Remark 2. Surmounting some technical difficulties, one can show that the question of the strong conditionality of a real trigonometric basis reduces furthermore to the question for the system of exponentials in the complex variant of the same functional space.

In conclusion I express my gratitude to S. V. Kislyakov for exceedingly useful consultations on the topic of this note.

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ALGEBRAIC POLYNOMIALS WITH INTEGER COEFFICIENTS DEVIATING LITTLE FROM ZERO ON AN INTERVAL

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Let π_n, n = 1, 2,..., be a linear space of algebraic polynomials of degree $\leqslant n$ with real coefficients, and let $\pi_n(\mathbf{Z})$ be a subset in π_n consisting of polynomials with integer coefficients.

According to a classical theorem of Chebyshev, for an arbitrary interval $[a, b] \subset R^1$ and n = 1, 2,...

$$\inf_{\pi_n \ni q(x)=x^n+\dots} \|q(x)\|_{C[a,b]} = 2\left(\frac{b-a}{4}\right)^n. \tag{1}$$

There is also, of long standing, a problem, considered in this paper, concerning the construction, for a given interval [a, b], of a polynomial $P \in \pi_n(\mathbf{Z})$, $P \not\equiv 0$, deviating little from zero in this or another metric (for details on this theme, see [1-4]). As long ago as 1894 Hilbert, using geometric considerations, showed that for an arbitrary interval [a, b]

$$\inf_{0 \not\equiv P \in \pi_n(\mathbf{Z})} \|P\|_{L^2(a,b)} \leqslant C \cdot n^{1/2}\left(\frac{b-a}{4}\right)^{n/2}, \quad n=1,\dots \tag{2}$$

(here and in the sequel c, C, C_1,... denote absolute positive constants).

Fekete [6] showed that for an arbitrary interval [a, b]

$$E_n^{\mathbf{Z}}(0,[a,b]) \equiv \inf_{0 \not\equiv P \in \pi_n(\mathbf{Z})} \|P\|_{C[a,b]} \leqslant 2^{1-\frac{1}{2n+1}}(n+1)\left(\frac{b-a}{4}\right)^{n/2}, \quad n=1,2,\dots \tag{3}$$

Proof of the estimate (3), given in [6], employs reasoning similar in many ways to that in [5]. We remark also that inequalities (2) and (3) are of interest only if b − a < 4 since, in the contrary case, the estimates required hold for P ≡ 1. Moreover, when $b-a \geqslant 4$ the quantities $E_n^{\mathbf{Z}}(0, [a, b])$ do not tend to zero as $n \to \infty$ [see Eq. (1)].

Fekete's theorem found application in the theory of algebraic numbers and the theory of integral functions (see, in particular, [7-9]).

It follows from relations (1) and (3) that the quantity

$$q[a,b] \equiv \overline{\lim_{n\to\infty}} \{E_n^{\mathbf{Z}}(0,[a,b])\}^{1/n} \tag{4}$$

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satisfies the inequalities ($b - a < 4$)

$$\frac{b-a}{4} \leqslant q[a, b] \leqslant \left(\frac{b-a}{4}\right)^{1/2}. \tag{5}$$

An exact value for $q[a, b]$ has not been found for any interval $[a, b]$ but it is known that the estimate (5) can, in a number of cases, be improved. Thus, in papers of Aparisio (see [1, 9]) it was shown that

$$(2.37686)^{-1} < q[0, 1] < (2.33071)^{-1}. \tag{6}$$

The method used in proving the right-hand inequality in (6) and in other known refinements of Fekete's theorem consists in the selection, for a given interval $[a, b]$, of a polynomial $R(x)$ with integer coefficients and of sufficiently small degree r (in [9] for $[a, b] = [0, 1]$ the degree r was taken equal to 17), so that when $n \geqslant n_0$

$$\| R^{[n/r]} \|_{C[a,b]} \leqslant \gamma^n, \quad \gamma < \left(\frac{b-a}{4}\right)^{1/2}$$

(here $[x]$ is the integer part of x). This type of approach to obtaining estimates from above for $q[a, b]$ ceases to work if the length of interval $[a, b]$ becomes close to 4. The degree of polynomial $R(x)$ with $\| R \|_{C[a,b]} < 1$ must then increase without bound, thereby making its selection very difficult. So far as is known to the author, for intervals of greater length ($b - a > 3$) there are no refinements for the estimate from above in (5) for $q[a, b]$, although possibly it is just this case which is of most interest from the point of view of applications. In the present paper we make the estimates for $E_n^{\mathbf{Z}}(0, [a, b])$, obtained by Fekete, more precise. In particular (see Theorem 1), for an arbitrary interval of the form $[-a, a]$, $a < 2$ we improve the upper estimate in (5) for $q[-a, a]$. Our discussion is of a geometric nature and (as in [5, 6]) involves Minkowski's theorem concerning lattice points lying in a convex body. First of all, we note, following [6] (see also [2]) but using instead of the trivial estimate for the volume of the set of coefficients of trigonometric polynomials bounded in absolute value by one, the estimate established by the author in [11, 12] (see, in particular, Lemma 2 and Proposition 1 in [11]); we obtain

Proposition 1. There exists an absolute constant C such that for an arbitrary interval $[a, b]$ and $n = 1, 2, \ldots$

$$E_n^{\mathbf{Z}}(0, [a, b]) \leqslant C \cdot n^{1/2} \cdot \left(\frac{b-a}{4}\right)^{n/2}.$$

Remark. In [4, p. 315] in presenting a modification of the proof of Fekete's theorem an error was committed: The analysis presented there can only lead to the estimate $E_n^{\mathbf{Z}}(0, [a, b]) \leqslant Cn^{t/2} \ln n\, [(b-a)/4]^{n/2}$, and not to the inequality $E_n^{\mathbf{Z}}(0, [a, b]) \leqslant C \cdot \ln n\, [(b-a)/4]^{n/2}$, as stated there.

If the interval $[a, b]$, $b - a < 4$ is fixed, and $n \to \infty$, the refinement of estimate (3), given in Proposition 1, is then insignificant. Indeed, the problem arises of making the value of the quantity $q[a, b]$ more precise. In this direction we have established

THEOREM 1. For arbitrary a, $0 < a < 2$ we have the strict inequality

$$q[-a, a] < (a/2)^{1/2}.$$

More precisely,

$$q[-a, a] \leqslant \inf_{0 \leqslant \alpha < 1} a^{\frac{1}{2}(1+\alpha)} 2^{-\frac{1}{2}(1-\alpha)^{-1}} \exp \frac{1}{4} W(\alpha), \tag{7}$$

where

$$W(\alpha) = (1-\alpha)^{-1} \cdot \{(\alpha+1)^2 \ln(1+\alpha) + (\alpha-1)^2 \ln(1-\alpha) - 2\alpha^2 \ln 2\alpha\}.$$

Moreover, for $1 \leqslant a < 2$

$$(a/2)^{1/2} - q[-a, a] \geqslant C(2-a)^2 \ln^{-1}(1 + (2-a)^{-1}), \tag{8}$$

where $C > 0$ is an absolute constant.

Remark. Calculations show, in connection with the interval $[0, 1]$, that when the equations $q\left[-\frac{1}{2}, \frac{1}{2}\right] = (q[0, {}^1/_4])^{1/2} = q[0, 1]$ (see [4, 1]) are taken into account the relation (7) yields:

$q[0,1] \leqslant (2.3263)^{-1}$, a somewhat poorer estimate than (6).

In what follows we use the following notation: If H is a Hilbert space with scalar product $(\cdot,\cdot)$, $H \supset L$ is a d-dimensional subspace, and $A \subset L$, then $\mathrm{Vol}_d(A)$ denotes d-dimensional volume of set A [standard Lebesgue measure, constructed starting from the scalar product $(\cdot,\cdot)$]. Further, if $Y = \{y_1, \ldots, y_d\} \subset H$, then let

$$|\det Y| \equiv \mathrm{Vol}_d\left\{\sum_{i=1}^d \lambda_i y_i;\ 0 \leqslant \lambda_i \leqslant 1,\ i = 1, \ldots, d\right\}$$

and let G(Y) be the Gram matrix of set Y, i.e.,

$$G(Y) = \{g_{ij}\},\ g_{ij} = (y_i, y_j),\ 1 \leqslant i,\ j \leqslant d.$$

In particular, we consider the case for which $H = L^2(-a, a)$, $(f, g) = \int_{-a}^{a} f\cdot g\,dx$.

For $a > 0$, $\alpha \in [0, 1)$ and n = 2, 3,... we define a set of elements of the space $L^2(-a, a)$:

$$E_{\alpha,n}(a) = \{x^{2k},\ \alpha n \leqslant 2k \leqslant n,\ k \in \mathbf{Z}\} \tag{9}$$

and we take $E_{\alpha,n}(1) \equiv E_{\alpha,n}$.

LEMMA 1. For fixed $\alpha \in [0, 1)$ we have as $n \to \infty$ the relation

$$|\det E_{\alpha,n}| = \left(\frac{1}{2}\right)^{n^2/4} \exp\left[\frac{n^2}{8}\tilde{Q}(\alpha) + o(n^2)\right],$$

where

$$\tilde{Q}(\alpha) = (1+\alpha)^2 \ln(1+\alpha) + (1-\alpha)^2 \ln(1-\alpha) - 2\alpha^2 \ln 2\alpha. \tag{10}$$

Proof. We determine the asymptotics for the determinant of the Gram matrix $G(E_{\alpha,n}) \equiv G(\alpha, n)$, and then use the equation

$$|\det E_{\alpha,n}| = \{\det G(\alpha, n)\}^{1/2}. \tag{11}$$

It is convenient to enumerate the elements of matrix $G(\alpha, n)$; thus:

$$G(\alpha, n) = \{g_{p,q}\},\ \alpha n \leqslant 2p,\ 2q \leqslant n, \tag{12}$$

and

$$g_{p,q} = \int_{-1}^{1} x^{2p} x^{2q} dx = 2\int_0^1 x^{2(p+q)} dx = \frac{1}{p+q+1/2}.$$

Applying Cauchy's identity (see, for example, [13], p. 104), we find

$$\det G(\alpha, n) = \prod_{\frac{\alpha}{2} n \leqslant p < p' \leqslant \frac{n}{2}} (p'-p) \prod_{\frac{\alpha}{2} n \leqslant q < q' \leqslant \frac{n}{2}} (q'-q) \cdot$$
$$\cdot\left[\prod_{\frac{\alpha}{2} n \leqslant p,\, q \leqslant \frac{n}{2}} (p+q+1/2)\right]^{-1} = \left\{\prod_{\frac{\alpha}{2} n \leqslant p < p' \leqslant \frac{n}{2}} (p'-p)\right\}^2 \cdot$$
$$\cdot\left\{\prod_{\frac{\alpha}{2} n \leqslant p,\, q \leqslant \frac{n}{2}} \left(p+q+\frac{1}{2}\right)\right\}^{-1} \equiv \{\pi_1(\alpha, n)\}^2 \cdot \{\pi_2(\alpha, n)\}^{-1}. \tag{13}$$

Calculations show that as $n \to \infty$

$$\{\pi_1(\alpha, n)\}^2 = \left\{\prod_{1 \leqslant s < s' \leqslant 1/2\cdot n(1-\alpha)} (s'-s)\right\}^2 \cdot \exp\{O(n \ln n)\} = r^{r^2} \cdot e^{-\frac{3}{2}r^2} \cdot \exp\{o(n^2)\};\quad r = 1/2\cdot n(1-\alpha). \tag{14}$$

The product $\pi_2(\alpha, n)$ can be written in the form

$$\pi_2(\alpha, n) = \left(\frac{n}{2}\right)^{r^2} \prod_{\frac{\alpha}{2} n \leqslant p,\, q \leqslant \frac{n}{2}} \left(\frac{p+q+1/2}{n/2}\right) \cdot \rho_{\alpha,n},$$

where $r = {}^1/_2 n(1-\alpha)$, $n^{-5n} \leqslant \rho_{\alpha,n} \leqslant n^{5n}$. Consequently, as $n \to \infty$

$$\ln \pi_2(\alpha, n) = (1-\alpha)^2 \frac{n^2}{4} \cdot \ln n/2 + \left(\frac{n}{2}\right)^2 \sum_{\frac{\alpha}{2} n \leqslant p,\, q \leqslant \frac{n}{2}} \left(\frac{n}{2}\right)^{-2} \ln \frac{p+q+1/2}{n/2} + o(n^2). \tag{15}$$

Further, as $n \to \infty$

$$\sum_{\frac{\alpha}{2} n\leqslant p,\, q\leqslant \frac{n}{2}} \left(\frac{n}{2}\right)^{-2} \ln \frac{p+q+1/2}{n/2} = I_\alpha + o(1),$$

where

$$I_\alpha = \int_\alpha^1 \int_\alpha^1 \ln(x+y)\,dx\,dy = 2\ln 2 - \frac{3}{2}(1-\alpha)^2 - (1+\alpha)^2 \ln(1+\alpha) + 2\alpha^2 \ln 2\alpha. \tag{16}$$

Therefore, as $n \to \infty$

$$\begin{gathered} \ln \pi_2(\alpha, n) = (1-\alpha)^2 \frac{n^2}{4} \ln \frac{n}{2} + \frac{n^2}{4} I_\alpha + o(n^2), \\ \pi_2(\alpha, n) = \left(\frac{n}{2}\right)^{\frac{n^2}{4}(1-\alpha)^2} e^{\frac{n^2}{4} I_\alpha + o(n^2)}. \end{gathered} \tag{17}$$

From relations (14) and (17), taking note of relations (13) and (16), we find that as $n \to \infty$

$$\det G(\alpha, n) = \frac{\left[\frac{n}{2}(1-\alpha)\right]^{\frac{n^2}{4}(1-\alpha)^2} e^{-\frac{3}{2}\left[\frac{n}{2}(1-\alpha)\right]^2 + o(n^2)}}{\left(\frac{n}{2}\right)^{\frac{n^2}{4}(1-\alpha)^2} e^{\frac{n^2}{4} I_\alpha + o(n^2)}} =$$

$$= \exp\left\{\frac{n^2}{4}\left[(1-\alpha)^2 \ln(1-\alpha) - \frac{3}{2}(1-\alpha)^2 - I_\alpha + o(1)\right]\right\} =$$

$$= \exp\left\{\frac{n^2}{4}\left[-2\ln 2 + (1-\alpha)^2 \ln(1-\alpha) + (1+\alpha)^2 \ln(1+\alpha) - 2\alpha^2 \ln 2\alpha + o(1)\right]\right\}. \tag{18}$$

Lemma 1 then follows from relations (18) and (11).

LEMMA 2. For fixed $a > 0$, $\alpha \in [0, 1)$, as $n \to \infty$ we have the relationship

$$|\det E_{\alpha, n}(a)| = |\det E_{\alpha, n}| \cdot a^{\frac{n^2}{4}(1-\alpha^2) + O(n)} \tag{19}$$

It is sufficient to verify [see relation (11)] that

$$\det G(E_{\alpha, n}(a)) = a^{\frac{n^2}{2}(1-\alpha^2) + O(n)} \det G(E_{\alpha, n}).$$

Let $G(a) \equiv G(E_{\alpha,n}(a))$. Then

$$(G(a))_{p,q} = \int_{-a}^{a} x^{2p} x^{2q}\,dx = \frac{a^{2(p+q+1/2)}}{p+q+1/2} = (G(1))_{p,q} \cdot a^{2(p+q+1/2)}$$

[we have used the same enumeration as was used in Lemma 1 (see relation (12)) for the elements $(G(a))_{p,q}$ of the matrix $G(a)$]. Therefore

$$\begin{gathered} \det G(a) = \det G(1) \cdot a^{2r}, \\ r = \sum_{\frac{\alpha}{2} n\leqslant p\leqslant \frac{n}{2}} p + \sum_{\frac{\alpha}{2}} \quad q + 1/2) = \frac{n^2}{4}(1-\alpha^2) + O(n), \end{gathered}$$

which is what we wished to show.

LEMMA 3. For $n = 1, 2, \ldots$ and $a > 0$ let $L_{n,a}$ be a subspace in $L^2(-a, a)$ spanned over the functions $1, x, \ldots, x^n$, and let

$$Q_{n,a} = \{P \in L_{n,a} \colon \|P\|_{C[a,b]} \leqslant 1\}. \tag{20}$$

Then for an arbitrary d-dimensional subspace $L \subset L_{n,a}$ $(1 \leqslant d \leqslant n+1)$ the volume

$$\mathrm{Vol}_d\{L \cap Q_{n,a}\} \geqslant c_1^n n^{-\frac{3}{2}n} a^{\frac{d}{2}}. \tag{21}$$

Remark. Lemma 3 can be strengthened (see Proposition 2 below); however, the crude estimate (21) suffices for our purposes.

Proof. Let $p_0, \ldots, p_n$ be a set of Legendre polynomials forming an orthonormalized basis in $L_{n,1}$. It is well known (see [14, pp. 80 and 171) that

$$\|p_j\|_{C[-1,1]} \leqslant C_2 j^{1/2}, \qquad j = 1, 2, \ldots \tag{22}$$

The set of functions

$$\{f_j(x)\}_{j=0}^n,$$
$$f_j(x) = f_j(x, a) = a^{-1/2} p_j(x/a),\ j = 0, \ldots, n$$

forms an orthonormalized basis in $L_{n,a}$. Here [see relations (22)] for arbitrary numbers b_j, $0 \leqslant j \leqslant n$

$$\left|\sum_{j=0}^n b_j f_j(x)\right| \leqslant \left(\sum_{j=0}^n b_j^2\right)^{1/2} \left(\sum_{j=0}^n f_j^2\right)^{1/2} \leqslant C_3 a^{-1/2} n \left(\sum_{j=0}^n b_j^2\right)^{1/2},$$

i.e.,

$$\left\{\sum_{j=0}^n b_j f_j, \left(\sum_{j=0}^n b_j^2\right)^{1/2} \leqslant r\right\} \subset Q_{n,a},\ r = C_3^{-1} a^{1/2} n^{-1}. \tag{23}$$

Inclusion (23) means that $Q_{n,a}$ contains a Euclidean ball of radius r, and hence

$$\mathrm{Vol}_d\{L \cap Q_{n,a}\} \geqslant r^d \omega_d \geqslant a^{d/2} C_4^{-d} n^{-d} d^{-d/2} \geqslant a^{d/2} c_1^n n^{-3/2n}.$$

(ω_d is the volume of the unit Euclidean ball in R^d). This establishes Lemma 3.

Proof of the Theorem. Let numbers $a > 0$ and $\alpha \in [0, 1)$ be fixed.

For n sufficiently large we consider subspaces $L_{n,a}$ [see relation (20)] and

$$L(n, a, \alpha) = \left\{\sum_{\alpha n \leqslant 2p \leqslant n} b_p x^{2p}\right\} \subset L_{n,a} \subset L^2(-a, a).$$

Also let

$$Q(n, a, \alpha) = Q_{n,\alpha} \cap L(n, a, \alpha).$$

According to Lemma 3

$$\mathrm{Vol}_d Q(n, a, \alpha) \geqslant c_1^n n^{-\frac{3}{2}n} a^{\frac{d}{2}},$$
$$d = \dim L(n, a, \alpha) = \frac{n}{2}(1 - \alpha) + O(1). \tag{24}$$

We consider now a convex centrally-symmetric set $\rho Q(n, a, \alpha)$ in $L(n, a, \alpha)$, and we determine for which ρ this set contains points of the lattice

$$\Lambda(n, a, \alpha) = \left\{\sum_{\alpha n \leqslant 2p \leqslant n} b_p x^{2p}, b_p \in \mathbf{Z}\right\}$$

(this indicates that there exists a polynomial P with integer coefficients and $\|P\|_{C[-a,a]} \leqslant \rho$, $P \in \pi_n$). From Minkowski's theorem (see [15], p. 87)

$$\Lambda(n, a, \alpha) \cap \rho Q(n, a, \alpha) \neq 0, \tag{25}$$

if [see relation (9)]

$$2^d |\det E_{\alpha,n}(a)| \leqslant \mathrm{Vol}_d[\rho Q(n, a, \alpha)] = \rho^d \cdot \mathrm{Vol}_d Q(n, a, \alpha).$$

Thus for relation (25) to hold it is sufficient [see also relation (24)] that $\rho = \rho(n, a, \alpha)$ satisfies the inequality

$$\rho \geqslant (c_1^{-1} n)^{c(\alpha)} \cdot a^{-1/2} \cdot |\det E_{\alpha,n}(a)|^{1/d} \geqslant 2 |\det E_{\alpha,n}(a)|^{1/d} \cdot \mathrm{Vol}_d^{-1/d} Q(n, a, \alpha). \tag{26}$$

For arbitrary fixed $\alpha \in [0, 1)$ for $n \to \infty$ we have the asymptotics [see Lemmas 1, 2 and relation (24)]:

$$|\det E_{\alpha,n}(a)|^{1/d} = \exp\left\{(\ln a)\left[\frac{n^2}{4}(1 - \alpha^2) + O(n)\right] \cdot \left[\frac{n}{2}(1 - \alpha) + O(1)\right]^{-1}\right\} 2^{-\frac{n^2}{4}\left[\frac{n}{2}(1-\alpha)+O(1)\right]^{-1}} \cdot$$
$$\cdot \exp\left\{\left[\frac{n^2}{8} \tilde{Q}(\alpha) + o(n^2)\right]\left[\frac{n}{2}(1 - \alpha) + O(1)\right]^{-1}\right\} = [U(a, \alpha)]^n \exp[o(n)],$$

where [see relations (10) and (7)]

$$U(a, \alpha) = \left(\frac{a}{2}\right)^{\frac{1}{2}(1-\alpha)^{-1}} a^{-\frac{1}{2}\frac{\alpha^2}{1-\alpha}} \exp\left[\frac{1}{4}\frac{\tilde{Q}(\alpha)}{1 - \alpha}\right] = a^{\frac{1}{2}(1+\alpha)} 2^{-\frac{1}{2}(1-\alpha)^{-1}} \exp\left[\frac{1}{4} W(\alpha)\right]. \tag{27}$$

Noting that $(c_1^{-1} n)^{c(\alpha)} = o\{(1 + \varepsilon)^n\}$ for arbitrary $\varepsilon > 0$ and using the fact that the parameter $\alpha \in [0, 1)$ can be chosen arbitrarily, we find that relation (25) holds for $n \geqslant n_0$ if

$$\rho = \gamma^n,\ \gamma > \gamma_0 \equiv \inf_{\alpha \in [0,1)} U(a, \alpha).$$

Consequently, $q[-a, a] \leqslant \gamma_0$ and one of the conclusions of the theorem is proved.

Further, $U((a, 0) = (a/2^{1/2}, *))$ and using relations $\tilde{Q}(\alpha) = o(\alpha)$ and $\alpha^2(1-\alpha)^{-1} = o(\alpha)$, we find, as $\alpha \to 0$, that for arbitrary $a < 2$ we have for $0 < \alpha < \alpha(a)$ the inequality

$$U(a, \alpha) < U(a, 0),$$

i.e., $q[-a, a] \leqslant \gamma_0 < (a/2)^{1/2}$. Finally, for $a \in [1, 2)$, using the relations

$$\exp\left[\frac{1}{4}\tilde{Q}(\alpha)(1-\alpha)^{-1}\right] \leqslant 1 + C_5\alpha^2 \ln 1/\alpha,$$

$$\left(\frac{a}{2}\right)^{\frac{1}{2}(1-\alpha)^{-1}} \leqslant \left(\frac{a}{2}\right)^{1/2}\left[1 - C_6\alpha \ln\left(\frac{2}{a}\right)\right], \quad 0 \leqslant \alpha \leqslant 1/10,$$

we obtain

$$\gamma_0 \leqslant \inf_{0 \leqslant \alpha < 1}\left(\frac{a}{2}\right)^{\frac{1}{2}(1-\alpha)^{-1}} \exp\left[\frac{1}{4}\tilde{Q}(\alpha)(1-\alpha)^{-1}\right] \leqslant$$

$$\leqslant \left(\frac{a}{2}\right)^{1/2} \min_{0 \leqslant \alpha \leqslant 1/10} [1 - C_6\beta\alpha][1 + C_5\alpha^2 \ln 1/\alpha]; \ \beta = \ln\frac{2}{a} > 0.$$

It is not difficult to verify that

$$\min_{0 \leqslant \alpha \leqslant 1/10} [1 - C_6\beta\alpha]\cdot[1 + C_5\alpha^2 \ln\frac{1}{\alpha}] \leqslant 1 - C_7\beta^2\left(\ln\frac{1}{\beta}\right)^{-1},$$

and hence that

$$\gamma_0 \leqslant \left(\frac{a}{2}\right)^{1/2}\left(1 - C_7\beta^2\left(\ln\frac{1}{\beta}\right)^{-1}\right) \leqslant \left(\frac{a}{2}\right)^{1/2} - C(2-a)^2 \ln^{-1}(1 + (2-a)^{-1}),$$

from which relation (8) follows. This establishes the completeness part of the theorem. We also have

Proposition 2. Let $\rho(x) \in L^1(-1, 1)$ and

$$0 < \delta \leqslant \rho(x) \text{ for arbitrary } x \in [-1, 1]. \tag{28}$$

Further, let $p_0, p_1, \ldots$ ($p_k \in \pi_k$) be a sequence of polynomials orthonormalized with weight $\rho(x)$. Then for n = 1, 2,...

$$\mathrm{Vol}_{n+1}(\Omega^n_{\rho(x)}) \equiv \mathrm{Vol}_{n+1}\{b = \{b_k\}_{k=0}^n \in R^{n+1}: \|\textstyle\sum_{k=0}^n b_k p_k(x)\|_{C[-1,1]} \leqslant 1\} \geqslant (c_\delta)^n \cdot n^{-n/2},$$

where the constant $c_\delta > 0$ depends only on δ.

Proof. Let

$$K_n(x) = \textstyle\sum_{k=0}^n p_k^2(x), \ n = 1, 2, \ldots, x \in [-1, 1].$$

Subject to condition (28) we have the estimate (see [14], p. 52 and 189)

$$K_n^{1/2}(x) \leqslant \min(A_\delta n, A'_\delta(1 - x^2)^{-1/4} n^{1/2}). \tag{29}$$

Further, for an arbitrary polynomial $q(x) \in \pi_n$

$$\|q\|_{C[-1,1]} \leqslant C_8 \max_{0 \leqslant j \leqslant 2n}\left|q\left(\cos\left\{\frac{\pi j}{2n}\right\}\right)\right|$$

(see, for example, [16]; the algebraic case follows from the trigonometric case after substituting $x = \cos z$). Therefore

$$\Omega^n_{\rho(x)} \supset W \equiv \{b \in R^{n+1}: |(b, e_j)| \leqslant C_8^{-1}, 0 \leqslant j \leqslant 2n\}, \tag{30}$$

where

$$e_j = \left\{p_k\left(\cos\left\{\frac{\pi j}{2n}\right\}\right)\right\}_{k=0}^n \in R^{n+1}, \quad j = 0, \ldots, 2n,$$

and where, by virtue of relation (29), for j = 0,...,2n

$$|e_j| \equiv (e_j, e_j)^{1/2} \leqslant \min(A_\delta n, A'_\delta n^{1/2} \cdot \sin^{-1/2}(\pi j/2n)). \tag{31}$$

*We note also that $U(a, \alpha) \to a$ if $\alpha \to 1$.

For a lower estimate of the volume of polyhedron W we apply Theorem 2 from Ball and Pajor's paper [17] (for the value of parameter $p = p_0 < 2$ (see [17])) to obtain:

$$\mathrm{Vol}_{n+1} W \geqslant C_8^{-(n+1)} (r_{p_0})^{-(n+1)}, \quad r_{p_0} = \left(\frac{1}{n+1} \sum_{j=0}^{2n} |e_j|^{p_0}\right)^{1/p_0}. \tag{32}$$

Since for $p_0 < 2$, by virtue of relation (31), $r_{p_0} \leqslant C_9 n^{1/2}$, $n = 1, 2, \ldots$, Proposition 2 then follows from relations (30) and (32).

We remark that Proposition 2 also allows us to obtain, using results given in [11], lower estimates of volumes of sections of the set $\Omega_0^n(x)$ by linear subspaces. Similar results for the trigonometric case have found application in analysis.

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