On the volume of the Set of Uniformly Bounded Polynomials

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Septembre 1992

IHES/M/92/72

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Let ω be a positive weight function on the interval $[-1,1]$. Corresponding orthogonal polynomials are denoted by $p_n(\omega)$. Let $W_{n,w}$ be the following subset of $\mathbb{R}^n$:

$$W_{n,w} = \left\{ x = (x_0, \cdots, x_{n-1}) \subset \mathbb{R}^n : \left\| \sum_{k=0}^{n-1} x_k p_k(\omega) \right\|_{C[-1,1]} \leq 1 \right\}.$$

We will estimate the volume of $W_{n,w}$. The set $W_{n,w}$ and the problem about its volume appeared in Kashin's paper [K] in connection with his study of the smallest deviation from zero of polynomials with integer coefficient (see also F. Amoroso [A]). Upper and lower bounds of $(\mathrm{vol}\, W_{n,w})^{1/n}$ coinciding up to a multiplicative constant are obtained in this paper for any weight function w of Szegö class. The same lower bound was obtained in [K] for weight functions bounded away from zero. Note that our method is simpler than that of [K].

Let $e_j (j = 1, \cdots, n)$ be the standard basis of $\mathbb{R}^n; \mathcal{D}_n$ be the Euclidean unit ball in $\mathbb{R}^n$; ℓ_∞^n be the space $\mathbb{R}^n$ equipped with the norm

$$\left\| \sum_{j=1}^{n} x_j e_j \right\| = \max_{i \leq j \leq n} |x_j|.$$

The k-dimensional counterpart of the octahedron is denoted by B_1^k, that is

$$B_1^k = \mathrm{conv}\{\pm e_1, \cdots, \pm e_k\}.$$

As the usual δ_{ij} is a Kronecker symbol ($\delta_{ij} = 1$ for $i = j$, otherwise $\delta_{ij} = 0$). $\mathcal{P}_n$ will be linear space of all polynomials of degree $\leq n$. c, c_1 and so on will be different universal positive constants.

A linear transform $\mathbb{R}^n \to \mathbb{R}^k$ is identified with its matrix throughout the paper.

Lemma (Vaaler, Ball). Let T be a linear transform from $\mathbb{R}^n$ to $\mathbb{R}^k$, $(k > n)$ and $V = V_T \subset \mathbb{R}^n$ be the following set

$$V = \left\{ x \in \mathbb{R}^n : \|Tx\|_{\ell_\infty^k} \leq 1 \right\}.$$

Then the following inequalities hold

$$1 \leq (\det(T^*T))^{1/2} (\mathrm{vol}\, V_T)^{1/n} \leq 2^{(k-n)/2n}. \tag{1}$$

The left-hand side inequality was proved by Vaaler [V] and the right-hand side one by Ball [B]. For our purpose the following weaker version of the lemma will suffice :

$$c_1(k/n)^{-1/2} \leq (\det(T^*T))^{1/2} (\mathrm{vol}\, V_T)^{1/n} \leq c_2(k/n)$$

For the reader's convenience we will quote a simple reduction (1) to the classical Santalo and inverse Santalo inequalities at the end of the paper.

Theorem. *Let a weight function* $w \in (-1,1)$ *be a non-negative and* $(1-t^2)^{1/2} |\log w(t)| \in L_1(-1,1)$. *Then*

$$(\mathrm{vol}\, W_{n,w})^{1/n} \asymp \frac{1}{\sqrt{n}} \left(-\frac{1}{2\pi} \int_{-1}^{1} \log \left(w(t) \sqrt{1-t^2} \right) \frac{dt}{\sqrt{1-t^2}} \right).$$

Proof. : Let $\kappa_n(w)$ be the leading coefficient of polynomial $p_n(w)$. Hence if w_0 is another weight function then the set $W_{n,w}$ is transformed to W_{n,w_0} by a linear operator with triangle matrix whose diagonal entries are $\kappa_k(w_0)/\kappa_k(w)$. So we have

$$\mathrm{vol}\, W_{n,w} = \left(\prod_{j=0}^{n-1} \frac{\kappa_j(w)}{\kappa_j(w_0)} \right) W_{n,w_0}.$$

If both w and w_0 satisfy conditions of the theorem, then from Szegö's theorem (see [S], 12,1,2) threre exists.[1]

$$\lim \frac{\kappa_n(w)}{\kappa_n(w_0)} = \exp \left(-\frac{1}{2\pi} \int_{-1}^{1} \log \frac{w(t)}{w_0(t)} dt \right).$$

The two last equalities reduce the problem to the case of one single weight. We'll deal with Tchebesheff's weight function $w_0(t) = (1-t^2)^{-1/2}$. Henceforth, $p_k(w_0)$ will be taken as p_n.

1 This asymptotic formula holds under far weaker conditions in weight functions (see Mathé, Nevai, Totik [M-N-T]. However, for our purpose Szegö's result is sufficient.

Being a consequence of Bernstein's inequality,

$$\frac{1}{c}\|p\|_{C[-1,1]} \leq \max_{0\leq j\leq 2n}\left|p\left(\cos\frac{\pi j}{2n}\right)\right| \leq \|p\|_{C[-1,1]} \tag{2}$$

for any polynomial $p \in \mathcal{P}_n$ (see e.g. [Z]).

Vectors $\varphi_j \in \mathbb{R}^n, j = 1, \cdots, 4n$ will be defined as follows

$$\varphi = \sum_{k=0}^{n-1} p_k\left(\cos\frac{2\pi j}{4n}\right) e_k.$$

The matrix with rows φ_j will be denoted by T.

Let $V \subset \mathbb{R}^n$ be the following set

$$V = \left\{x \in \mathbb{R}^n : \|Tx\|_{\ell_n^{4n}} \leq 1\right\}.$$

The inequality (2) gives

$$W_{n,w_0} \subset V \subset cW_{n,w_0} \tag{3}$$

It follows from (1)

$$(c_1/2)^n \leq (\det(T^*T))^{1/2}\,\mathrm{vol}\,V_n \leq (2c_2)^n. \tag{4}$$

It is well known that for any $p \in \mathcal{P}_m$ the following equality holds

$$\sum_{j=0}^{2m-1} p\left(\cos\frac{\pi j}{m}\right) = \frac{2m}{\pi}\int_0^{\pi} p(\cos t)dt = \frac{2m}{\pi}\int_{-1}^{1} p(s)\frac{ds}{\sqrt{1-s^2}}.$$

So for entries of the matrix T^*T we have :

$$(T^*Te_i, e_j) = (\varphi_i, \varphi_j) = \sum_{k=1}^{4n} p_i\left(\cos\frac{\pi k}{2n}\right) p_j\left(\cos\frac{\pi k}{2n}\right) =$$

$$\frac{4n}{\pi}\int_{-1}^{1} p_i(s)p_j(s)\frac{ds}{\sqrt{1-s^2}} = \frac{4n}{\pi}\delta_{i,j}.$$

In particular $T^*T = (4n/\pi)^n$ and the theorem follows from (3) and (4). ∎

Proof of the inequality (1) We use the following result. Let K be a convex symmetric body with center of symmetry at zero and K^0 be its polar body :

$$K^0 = \{x \in \mathbb{R}^n : (x, y) \leq 1 \text{ for any } y \in K\}.$$

Then the following inequalities hold

$$1 \leq \frac{\operatorname{vol} K}{\operatorname{vol} D_n} \cdot \frac{\operatorname{vol} K^0}{\operatorname{vol} D_n} \leq c^n. \tag{5}$$

This remarkable inequality was obtained by Santalo [S] (left-hand side) and by Bourgain and Milman [B-M] (right-hand side).

Let $\mathcal{A}$ be the collection of all n-element subsets of the set $\{1, 2, \cdots, k\}$. For $\alpha \in \mathcal{A}$ let T_α be a $n \times n$ matrix with rows Te_i for $i \in \alpha$.

By definition of V

$$V^0 = T^* B_1^k.$$

Therefore by Caratheodory's theorem

$$V^0 = \bigcup_{\alpha \in \mathcal{A}} T_\alpha B_1^n.$$

It follows that

$$\max_{\alpha \in \mathcal{A}} \operatorname{vol} T_\alpha B_1^n \leq \operatorname{vol} V^0 \leq \sum_{\alpha \in \mathcal{A}} \operatorname{vol} T_\alpha B_1^n. \tag{6}$$

From Binet-Cauchy formula

$$\sum (\operatorname{vol} T_\alpha B_1^n)^2 = (\operatorname{vol} B_1^n)^2 \sum (\det(T_\alpha))^2 = (\operatorname{vol} B_1^n)^2 \det(T^* T).$$

Inequality (1) now follows from (5), (6) and standard estimate

$$\operatorname{card} \mathcal{A} = \binom{k}{n} \leq \left(\frac{ek}{n}\right)^n.$$

■

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