ANALOGUE OF MEN'SHOV's THEOREM "ON CORRECTION" FOR DISCRETE ORTHONORMAL SYSTEMS

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In 1940 D. E. Men'shov obtained the following result which shows the possibility of an improvement of the convergence properties of the Fourier series of an arbitrary function by means of alteration of its values on a set of small measure.

THEOREM A (see [1, p. 448]). For any $\varepsilon > 0$ one can find a constant C_ε such that for any function f continuous on $[0, 2\pi]$ there can be found a function $\tilde{f}$ such that its Fourier series is uniformly convergent and

$$1)\ m\{x \in (0, 2\pi): f(x) \neq \tilde{f}(x)\} \leqslant \varepsilon;$$

$$2)\ \|\tilde{f}\|_U \equiv \sup_n \|\sigma_n(\tilde{f})\|_\infty \leqslant C_\varepsilon \|f\|_\infty$$

(here $\sigma_n(\tilde{f})$ are the partial sums of the Fourier series of the function $\tilde{f}$, m{E} is the Lebesgue measure of a set E).

By the present time, extensions and analogues of Theorem A for some other orthonormal systems have been established, and modifications of its original proof have been proposed (see the comprehensive survey [2]).

In [3, 4] "discrete" analogues of Theorem A were obtained. Later [5] it was found that these "discrete" results were making it possible to obtain new theorems on correction of the traditional (i.e., not "discrete") type. An application of results on discrete orthonormal systems to proving assertions of an opposite character, theorems on "incorrigibility," had been presented earlier in [6].

In [3], a discrete trigonometric system was considered. Olevskii [2, p. 172] raised the question of validity of an analogue of Men'shov's theorem for arbitrary discrete complete orthonormal systems. Theorem 1 established below answers this question in the affirmative. Before presenting its formulation, let us introduce some notation.

We consider orthonormal bases (o.n.b.) $\Phi = \{\varphi_i\}_{i=1}^n$ in an n-dimensional real Euclidean space $\mathbf{R}^n$ with its scalar product $(\cdot, \cdot)$. The coordinates of vectors $\varphi \in \mathbf{R}^n$ in the standard basis $\{e_j\}$ are denoted by $(\varphi)_j$, $1 \leqslant j \leqslant n$. For a fixed basis Φ and $g \in \mathbf{R}^n$ we put

$$\|g\|_U \equiv \|g\|_{U(\Phi)} \equiv \max_{1 \leqslant \nu \leqslant n} \left\|\sum_{i=1}^{\nu} (g, \varphi_i)\varphi_i\right\|_{l_\infty^n}. \tag{1}$$

Clearly,

$$\|g\|_{l_2^n} \geqslant \|g\|_U \geqslant \|g\|_{l_\infty^n} \geqslant n^{-1/2}\|g\|_{l_2^n}, \tag{2}$$

where

$$\|g\|_{l_\infty^n} \equiv \|g\|_\infty = \max_{1 \leqslant j \leqslant n} |(g)_j|; \quad \|g\|_{l_2^n} \equiv \|g\|_2 = [(g, g)]^{1/2}.$$

By $|\Lambda|$ we denote the number of elements of a finite set Λ of natural numbers; at that, if $\Lambda \subset \{1, 2, \ldots, n\}$, then $\mu_n(\Lambda) \equiv n^{-1}|\Lambda|$. Finally, for $g \in R^n$ $\operatorname{supp} g \equiv \{j: (g)_j \neq 0\}$.

THEOREM 1. For any $\varepsilon > 0$ there exists a constant C_ε such that for n = 1, 2, ..., any orthonormal basis Φ in $\mathbf{R}^n$, and any vector $g \in \mathbf{R}^n, \|g\|_2 = 1,$ there can be found a vector $\tilde{g} \in \mathbf{R}^n$ such that

$$1)\ |\{j: (g)_j \neq (\tilde{g})_j\}| \leqslant \varepsilon n;$$

$$2)\ \|\tilde{g}\|_{U(\Phi)} \leqslant C_\varepsilon n^{-1/2}.$$

The central role in the proof of Theorem 1 is played by the following

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LEMMA. There exists an absolute constant n_0 such that for any $\rho \in (0, 1]$ there can be found a constant K_ρ possessing the following property:

for any o.n.b. Φ in $\mathbf{R}^n$ and any vector $g \in \mathbf{R}^n$, $\|g\|_\infty \leqslant 1$, $\operatorname{supp} g = \Lambda$, $|\Lambda| \geqslant \rho n \geqslant n_0$, there can be found a vector v, supp $v \subset \Lambda$, such that

$$\begin{aligned} &1)\ (v)_j = \lambda_j (g)_j,\ \ \lambda_j \in \{0, +1, -1\},\ 1 \leqslant j \leqslant n; \\ &2)\ |\operatorname{supp}(g - v)| \leqslant 0{,}95\,|\Lambda|; \\ &3)\ \|v\|_{U(\Phi)} \leqslant K_\rho. \end{aligned} \tag{3}$$

We shall first deduce the assertion of Theorem 1 from the lemma and then prove the latter. It is easily seen that in proving Theorem 1 there is no restriction of generality in assuming that $\varepsilon \in (0, 1)$, that the number n is sufficiently large, $n \geq n(\varepsilon)$, and that $(g)_j \neq 0$ for $j = 1, 2, \ldots, n$. Furthermore, it immediately follows from Chebyshev's inequality that for any $\varepsilon > 0$ and any $g \in \mathbf{R}^n$, $\|g\|_2 \leqslant 1$, $(g)_j \neq 0$, $1 \leqslant j \leqslant n$, there can be found a vector g' such that $\|g'\|_\infty \leq (2/\varepsilon)^{1/2} n^{-1/2}$ and

$$|\Omega| \equiv |\{j\colon (g)_j \neq (g')_j\}| \leqslant (\varepsilon/2)\, n,\quad 0 < |(g')_j| \leqslant |(g)_j|,$$
$$1 \leqslant j \leqslant n,$$

i.e., the vector g can be "corrected" on the set Ω with $\mu_n(\Omega) \leq \varepsilon/2$ into the vector g' with $\|g'\|_\infty \leq (2/\varepsilon)^{1/2} n^{-1/2}$. Summarizing these observations, we find that it suffices for each $g \in (0, 1)$ to find a constant C^0_ε such that for $n \geq n(\varepsilon)$, any o.n.b. Φ in $\mathbf{R}^n$ and any vector

$$g \in \mathbf{R}^n,\ \|g\|_\infty \leqslant 1,\ (g)_j \neq 0,\ 1 \leqslant j \leqslant n, \tag{4}$$

there will be found a vector $\tilde{g} \in \mathbf{R}^n$ such that

$$\text{a) } \|\tilde{g}\|_{U(\Phi)} \leqslant C^0_\varepsilon;\quad \text{b) } \mu_n\{\operatorname{supp}(g - \tilde{g})\} \leqslant \varepsilon. \tag{5}$$

Now fix $\varepsilon \in (0, 1)$, $n \geq n_0\varepsilon^{-1}$, an o.n.b. Φ, and a vector g satisfying relations (4). Let an integer s_0 be such that

$$(0{,}95)^{s_0} < \varepsilon \leqslant (0{,}95)^{s_0 - 1} \text{ and } C^0_\varepsilon \equiv \left(\sum\nolimits_{\nu=0}^{s_0} 2^\nu\right) K_\varepsilon, \tag{6}$$

where the constant K_ε is defined in the lemma. We shall show that with such a choice of C^0_ε a vector $\tilde{g}$ with properties (5) can be found. Applying the lemma with $\rho = \varepsilon$ to the vector $g^0 \equiv g$, supp $g^0 = \Lambda_0 = \{1, \ldots, n\}$, we find a vector $v^0 \equiv v$ for which relations 1)-3) in (3) hold. If $|\Lambda_1| \equiv |\operatorname{supp}(g - v^0)| \leq \varepsilon n$, then (4) is established for $\tilde{g} = v^0$, since obviously $K_\varepsilon \leq C^0_\varepsilon$. Otherwise, let $g^1 = g^0 - v^0$. Then (see (3))

$$\Lambda_1 = \{j\colon \lambda_j \neq 1\},\quad (g^1)_j = \begin{cases} (g^0)_j, & \text{if } \lambda_j = 0, \\ 2(g^0)_j, & \text{if } \lambda_j = -1, \end{cases}$$

i.e., $\|g^1\|_\infty \leq 2$. Applying the lemma with $\rho = \varepsilon$ to the vector g^1, we find v^1 with $\|v^1\|_{U(\Phi)} \leq 2K_\varepsilon$, for which relations 1) and 2) in (3) hold (v being replaced by v^1 and g by g^1). Then $\|v^0 + v^1\|_{U(\Phi)} \leqslant 3K_\varepsilon$, $|\Lambda_2| \equiv |\operatorname{supp}(g - v^0 - v^1)| \leqslant (0{,}95)^2 |\Lambda_0|$. If $|\Lambda_2| \leq \varepsilon n$, then (4) holds for $\tilde{g} = v^0 + v^1$, otherwise put $g^2 = g^0 - v^0 - v^1$, $\|g^2\|_\infty \leq 4$ .... Continuing this process, we determine vectors v^2, g^3, v^3, ..., g^s, where $s \leq s_0$; at that, for $\nu = 1, 2, \ldots$

$$|\operatorname{supp} g^\nu| \leqslant (0{,}95)^\nu n,\quad \|g^\nu\|_\infty \leqslant 2^\nu,\ \|v^\nu\|_{U(\Phi)} \leqslant 2^\nu K_\varepsilon, \tag{7}$$
$$g^\nu = g^{\nu-1} - v^{\nu-1} = g^0 - \sum\nolimits_{r=0}^{\nu-1} v^r.$$

The time to stop (the number s) is chosen such that $|\operatorname{supp} g^\nu| \leq \varepsilon n$. Then, putting $\tilde{g} = \sum_{r=0}^{s-1} v^r$, we have (see (7) and (6)) $\|\tilde{g}\|_{U(\Phi)} \leq C^0_\varepsilon$, $|\operatorname{supp}(g - \tilde{g})| \leq \varepsilon n$, therefore (4) holds. Thereby we have completed the deduction of Theorem 1 from the lemma.

The proof presented below utilizes the methods introduced by the present author in [7] (see also [8, 9]) and the estiamtes of the volumes of polyhedrons established by Gluskin [10].

Proof of the Lemma. Let there be given an o.n.b. $\Phi = \{\varphi_i\}_{i=1}^n$, and let a vector g satisfy the hypotheses of the lemma. With no restriction of generality (permuting if necessary columns of the matrix $\{(\varphi_i)_j\}$), we can assume that

$$\operatorname{supp} g = \Lambda = \{1, 2, \ldots, d\}, \quad d \geqslant \rho n. \tag{8}$$

Denoting by $\mathbf{R}^d$ the subspace of $\mathbf{R}^n$ spanned by the vectors e_j, $1 \le j \le d$, consider in $\mathbf{R}^d$ the lattice

$$L = \{\sum_{j=1}^{d} k_j (g)_j e_j\}, \quad \{k_j\} \in \mathbf{Z}^d,$$

and put

$$B = B(\Phi, \Lambda) = \{g \in \mathbf{R}^d : \| g \|_{U(\Phi)} \leqslant 1\}.$$

We shall show that for $d \ge n_0$ and a sufficiently large constant $K = K_\rho$ the convex body KB contains points of the lattice L of the form

$$w = \sum_{j=1}^{d} k_j (g)_j e_j; \quad k_j \in \{0, +1, -1\}, \quad j = 1, \ldots, d,$$

$$\sum_{j=1}^{d} |k_j| \geqslant 0,1d = 0,1|\Lambda|.$$

Then, as is easily seen, we can take either w or −w as the vector sought, i.e., the assertion of the lemma will be established.

Define a linear operator $T: \mathbf{R}^d \to \mathbf{R}^d$ by the relation

$$T(\Sigma \alpha_j e_j) = \Sigma \alpha_j [(g)_j]^{-1} e_j.$$

Then $T(L) = \mathbf{Z}^d$ and it will suffice to show that there exists a vector

$$z \in \mathbf{Z}^d, \ \| z \|_\infty \leqslant 1, \ |\operatorname{supp} z| \geqslant 0,1d, \tag{9}$$

such that

$$z \in T(K_\rho B(\Phi, \Lambda)) = K_\rho T(B). \tag{9'}$$

Let

$$W = W(K_\rho) = 1{,}99 Q^d \cap K_\rho T(B), \tag{10}$$

where $Q^d = \{x \in \mathbf{R}^d : \| x \|_\infty \leqslant 1\}$ is a cube in $\mathbf{R}^d$. In [7] (see also [10]) the fact was utilized that for sufficiently large d ($d \ge n_0$) the existence of a vector z with properties (9), (9') follows from this estimate of the d-dimensional volume of the set $W(K_\rho)$:

$$V_d(W(K_\rho)) \geqslant 2^d (1{,}9)^d. \tag{11}$$

Let us establish that (11) holds indeed, if the constant K_ρ is sufficiently large. Let $\mathcal{F} \equiv \mathcal{F}_{\Phi,\Lambda}: \mathbf{R}^d \to \mathbf{R}^n$ be an operator assigning to a vector $z \in \mathbf{R}^d$ a sequence of Fourier coefficients. More precisely, if

$$\tilde{z} \in \mathbf{R}^n, \quad (\tilde{z})_j = \begin{cases} (z)_j, & 1 \leqslant j \leqslant d, \\ 0, & j > d \end{cases} \quad \text{and} \quad \tilde{z} = \sum_{i=1}^{n} y_i \varphi_i,$$

then $\mathcal{F}(z) = y = \{(y)_i\} \in \mathbf{R}^n$. Taking into account the definition of the norm $\|\cdot\|_{U(\Phi)}$, we have

$$W(K_\rho) = 1{,}99 Q^d \cap \{z \in \mathbf{R}^d : |(\mathcal{F} T^{-1}(z), w_{j,s})| \leqslant K_\rho, \ 1 \leqslant j, s \leqslant n\}, \tag{12}$$

where $w_{j,s} \in \mathbf{R}^n$,

$$(w_{j,s})_i = \begin{cases} (\varphi_i)_j, & 1 \leqslant i \leqslant s, \\ 0, & \text{if } i > s. \end{cases} \tag{13}$$

But for any $w \in \mathbf{R}^n$ $(\mathcal{F} T^{-1} z, w) = (z, Gw)$, where $G = (T^{-1})^* \mathcal{F}^*$, and $\mathcal{F}^*: \mathbf{R}^n \to \mathbf{R}^d$, $(T^{-1})^*: \mathbf{R}^d \to \mathbf{R}^d$ are the adjoints. At that for any $w \in \mathbf{R}^n$, $x \in \mathbf{R}^d$

$$\| \mathcal{F}^* w \|_{l_2^d} \leqslant \| w \|_{l_2^n},$$

$$\| (T^{-1})^* x \|_{l_2^d} \leqslant \max_j |(g)_j| \| x \|_{l_2^d} \leqslant \| x \|_{l_2^d},$$

and, consequently,

$$\| Gw \|_{l_2^d} \leqslant \| w \|_{l_2^n}. \tag{14}$$

Thus,

$$W(K_\rho) = 1{,}99Q^d \cap \{z \in \mathbf{R}^d: |(z, Gw_{j,s})| \leqslant K_\rho, 1 \leqslant j, s \leqslant n\}. \tag{15}$$

Consider the following decomposition of the vectors $\varphi(j) \equiv \{(\varphi_i)_j\}_{i=1}^n, 1 \leqslant j \leqslant n,$ into the binary blocks (this decomposition is often used in the theory of orthogonal series, for more details see [11, p. 325]):

$$\varphi(j) = \sum_{0 \leqslant \nu \leqslant 2^p - 1} r_\nu^p(j) \equiv \sum r_\nu^p, \quad p = 0, 1, \ldots, p_0(j),$$

where $r_0^0 = \varphi(j)$, for p = 1, 2, ...

$$\|r_\nu^p\|_2 \leqslant 2^{-p/2} \|\varphi(j)\|_2 = 2^{-p/2}, \quad 0 \leqslant \nu \leqslant 2^p - 1, \tag{16}$$

and the number $p_0 = p_0(j)$ is chosen so large that $|\operatorname{supp} r_\nu^{p_0}| \leqslant 2,\ 0 \leqslant \nu \leqslant 2^{p_0} - 1$. Then (see [11, p. 326] and also (13)) for each pair (j, s), j, s = 1, ..., n, there can be found a set $\{\nu(p)\}_{p=1}^{p_0}$ such that

$$w_{j,s} = \sum_{p=1}^{p_0} r_{\nu(p)}^p(j) + \Delta, \tag{17}$$

$$|\operatorname{supp} \Delta| \leqslant 2, \ \|\Delta\|_\infty \leqslant \|w_{j,s}\|_\infty \leqslant 1.$$

It follows from (17) that for $z \in \mathbf{R}^d$

$$|(z, Gw_{j,s})| \leqslant \sum_{p=1}^{p_0(j)} \max_{0 \leqslant \nu \leqslant 2^p - 1} |(z, f_\nu^p(j))| + 2 \max_{1 \leqslant r \leqslant n} |(\psi_r, z)|, \tag{17'}$$

where $f_\nu^p(j) = Gr_\nu^p(j)$, $\psi_r = Ge_r$. At that (see (14) and (16)),

$$\|\psi_r\|_2 \leqslant 1, \quad 1 \leqslant r \leqslant n; \ \|f_\nu^p(j)\|_2 \leqslant 2^{-p/2}, \quad 0 \leqslant \nu \leqslant 2^p - 1, \quad 1 \leqslant j \leqslant n. \tag{18}$$

Consequently (see (15), (17'), (18)),

$$W(K_\rho) \supset 1{,}99Q^d \cap \{z \in \mathbf{R}^d: |(z, \psi_r)| \leqslant K_\rho/10, \ 1 \leqslant r \leqslant n\} \cap$$
$$\bigcap_{j=1}^n \bigcap_{p=1}^{p_0(j)} \bigcap_{\nu=0}^{2^p-1} \{z \in \mathbf{R}^d: |(z, f_\nu^p(j))| \leqslant K_\rho/(10p^2)\}. \tag{19}$$

Putting $\tilde{\psi}_r = (10/K_\rho)\psi_r$, $\tilde{f}_\nu^p(j) = (10p^2/K_\rho) f_\nu^p(j)$, transform relation (19):

$$(1{,}99)^{-1} W(K_\rho) \supset \{z \in \mathbf{R}^d: |(z, e_i)| \leqslant 1, 1 \leqslant i \leqslant d\} \cap$$
$$\cap \{z \in \mathbf{R}^d: |(z, 1{,}99\tilde{\psi}_r)| \leqslant 1, \quad 1 \leqslant r \leqslant n\} \bigcap_{j=1}^n \bigcap_{p=1}^{p_0(j)} \bigcap_{\nu=0}^{2^p-1} \{z \in \mathbf{R}^d: |(z, 1.99\tilde{f}_\nu^p(j))| \leqslant 1\}, \tag{20}$$

where

$$\|\tilde{\psi}_r\|_2 \leqslant 10/K_\rho, \quad \|\tilde{f}_\nu^p(j)\|_2 \leqslant \frac{10p^2}{K_\rho} \cdot 2^{-p/2}. \tag{21}$$

It suffices to show (see (11)) that the d-th root of the volume of the set in the left-hand member of inclusion relation (20), which set is below denoted by $A = A(K_\rho)$, satisfies the inequality

$$[V_d(A)]^{1/d} \geqslant \frac{1{,}9}{1{,}99} \cdot 2, \tag{22}$$

if the constant K_ρ is sufficiently large. Denote by $\{x_\alpha\}_{\alpha=1}^{\alpha_0}$ the vectors which are not identically zero, are contained in the sets $\{e_i\}$, $\{1.99\tilde{\psi}_r\}$, $\{1.99\tilde{f}_\nu^p\}$, and numbered in such a way that $x_\alpha = e_\alpha$, $1 \leq \alpha \leq d$. Applying to the set $\chi = \{x_\alpha\}$ Corollary 2 in [10] (in whose formulation we set n = d, q = d − 1), we obtain this lower estimate of the volume $V_d(A)$:

$$[V_d(A)]^{1/d} \geqslant \sqrt{2}\,[V(B_2^d)]^{1/d} \left[\frac{\Gamma(d/2)}{\Gamma(1/2)}\right]^{1/(d-1)} \left[(d-1)\int_0^\infty t^{-d} \prod_{\alpha=1}^{\alpha_0} (2\Phi_0(t/\|x_\alpha\|_2))\,dt\right]^{1/(d-1)}, \tag{23}$$

where $V(B_2^d) = \pi^{d/2}[\Gamma(1 + d/2)]^{-1}$ is the volume of the Euclidean unit ball in $\mathbf{R}^d$, and $\Phi_0(t) = (2\pi)^{-1/2}\int_0^t e^{-x^2/2}\,dx$. It follows from (23) that

$$[V_d(A)]^{1/d} \geqslant [1 + o_d(1)](2\pi)^{1/2} I^{1/(d-1)}, \tag{24}$$

where

$$I=\int_0^\infty t^{-d}\prod_{\alpha=1}^{\alpha_0}(2\Phi_0\{t/\|x_\alpha\|_2\})\,dt,$$

and $o_d(1)$ is a quantity tending to zero as $d \to \infty$. Note that

$$\text{a)}\ 2\Phi_0(z) \geqslant 2(2\pi)^{-1/2}e^{-z^2/2};$$

$$\text{b)}\ 2\Phi_0(z) \geqslant \exp(-2\exp(-z^2/2)) \text{ for } z \geqslant 1. \tag{25}$$

From (25a) and the equalities $\|x_\alpha\|_2 = 1$, $1 \le \alpha \le d$, we obtain

$$I \geqslant (2/\pi)^{d/2}\int_0^\infty \exp(-dt^2/2)\prod_{\alpha=d+1}^{\alpha_0}(2\Phi_0\{t/\|x_\alpha\|_2\})\,dt. \tag{26}$$

Let it be known that the constant K_ρ is so large that for $t \ge \gamma \equiv 10^{-5}\,t(\|x_\alpha\|_2)^{-1} \ge 1$, $\alpha = d+1, \ldots, \alpha_0$ (see (21)). Then, taking (25b) into account, we derive from (26)

$$I \geqslant (2/\pi)^{d/2}\int_\gamma^{2\gamma}\exp(-dt^2/2)\times$$

$$\times\exp\left[-2\sum_{\alpha=d+1}^{\alpha_0}\exp\left(-\frac{1}{2}t^2/\|x_\alpha\|_2^2\right)\right]dt \geqslant (2/\pi)^{d/2}e^{-2d\gamma^2}\int_\gamma^{2\gamma}\exp(-2S(t))\,dt, \tag{27}$$

where

$$S(t)\equiv\sum_{\alpha=d+1}^{\alpha_0}\exp\left(-\frac{1}{2}t^2/\|x_\alpha\|_2^2\right).$$

Estimating the last sum with the use of inequalities (21) (see also (20) and the definition of the system of vectors $\{x_\alpha\}$), we find that for $t \in (\gamma, 2\gamma)$

$$S(t)\leqslant n\exp\left\{-\frac{t^2}{2}(K_\rho/20)^2\right\}+n\sum_{p=1}^{\infty}2^p\exp\left\{-\frac{t^2}{2}\frac{2^pK_\rho^2}{400p^4}\right\} \leqslant n\exp(-10^{-13}K_\rho^2)+nC_1\exp(-10^{-15}K_\rho^2) \tag{28}$$

(it is assumed here that K_ρ is sufficiently large: $K_\rho \ge C_2$; $C_1, C_2, \ldots$ are absolute positive constants).

Since $d \ge \rho n$ (see (8)), it follows from (28) that $S(t) \le 10^{-5}d$, if $K_\rho \ge C_3 \ln^{1/2}(2/\rho)$. As a result, for $K_\rho \ge C_4 \ln^{1/2}(2/\rho)$ we have

$$I \geqslant (2/\pi)^{d/2}e^{-2d\gamma^2}\gamma\exp(-2d\cdot 10^{-5}). \tag{29}$$

Finally, it follows from (24) and (29) that

$$[V_d(A)]^{1/d} \geqslant 2(1+o_d(1))\exp(-3\cdot 10^{-5}) \geqslant 2\exp(-4\cdot 10^{-5}) \geqslant 2\cdot(1{,}9/1{,}99),$$

if d is sufficiently large: $d \ge n_0$. Estimate (22) and, consequently, the assertion of the lemma are established.

Remark. The estimating technique for ball volumes in the spaces $U(\Phi)$ employed in the proof of the lemma is applicable not only to the discrete systems Φ but also to the quasimatrix systems introduced in [12]. We call a system of functions $\{\varphi_i(\theta)\}_{i=1}^\infty$ defined on some set G a quasimatrix one if for $n = 1, 2, \ldots$ there can be found a set $\Omega_n \subset G$ with the number of its elements $\le C_5 n$ such that for any polynomial $P(\theta)=\sum_{i=1}^n a_i\varphi_i(\theta)$ the inequality

$$\sup_{\theta\in G}|P(\theta)|\leqslant C_6\max_{\theta\in\Omega_n}|P(\theta)|$$

holds.

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RELATIONS IN THE PIECEWISE-LINEAR COBORDISM RING

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INTRODUCTION

Let p be an odd prime and let CP^{p-1} be the complex projective space of complex dimension p − 1. In addition, let $MSPL^{CP^{p-1}}$ be the theory of oriented piecewise-linear cobordism and singularity CP^{p-1} (and also the cohomology spectrum of this theory). In [1] the author proved that the spectrum $MSPL^{CP^{p-1}}_{(p)}$ is a graded Eilenberg–MacLane spectrum (the subscript (p) here denotes localization). Here we derive the following theorem from this result.

THEOREM A. In the ring $\pi_*(MSPL)$ one has the relation $[CP^{p-1}]x = 0$, where x is the lowest dimensional p-torsion element of the ring $\pi_*(MSPL)$, dim $x = 2p^2 - 2p - 1$.

As a matter of fact this theorem is a consequence of some integrality relations in the SPL-cobordism ring found by Brumfiel, Madsen, and Milgram (cf., e.g., [2]) and Theorem B formulated below. Let f(x, y) be a universal formal group over the ring $\pi_*(MU)$ (cf., e.g., [3-5]), and let $[p]_f(x) = px + \sum_{k>0} \alpha_k x^{k+1}$ be the formal series corresponding to raising to the p-th power in this formal group. Let φ: $MU \to MSPL^{CP^{p-1}}$ be the obvious foregetful morphism.

THEOREM B. The element $\varphi_*(\alpha_k) \in \pi_{2k}(MSPL^{CP^{p-1}})$ is divisible by p.

It turns out (cf. Theorem 3.3) that in the ring $\pi_*(MSPL)/([CP^{p-1}])$ the images of the elements $\alpha_{p^{i}-1}$, $i > 1$, are not divisible by p. Hence the groups $\pi_*(MSPL^{CP^{p-1}})$ and $\pi_*(MSPL)/([CP^{p-1}])$ do not coincide and it obviously follows from this that $[CP^{p-1}]$ is a divisor of zero in the ring $\pi_*(MSPL)$. From this and the assertion that p-torsion first appears in the ring $\pi_*(MSPL)$ in dimension $2p^2 - 2p - 1$ (cf. Proposition 1.4), Theorem A follows.

The next theorem follows from Theorem B and the elementary theory of formal groups.

THEOREM C. In the ring $\pi_*(MSPL^{CP^{p-1}}_{(p)})$ the element $[CP^n]$ is divisible by n + 1.

(The fact that $MSPL^{CP^{p-1}}_{(p)}$ is a commutative ring spectrum is proved separately in Theorem 1.7.)

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