**Lemma 2.** Let us assume that \( \{\varphi_i\}_{i=1}^m \) is a collection of elements of a Hilbert space \( H \) such that for some \( 0 < \varepsilon < 1 \) and any vector \( c = (c_1, c_2, \ldots, c_m) \) one has

\[ (1 - \varepsilon)||c||_2 \leq \sum_{i=1}^m c_i |\varphi_i| < (1 + \varepsilon)||c||_2. \]

Then

\[ (1 - \varepsilon)^4 ||c||_2^2 \leq \sum_{k=1}^m \left( \sum_{i=1}^m c_i |\varphi_i, \varphi_k| \right)^2 \leq (1 + \varepsilon)^4 ||c||_2^2 \]

for the same \( \varepsilon > 0 \) and each \( c \in l^2_m \).

**Proof.** Our assumption implies that

\[ (1 - \varepsilon)^2 ||c||_2^2 \leq \sum_{i,j=1}^m c_i c_j (|\varphi_i, \varphi_j| \leq (1 + \varepsilon)^2 ||c||_2^2 \]

for any \( c \in l^2_m \), i.e., the matrix \( G = \{ (\varphi_i, \varphi_j) \}_{i,j=1}^m \) is positive definite and its eigenvalues \( (\lambda_1, \ldots, \lambda_m) \) satisfy the inequalities

\[ (1 - \varepsilon)^2 \leq \lambda_i \leq (1 + \varepsilon)^2; \quad 1 \leq i \leq m. \]

Hence the eigenvalues of the matrix \( GG^* = G^2 \) lie between \((1 - \varepsilon)^4 \) and \((1 + \varepsilon)^4 \). In particular,

\[ (1 - \varepsilon)^4 ||c||_2^2 \leq (c, GG^* c) \leq (1 + \varepsilon)^4 ||c||_2^2 \]

for any \( c \in l^2_m \). This finishes the proof since

\[ \sum_{k=1}^m \left( \sum_{i=1}^m c_i |\varphi_i, \varphi_k| \right)^2 = \sum_{k=1}^m \sum_{i,j=1}^m c_i c_j (\varphi_i, \varphi_k)(\varphi_j, \varphi_k) = (c, GG^* c). \]

**Lemma 3.** Under the hypotheses of Lemma 2, for the element \( f = \sum_{i=1}^m c_i \varphi_i \) one has

\[ \|f - \sum_{i=1}^m (f, \varphi_i) \varphi_i\| \leq 3 \sqrt{3} \left( \sum_{i=1}^m |(f, \varphi_i)|^2 \right)^{1/2}, \]

if \( \varepsilon \) is sufficiently small.

**Proof.** By Lemma 2

\[ 0 \leq \left\| f - \sum_{i=1}^m (f, \varphi_i) \varphi_i \right\|^2 = \|f\|^2 + \left\| \sum_{i=1}^m (f, \varphi_i) \varphi_i \right\|^2 - 2 \left( \sum_{i=1}^m |(f, \varphi_i)|^2 \right)^2 \]

\[ \leq \|f\|^2 - (1 - \varepsilon)^4 ||c||_2^2 + (2 \varepsilon + \varepsilon^2) \sum_{i=1}^m |(f, \varphi_i)|^2 \leq \|(1 + \varepsilon)^2 - (1 - \varepsilon)^4 + (2 \varepsilon + \varepsilon^2) (1 + \varepsilon)^4 \| ||c||_2^2. \]

Hence, if \( \varepsilon \) is sufficiently small, we get

\[ 0 \leq \left\| f - \sum_{i=1}^m (f, \varphi_i) \varphi_i \right\|^2 \leq 8.5 \varepsilon ||c||_2^2 \leq 9 \varepsilon \sum_{i=1}^m |(f, \varphi_i)|^2. \]
LEMMA 4. There exists an $\varepsilon_0 > 0$ such that as soon as a collection $\{\varphi_i\}_{i=1}^m$ of elements of the Hilbert space $H$ is given satisfying the condition

$$(1 - \varepsilon)\|c\|_2 \leq \left\| \sum_{i=1}^m c_i \varphi_i \right\|_H \leq (1 + \varepsilon)\|c\|_2,$$

for some $0 < \varepsilon < \varepsilon_0$ and any $c = (c_1, \ldots, c_m) \in l_2^m$, one has that for any $z \in H$ and any vector $c = (c_1, \ldots, c_m)$

$$\left\| z - \sum_{i=1}^m c_i \varphi_i \right\| \geq (1 - \varepsilon) \left[ \sum_{i=1}^m ((z, \varphi_i) - c_i)^2 \right]^{1/2} - 3\sqrt{\varepsilon} \left( \sum_{i=1}^m ((z, \varphi_i)^2) \right)^{1/2}.$$

Proof. Let $R$ be the orthogonal projection from $H$ to $[\varphi_i]_{i=1}^m$. Then

$$\left\| z - \sum_{i=1}^m c_i \varphi_i \right\| \geq \left\| Rz - \sum_{i=1}^m c_i \varphi_i \right\|$$

and $(Rz, \varphi) = (z, \varphi)$, $1 \leq i \leq m$, i.e., it is enough to establish the inequality for $Rz$ instead of $z$. In other words we can assume that $z \in L$. Then by our hypotheses and Lemma 3 we find that

$$\left\| z - \sum_{i=1}^m c_i \varphi_i \right\| \geq \left\| Rz - \sum_{i=1}^m c_i \varphi_i \right\| \geq (1 - \varepsilon) \left[ \sum_{i=1}^m ((z, \varphi_i) - c_i)^2 \right]^{1/2} - 3\sqrt{\varepsilon} \left( \sum_{i=1}^m ((z, \varphi_i)^2) \right)^{1/2}.$$

To estimate $U(\sigma)$ we need

LEMMA 5. Let $\{W_j\}_{j=1}^{N=2^r}$ be the first $N$ Walsh functions defined on $[0, 1]$, and let $\sigma \subset \{1, 2, \ldots, N\}$. Then

$$U(\sigma) = \max_{1 \leq p \leq N} \max_{z \in Z(\sigma)} |(v_p, z)|,$$

where $v_p = \underbrace{(1, 1, \ldots, 1, 0, \ldots, 0)}_{p \text{ times}}$ and

$$Z(\sigma) = \left\{ z = (z_1, \ldots, z_N) \in \mathbb{R}^N, \supp(z) \subset \sigma, \left\| \sum_{i=1}^N z_i W_i \right\|_\infty \leq 1 \right\}.$$

Proof. We fix $1 \leq p \leq N$ and $z \in Z(\sigma)$. Then for any $0 < \tau < 1/N$

$$|(v_p, z)| = \left| \sum_{j=1}^p z_j \right| = \left| \sum_{j=1}^p z_j W_j(\tau) \right| \leq \left\| \sum_{j=1}^p z_j W_j \right\|_\infty \leq U(\sigma).$$

To prove the opposite inequality we fix a function $f = \sum_{k=1}^N \hat{f}(k) W_k$ such that $\{\hat{f}(k)\} \subset \sigma$, $\|f\|_\infty = 1$ and for some $1 \leq p \leq N$ and $0 \leq \tau_0 \leq 1$ one has

$$U(\sigma) = \sum_{k=1}^p \hat{f}(k) W_k(\tau_0).$$

Let

$$g(x) = \sum_{k=1}^N \hat{f}(k) W_k(\tau_0) W_k(x), \quad x \in [0, 1],$$

and we note that $g(x) = \sum_{k=1}^N \hat{f}(k) W_k(x \oplus_\delta \tau_0)$, where $x \oplus_\delta \tau_0$ means addition of $x$ and $\tau_0$ modulo 2 (cf. [7, p. 135]). Hence

$$g(x) = f(x \oplus_\delta x_0)$$

and so $\|g\|_\infty \leq 1$. Consequently, $z^0 = \{\hat{f}(k) W_k(\tau_0)\}_{k=1}^N \in Z(\sigma)$ and thus

$$U(\sigma) = (v_p, z^0) \leq \max_{1 \leq p \leq N} \max_{z \in Z(\sigma)} |(v_p, z)|.$$
LEMMA 6. For $N = 2^r$ we consider the discrete Walsh system $W_i = (w_{i,j})_{j=1}^N$, $1 \leq i \leq n$, as a collection of vectors normalized in $l^\infty_\mathbb{N}$, i.e., such that $|w_{i,j}| = 1$ for all $1 \leq i, j \leq N$. In addition let $\{W^{(j)}\}_{j=1}^N$ be the columns of the Walsh matrix. Then for any $\sigma \subset \{1, 2, \ldots, N\}$

$$U(\sigma) = \inf \left\{ \lambda; \forall 1 \leq p \leq N, R_\sigma v_p = \sum_{j=1}^{N} \lambda_j R_\sigma W^{(j)} \quad \text{c} \quad \sum_{j=1}^{N} |\lambda_j| \leq \lambda \right\},$$

where $v_p = (1, \ldots, 1, 0, \ldots, 0)$ and $R_\sigma$ is the orthogonal projection operator to $[e_i]_{i \in \sigma} \{e_i\}_{i=1}^N$ denotes the canonical basis in $\mathbb{R}^N$.

Proof. First we show that for any $1 \leq p \leq N$ and $\sigma \subset \{1, 2, \ldots, N\}$

$$R_\sigma v_p \in \text{conv}\{\pm U(\sigma) R_\sigma W^{(j)}; 1 \leq j \leq N\}.$$ 

Indeed if this conclusion failed for some $1 \leq p \leq N$, then the standard argument based on the property of separability of convex sets would imply the existence of a vector $b = (b_1, \ldots, b_N)$ such that $(b, R_\sigma v_p) > 1$, but $|b, U(\sigma) R_\sigma W^{(j)}| \leq 1$ for all $1 \leq j \leq N$. Then we set $z = U(\sigma) R_\sigma b$ and note that for the $p$ considered $|z, v_p| > U(\sigma)$. On the other hand, $|z, W^{(j)}| \leq 1$ for all $1 \leq j \leq N$, i.e., $z \in Z(\sigma)$, which by Lemma 5 implies $|z, v_p| \leq U(\sigma)$, and we arrive at a contradiction.

As an immediate corollary we have

$$R_\sigma v_p = \sum_{j=1}^{N} \mu_j U(\sigma) R_\sigma W^{(j)},$$

for any $1 \leq p \leq N$ with $\Sigma_{j=1}^{N} |\mu_j| \leq 1$. Consequently,

$$\inf \left\{ \lambda; \forall 1 \leq p \leq N, R_\sigma v_p = \sum_{j=1}^{N} \lambda_j R_\sigma W^{(j)} \quad \text{c} \quad \sum_{j=1}^{N} |\lambda_j| \leq \lambda \right\} \leq U(\sigma).$$

On the other hand, if for some $1 \leq p \leq N$ we have that

$$R_\sigma v_p = \sum_{j=1}^{N} \lambda_j R_\sigma W^{(j)}$$

with $\Sigma_{j=1}^{N} |\lambda_j| \leq \lambda$, then for $z \in Z(\sigma)$

$$|(v_p, z)| = |(R_\sigma v_p, z)| = \sum_{j=1}^{N} \lambda_j (z, W^{(j)}) \leq \lambda \max_{1 \leq j \leq N} |(z, W^{(j)})| \leq \lambda,$$

i.e., $U(\sigma) \leq \lambda$.

For completeness of the exposition we also cite the following familiar probabilistic result.

LEMMA 7. Let $0 < \delta \leq 1/2$ and let $\{\xi_k\}_{k=1}^{N}$ be a collection of independent random variables defined on a probability space $(\Omega, \Sigma, \mu)$ and assuming values 0 or 1 with mean $\delta$. Then for any $|a_k| \leq 1$, $1 \leq k \leq N$, and $0 \leq \gamma \leq \delta N$ we have

$$\mu \left\{ \omega \in \Omega; \left| \sum_{k=1}^{N} a_k (\xi_k(\omega) - \delta) \right| \geq \gamma \right\} \leq 2e^{-\gamma^2/(\delta N)}.$$

Proof. We fix $1 \leq k \leq N$ and let $X_k(\omega) = \xi_k(\omega) - \delta$ and we note that for $0 \leq t \leq 1$

$$\int_{\Omega} e^{t X_k(\omega)} d\mu(\omega) = \int_{\Omega} \left[ 1 + t X_k(\omega) + \sum_{j=2}^{\infty} \frac{t^j}{j!} X_k^j(\omega) \right] d\mu(\omega)$$

$$\leq 1 + t^2 \int_{\Omega} X_k^2(\omega) d\mu(\omega) \sum_{j=2}^{\infty} \frac{1}{j!} = 1 + t^2 \delta(1 - \delta)(e - 2) \leq e^{t^2(1 - \delta)}.$$ 

In particular

$$\int_{\Omega} e^{t a_k X_k(\omega)} d\mu(\omega) \leq e^{t^2(1 - \delta)},$$

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provided that $0 \leq t \leq 1$. Consequently, by Theorem 15 of [8, p. 52] we get that

$$\mu\left\{\omega \in \Omega; \sum_{k=1}^{N} a_k(\xi_k(\omega) - \delta) \geq \gamma\right\} \leq e^{-\gamma^2/(4\delta(1-\delta)N)} \leq e^{-\gamma^2/(4\delta N)}$$

provided $0 \leq \gamma \leq 2\delta(1-\delta)N$, and in particular if $0 \leq \gamma \leq \delta N$. Lemma 7 is proved.

Before proving Theorem 1 we note that for $1 \leq j \leq N$

$$|\{v_p, W^{(j)}\}| \leq \frac{N}{j} \quad \text{(i)}$$

for each $1 \leq p \leq N$, and if

$$p = \lfloor N/3 \rfloor \quad \text{and} \quad b_j = \frac{\{v_p, W^{(j)}\}}{N}, \quad 1 \leq j \leq N, \quad \text{(ii)}$$

then the cardinality of the set

$$\Delta_s = \left\{k; \frac{1}{2^{s + 1}} < |b_k| \leq \frac{1}{2^s}\right\}$$

satisfies $|\Delta_s| \geq 2^{s/4}$ for all $0 < s < \log_2 N$.

For $\sigma \subset \{1, 2, \ldots, N\}$ and $p = \lfloor N/3 \rfloor$ it follows from Lemma 6 that

$$R_{\sigma} v_p = \sum_{j=1}^{N} \nu_j^p(\sigma) R_{\sigma} W^{(j)},$$

where $\sum_{j=1}^{N} |\nu_j^p(\sigma)| \leq U(\sigma)$.

For fixed $\delta$ and $N$ such that $\delta N \geq 20 \log N$, let

$$m = \left\lfloor \left(\frac{\delta N}{20 \log N}\right)^{3/8}\right\rfloor,$$

and we note here that

$$\|R_{\sigma} v_p - \sum_{j=1}^{m} \nu_j^p(\sigma) R_{\sigma} W^{(j)}\|_2^2 = \left(\sum_{h=m+1}^{N} \nu_h(\sigma) R_{\sigma} W^{(h)}, R_{\sigma} v_p - \sum_{j=1}^{m} \nu_j^p(\sigma) R_{\sigma} W^{(j)}\right)$$

$$\leq U(\sigma) \max_{m < h \leq N} |\{v_p, R_{\sigma} W^{(h)}\}| + U(\sigma)^2 \max_{1 \leq j \leq m} \|W^{(h)} - R_{\sigma} W^{(j)}\|.$$

To estimate the right side of the last inequality from above we first apply Lemma 7 for fixed $h$, $m < h \leq N$, and $0 \leq \gamma \leq \delta N$, and we get

$$\mu\left\{\omega \in \Omega; \left|\sum_{i=1}^{p} w_{i,h}(\xi_i(\omega) - \delta)\right| \geq \gamma\right\} \leq 2e^{-\gamma^2/(4\delta N)},$$

where $\{\xi_i\}_{i=1}^{N}$, as usual, are $(0, 1)$-valued independent random variables with mean $\delta$ for some $0 < \delta < 1$, defined on a probability space $(\Omega, \Sigma, \mu)$. Consequently,

$$\mu\left\{\omega \in \Omega, \max_{m < h \leq N} \left|\sum_{i=1}^{p} w_{i,h}(\xi_i(\omega) - \delta)\right| \geq \gamma\right\} \leq 2Ne^{-\gamma^2/(4\delta N)},$$

and so with probability $\geq 1 - 2Ne^{-\gamma^2/(4\delta N)}$ we have

$$\max_{m < h \leq N} |(R_{\sigma} v_p, W^{(h)}) - \delta(v_p, W^{(h)})| \leq \gamma.$$

By remark (i) above
\[
\max_{m < h \leq N} |(R_{\sigma} v_p, W^{(h)})| \leq \frac{\delta N}{m} + \gamma
\]

with probability \(\geq 1 - 2Ne^{-\gamma^2/(4\delta N)}\).

Let \(\gamma = \sqrt{20\delta N \log N}\) and we note that \(0 < \gamma \leq \delta N\). Then for \(\varepsilon = (\delta N/20 \log N)^{-1/16}\) one has

\[
m = \left\lceil \frac{\varepsilon^2}{\sqrt{20 \log N}} \right\rceil,
\]

using which, we easily derive that

\[
\max_{m < h \leq N} |(R_{\sigma} v_p, W^{(h)})| \leq \left(1 + \frac{1}{\varepsilon^2}\right) \sqrt{20\delta N \log N} \leq \frac{2}{\varepsilon^2} \sqrt{20\delta N \log N}
\]

with probability \(\geq 1 - 2/N^4\).

An analogous calculation using Lemma 7 and the orthogonality of the Walsh matrix shows that

\[
\mu\left\{ \omega \in \Omega; \left| \sum_{i=1}^{N} w_{i,h} w_{i,j} \xi_i(\omega) \right| \geq \gamma \right\} \leq 2e^{-\gamma^2/(4\delta N)}
\]

for any \(0 < \gamma \leq \delta N\), \(m < h \leq N\), and \(1 \leq j \leq m\). Hence with probability \(\geq 1 - 2/N^3\) we have

\[
\max_{1 \leq j \leq m, m < h \leq N} |(W^{(h)}, R_{\sigma} W^{(j)})| \leq \sqrt{20\delta N \log N}.
\]

In sum we get that

\[
\|R_{\sigma} v_p - \sum_{j=1}^{m} \nu_j^{(p)}(\sigma) R_{\sigma} W^{(j)}\|_2^2 \leq \frac{2}{\varepsilon^2} U(\sigma) \sqrt{20\delta N \log N} + U(\sigma)^2 \sqrt{20\delta N \log N}
\]

\[
\leq \left(\frac{2}{\varepsilon^2} + U(\sigma)\right) U(\sigma) \sqrt{20\delta N \log N}
\]

with probability \(\geq 1 - 4/N^3\).

To estimate the left side of the last inequality from below we set

\[
\varphi_j = \frac{R_{\sigma} W^{(j)}}{\sqrt{\delta N}}; \quad 1 \leq j \leq m,
\]

and we note that arguing as above we get

\[
\mu\left\{ \omega \in \Omega; \left| \sum_{i=1}^{N} w_{i,j} w_{i,l} (\xi_i(\omega) - \delta) \right| \geq \gamma \right\} \leq 2e^{-\gamma^2/(4\delta N)}
\]

for any \(1 \leq j, l \leq m\). This implies that with probability \(\geq 1 - 2/N^3\)

\[
|(R_{\sigma} W^{(j)}, R_{\sigma} W^{(l)}) - \delta(W^{(j)}, W^{(l)})| \leq \sqrt{20\delta N \log N}, \quad 1 \leq j, l \leq m.
\]

Hence, with the same probability \(\geq 1 - 2/N^3\)

\[
|\langle \varphi_j, \varphi_l \rangle - \langle W^{(j)}, W^{(l)} \rangle| \leq \sqrt{\frac{20 \log N}{\delta N}}, \quad 1 \leq j, l \leq m.
\]

Hence for each vector \(c = (c_j)_{j=1}^{m}\)
\[
\left\| \sum_{j=1}^{m} c_j \varphi_j \right\|^2 - \|c\|^2 \leq \sum_{j,i=1}^{m} |c_j||c_i| |(\varphi_j, \varphi_i) - \left( \frac{W(j)}{\sqrt{N}}, \frac{W(i)}{\sqrt{N}} \right)| \leq \left( \sum_{j=1}^{m} |c_j|^2 \right)^2 \sqrt{\frac{20 \log N}{\delta N}} \leq m\|c\|^2 \sqrt{\frac{20 \log N}{\delta N}} \leq \varepsilon^2 \|c\|^2 ,
\]
i.e.,

\[
(1-\varepsilon)\|c\| \leq \left\| \sum_{j=1}^{m} c_j \varphi_j \right\| \leq (1+\varepsilon)\|c\| ,
\]

Consequently, using Lemma 4 applied when \( z = R_v v_p / \sqrt{\delta N} \) and \( c_j = v_j^p(\sigma), \ 1 \leq j \leq m \), we get that

\[
\left\| \frac{R_v v_p}{\sqrt{\delta N}} - \sum_{j=1}^{m} v_j^p(\sigma) \frac{R_v W(j)}{\sqrt{\delta N}} \right\|_2 \geq (1-\varepsilon)\left( \sum_{j=1}^{m} \left| \left( \frac{R_v v_p}{\sqrt{\delta N}}, \frac{W(j)}{\sqrt{\delta N}} \right) - v_j^p(\sigma) \right|^2 \right)^{1/2}
\]

\[
= 3\sqrt{\varepsilon} \left( \sum_{j=1}^{m} \left| \left( \frac{R_v v_p}{\sqrt{\delta N}}, \frac{R_v W(j)}{\sqrt{\delta N}} \right) \right|^2 \right)^{1/2} ,
\]
i.e.,

\[
\left\| \frac{R_v v_p}{\sqrt{\delta N}} - \sum_{j=1}^{m} v_j^p(\sigma) R_v W(j) \right\|_2 \geq (1-\varepsilon)\sqrt{\delta N} \left( \sum_{j=1}^{m} \left| \left( \frac{R_v v_p}{\sqrt{\delta N}}, \frac{W(j)}{\sqrt{\delta N}} \right) - v_j^p(\sigma) \right|^2 \right)^{1/2}
\]

\[
= 3\sqrt{\varepsilon} \left( \sum_{j=1}^{m} \left| \left( \frac{R_v v_p}{\sqrt{\delta N}}, \frac{R_v W(j)}{\sqrt{\delta N}} \right) \right|^2 \right)^{1/2} .
\]

Previously we made the assumption that \( q(\log N)^{-1} = \delta N(\log N)^{-1} \) is large enough (or equivalently that \( \varepsilon \) is small enough) that we can use Lemma 4. Repeating calculations made earlier we conclude that

\[
|\langle v_p, R_v W(j) \rangle - \delta(v_p, W(j))| \leq \sqrt{20 \delta N \log N} \quad 1 \leq j \leq m ,
\]

with probability \( \geq 1 - 2/N^4 \). This implies that

\[
\left\| \frac{R_v v_p}{\sqrt{\delta N}} - \sum_{j=1}^{m} v_j^p(\sigma) R_v W(j) \right\| \geq (1-\varepsilon)\sqrt{\delta N} \left( \sum_{j=1}^{m} \left| \left( \frac{v_p}{\sqrt{N}}, \frac{W(j)}{\sqrt{N}} \right) - v_j^p(\sigma) \right|^2 \right)^{1/2}
\]

\[
= (1-\varepsilon)\sqrt{\delta N} m^{1/2} \sqrt{\frac{20 \log N}{\delta N}} - \frac{3\sqrt{\varepsilon}}{\sqrt{\delta N}} \left( \sum_{j=1}^{m} |\langle v_p, W(j) \rangle|^2 \right)^{1/2}
\]

\[
= 3\sqrt{\varepsilon} \left( \sum_{j=1}^{m} \left| \left( \frac{v_p}{\sqrt{N}}, \frac{W(j)}{\sqrt{N}} \right) \right|^2 \right)^{1/2} - \frac{3\sqrt{\varepsilon}}{\sqrt{\delta N}} \left( \sum_{j=1}^{m} |\langle v_p, W(j) \rangle|^2 \right)^{1/2}
\]

\[
\geq (1-\varepsilon)\sqrt{\delta N} \left( \sum_{j=1}^{m} \left| \left( \frac{v_p}{\sqrt{N}}, \frac{W(j)}{\sqrt{N}} \right) - v_j^p(\sigma) \right|^2 \right)^{1/2}
\]

\[
= 3\sqrt{\varepsilon} \left( \sum_{j=1}^{m} \left| \left( \frac{v_p}{\sqrt{N}}, \frac{W(j)}{\sqrt{N}} \right) \right|^2 \right)^{1/2} - (1-\varepsilon)\varepsilon^2 (20 \delta N \log N)^{1/4} - 3\varepsilon^{3/2} (20 \delta N \log N)^{1/4} .
\]
By remark (ii) above and Lemma 1, 
\[ \left( \sum_{j=1}^{m} \left( \frac{v_{p}}{\sqrt{N}} - \frac{W(j)}{\sqrt{N}} \right) \right)^{1/2} \geq \frac{1}{16} \left[ \frac{1}{216U(\sigma)} - \frac{1}{N^{1/3}} \right] \]
provided that \( U(\sigma) \leq (r - 2)/32 \), which lets us use Lemma 1. Now if \( U(\sigma) > (r - 2)/32 \), then the last inequality obviously holds since the right side becomes negative. In addition, by Bessel's inequality, 
\[ \left( \sum_{j=1}^{m} (v_{p}, W(j))^2 \right)^{1/2} \leq N. \]
Hence with probability \( \geq 1 - 5/N^3 \)
\[ \left( \frac{2}{\varepsilon^2} + U(\sigma) \right) (20\delta N \log N)^{1/4} \geq \left( \frac{2}{\varepsilon^2} + U(\sigma) \right)^{1/2} U(\sigma)^{1/2} (20\delta N \log N)^{1/4} \]
\[ \geq \frac{(1 - \varepsilon)(\delta N)^{1/2}}{16 \cdot 2^{16}U(\sigma)} - 3(\varepsilon \delta N)^{1/2} - 4\varepsilon (20\delta N \log N)^{1/4} \]
\[ \geq \frac{(\delta N)^{1/2}}{30 \cdot 2^{16}U(\sigma)} - 4(\varepsilon \delta N)^{1/2}. \]
To finish the proof of (*) and hence of Theorem 1 it is enough to verify that the inequalities given above imply that 
\[ U(\sigma) \geq c \log \left( \frac{\delta N}{\log N} \right) \]
for some absolute positive constant \( c \). Indeed if \( \frac{1}{30 \cdot 2^{16}U(\sigma)} \leq 10\varepsilon^{1/2} \), then \( 30 \cdot 2^{16}U(\sigma) \geq \frac{1}{10} \left( \frac{\delta N}{20 \log N} \right)^{1/32} \), i.e.,
\[ \log 30 + 16U(\sigma) \log 2 \geq \frac{1}{32} \log \left( \frac{\delta N}{20 \log N} \right) - \log 10. \]
Consequently,
\[ U(\sigma) \geq c \log \left( \frac{\delta N}{\log N} \right), \]
where \( c > 0 \) is an absolute constant. On the other hand, if \( \frac{1}{30 \cdot 2^{16}U(\sigma)} \geq 10\varepsilon^{1/2} \), then since
\[ \frac{2}{\varepsilon^2} = 2 \left( \frac{\delta N}{20 \log N} \right)^{1/8} \gg U(\sigma), \]
we get that
\[ 3 \cdot \left( \frac{\delta N}{20 \log N} \right)^{1/8} (20\delta N \log N)^{1/4} \geq \frac{(\delta N)^{1/2}}{60 \cdot 2^{16}U(\sigma)} \]
or
\[ 180 \cdot 2^{16}U(\sigma) \geq \left( \frac{\delta N}{20 \log N} \right)^{1/8}, \]
i.e., again
\[ U(\sigma) \geq c \log \left( \frac{\delta N}{\log N} \right). \]
Now we consider the case of a trigonometric system. For \( U(\sigma) \) defined at the beginning of the paper one has an estimate analogous to Theorem 1.

**THEOREM 2.** There exists an absolute constant \( b > 0 \) such that for \( N = 2, 3, \ldots \) and \( 1 \leq q \leq N \)

\[
\nu\left\{ \sigma \in S_{2N+1}^q; U(\{\sigma - N - 1\}) \leq b \log \left( 2 + \frac{q}{\log N} \right) \right\} < \frac{1}{N^2}.
\]

We have used the usual notation above: the set

\[
\sigma - N - 1 = \{ k - N - 1; \ k \in \sigma \}
\]

is a subset of the set

\[
\{-N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N\}.
\]

**Proof of Theorem 2.** The first step in the proof of Theorem 2 is to consider for given \( N \) the discrete trigonometric system

\[
\varphi_k = \left\{ \varphi_{k,j} = e^{2\pi i k j / (2N+1)} \right\}_{j=0}^{2N} \in \mathbb{C}^{2N+1}, \quad -N \leq k \leq N.
\]

As in the previous case we denote by \( \varphi^{(0)}_0, 0 \leq j \leq 2N \), the columns of the matrix \( \{ \varphi_{k,j} \}_{k=-N}^{N, j=0} \).

Repeating the proof of Theorem 1 for the system \( \{ \varphi_k \}_{k=-N}^{N} \) in place of the Walsh system \( \{ W_k \}_{k=1}^{N} \) and for \( p = \lfloor N/2 \rfloor \) instead of \( p = \lfloor N/3 \rfloor \), we find that \( \nu(A) \geq 1 - 1/N^2 \), where \( A \) is the set of all subsets \( \sigma \subset S_{2N+1}^q \) for which one can find a sequence \( \sigma^a = (a_k^a)_{k=-N}^{N} \) with support in \( \sigma - N - 1 \) such that

\[
\left\| \sum_{k=-N}^{N} a_k^a \varphi_k \right\|_{\infty} = 1 \quad \text{and} \quad \left\| \sum_{k=-p}^{p} a_k^a \varphi_k \right\|_{\infty} \geq c \log \left( 2 + \frac{q}{\log N} \right),
\]

where \( c > 0 \) is an absolute constant. Here, in contrast with the Walsh system, we consider the symmetric vectors

\[
v_p = (0, 0, \ldots, 0, 1, \ldots, 1, 0, 0, \ldots, 0) \in \mathbb{R}^{2N+1},
\]

which obviously satisfy the relation

\[(v_p, \varphi^{(j)}) = D_p(2\pi j / (2N + 1))\]

for all \( 0 \leq j \leq 2N \), where \( D_p \) denotes the usual complex Dirichlet kernel.

To derive Theorem 2 from the discrete case analyzed above, for any \( \sigma \in A \) we consider the polynomial \( T^a(x) = \sum_{k=-N}^{N} a_k^a e^{2\pi i k x} \) and its de la Vallee Poussin mean:

\[
T^a(x) = \frac{1}{p} \sum_{n=p}^{2p-1} \sum_{k=-n}^{n} a_k^a e^{2\pi i k x}.
\]

Then, using standard properties of the de la Vallee Poussin kernel, we easily get that \( \left\| T^a \right\|_{\infty} \leq 10 \). On the other hand,

\[
\left\| \sum_{k=-p}^{p} \tilde{T}^a(n) e^{2\pi i k x} \right\|_{\infty} = \left\| \sum_{k=-p}^{p} a_k^a e^{2\pi i k x} \right\|_{\infty} \geq \left\| \sum_{k=-p}^{p} a_k^a \varphi_k \right\|_{\infty} \geq c \log \left( 2 + \frac{q}{\log N} \right).
\]

Consequently, for any \( \sigma \in A \)

\[
U(\sigma) \geq \frac{c}{10} \log \left( 2 + \frac{q}{\log N} \right).
\]

**Remark.** The approach used in the proof of Theorem 1 can also be applied for other orthogonal systems. In particular, without changing the proof for each permutation of a discrete trigonometric system.
REFERENCES


