Dokl. Akad. Nauk SSSR
Tom 214 (1974), No. 5

Soviet Math. Dokl.
Vol. 15 (1974), No. 1

ON KOLMOGOROV DIAMETERS OF OCTAHEDRA

UDC 513.82

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Let X be a Banach space and let K be a bounded centrally symmetric set in X. The quantity

$$d_m(K, X) = \inf \operatorname*{Sup}_{x\in K} \min_{y\in L_m} \|x-y\|,$$

where the infimum is taken with respect to all m-dimensional hyperplanes L_m, is called the *Kolmogorov m-diameter* of the set K in the space X.

Let R^n be the n-dimensional space of real vectors $x = (x_1, \cdots, x_n)$, and let X be the Banach space obtained by introducing a norm in R^n.

In this paper only the l_p^n norms, $1 \leq p \leq \infty$, defined by

$$\|x\|_{l_p^n} = \left(\sum_{i=1}^n x_i^p\right)^{1/p}, \quad \|x\|_{l_\infty^n} = \max_{1\leqslant i\leqslant n} |x_i|,$$

are considered.

In what follows, l_p^n will denote the space R^n with the l_p^n norm. Let S_p^n be the unit sphere in the space l_p^n. In what follows, we denote the diameter $d_m(S_p^n, l_q^n)$ by $d_m(l_p^n, l_q^n)$. Obtaining an estimate of the diameters of certain function classes necessitates sufficiently precise estimates of $d_m(l_p^n, l_\infty^n)$. Thus in [1] the evaluation of $d_m(l_2^n, l_\infty^n)$ was proposed.

It is not difficult to prove that

$$d_m(l_p^n, l_\infty^n) \leqslant [d_m(l_1^n, l_\infty^n)]^{1/p}, \quad 1\leqslant p\leqslant\infty.$$

Therefore it is entirely sufficient to estimate the diameter of the octahedron $d_m(l_1^n, l_\infty^n)$.

R. S. Ismagilov obtained an estimate of the diameter $d_m(l_1^n, l_\infty^n)$ which is sharp for $m < n < Cm$ (where C is an absolute constant). However, in applications a more precise estimate of $d_m(l_1^n, l_\infty^n)$ is necessary when m is much less than n.

This paper establishes the

Theorem. $d_m(l_1^n, l_\infty^n) \leq 2m^{-1/2}\ln^{1/2} n$.

We need a preliminary lemma.

AMS (MOS) subject classifications (1970). Primary 46B99.

Lemma. *There exists a system of vectors* $\{\bar{y}_i\}_1^n \subset R^m$, $\bar{y}_i = (y_{1i}, \cdots, y_{mi})$, *such that*

1) $\|\bar{y}_i\|_{l_2^m} = m^{1/2}$, $i = 1, 2, \ldots, n$;
2) $(\bar{y}_i, \bar{y}_j) \leq 2m^{1/2} \ln^{1/2} n$ *for* $i \neq j$.

Proof of the Lemma. All vectors in the desired system to be constructed will have coordinates 1 and -1. Then condition 1) is ensured.

Let $\bar{\epsilon}_1, \bar{\epsilon}_2, \cdots, \bar{\epsilon}_{2^m}$ be an enumeration in an arbitrary manner of the vectors in R^m with coordinates $+1$ and -1. Take $\bar{\epsilon}_1$ to be the vector with all coordinates equal to $+1$. Form the matrix $B = \{\phi_{ij}\}_{i=1, j=1}^{2^m, 2^m}$, where $\phi_{ij} = (\bar{\epsilon}_i, \bar{\epsilon}_j)$. The subsequent argument is divided into several parts.

I. It is easy to see that each row of the matrix B consists of the same numbers to within a permutation.

II. Let a be a fixed number, and let $\alpha_i(a)$ denote the number of entries ϕ_{ij}, $j = 1, \cdots, 2^m$, which satisfy the inequality $|\phi_{ij}| > a$. By I, $\alpha_i(a) \equiv \alpha(a)$ (independent of i).

III. Suppose we have found a number a such that

$$\alpha(a) < 2^m/n. \tag{1}$$

Then there exists a collection of n vectors $\bar{\epsilon}_{i_1}, \bar{\epsilon}_{i_2}, \cdots, \bar{\epsilon}_{i_n}$ such that

$$(\bar{\varepsilon}_{i_k}, \bar{\varepsilon}_{i_r}) \leq a \quad \text{for} \quad k \neq r.$$

Indeed, $\bar{\epsilon}_{i_1}$ is arbitrarily selected, and $\bar{\epsilon}_{i_2}$ is selected so that

$$|\varphi_{i_1 i_2}| = |(\bar{\varepsilon}_{i_1}, \bar{\varepsilon}_{i_2})| \leq a.$$

This is possible by formula (1).

Furthermore, a vector $\bar{\epsilon}_{i_3}$ can be found such that

$$|(\bar{\varepsilon}_{i_1}, \bar{\varepsilon}_{i_3})| \leq a, \quad |(\bar{\varepsilon}_{i_2}, \bar{\varepsilon}_{i_3})| \leq a. \tag{2}$$

Indeed, by (1) the number of vectors $\bar{\epsilon}_0$ such that $|(\bar{\epsilon}_0, \bar{\epsilon}_{i_1})| > a$ is less than $2^m/n$, and similarly the number of vectors $\bar{\epsilon}_0'$ such that $|(\bar{\epsilon}_0', \bar{\epsilon}_{i_2})| > a$ is less than $2^m/n$. For $n \geq 3$, $2^m/n + 2^m/n < 2^m$, and thus (2) is established.

Subsequent construction of the desired collection is carried out similarly.

Part III is established.

IV. Directly from the definition of the Rademacher system of functions $r_i(x)$, $i = 0, 1, \cdots$ (see [2], pp. 42–43), it is clear that

$$\alpha(a)/2^m = \alpha_1(a)/2^m = \mu\left\{x : x \in [0, 1], \left|\sum_{i=1}^m r_i(x)\right| > a\right\}.$$

Thus, by part III, for the proof of the lemma it is sufficient to show that the measure

$$\mu\left\{x: x\in[0,1],\ \left|\sum_{i=1}^{m} r_i(x)\right| > 2m^{1/2}\ln^{1/2} n\right\} < \frac{1}{n}. \tag{3}$$

But the estimate

$$\mu\left\{x: x\in[0,1],\ \left|\sum_{i=1} r_i(x) > b\right\} < \exp\left\{-\frac{b^2}{4m}\right\}\right.$$

is well known (see [3], for example). By substituting the number $2m^{1/2}\ln^{1/2} n$ for b, we obtain formula (3), and therefore the lemma is proved.

Proof of the theorem. Let $\bar{l}_i = \{0, \cdots, \underset{i}{1}, \cdots, 0\} \in R^n$, $i = 1, \cdots, n$, be the natural basis vectors in R^n. Since the octahedron is the convex closure of the vectors $\bar{l}_i$, it is necessary for the proof of the theorem to find, for each m, a subspace L, $L \subset R^n$, $\dim L = m$, such that for each $i = 1, 2, \cdots, n$, there is a $\bar{z}_i \in L$ for which

$$\|\bar{l}_i - \bar{z}_i\|_{l_\infty^n} \leqslant 2m^{-1/2}\ln^{1/2} n.$$

For a fixed m, we shall construct the desired subspace $L \subset R^n$. Take a system of vectors $\{\bar{y}_i\}_1^n \subset R^m$ satisfying the conditions of the Lemma. Construct a matrix A which has n columns and m rows, where the ith column of the matrix A coincides with the vector $\bar{y}_i$.

Let $\bar{z}_j \in R^n$, $j = 1, 2, \cdots, m$, be the jth row of the matrix A. We shall show that the subspace L spanned by the vectors $\bar{z}_j$, $j = 1, 2, \cdots, m$, is the desired subspace.

Fix i. We shall approximate the vector $\bar{l}_i$ by an element from L. Consider the vector

$$l_i - \left(\sum_{k=1}^{m} y_{ki}\bar{z}_k\right) \Big/ m;$$

its ith coordinate is equal to

$$1 - \left(\sum_{k=1}^{m} y_{ki}^2\right) \Big/ m = 0$$

by property 1) of the system $\{\bar{y}_i\}_1^n$ (see the Lemma), and the sth coordinate ($s \neq i$) of this vector is equal to

$$\left(\sum_{k=1}^{m} y_{ki}y_{ks}\right) \Big/ m \leqslant \frac{2m^{1/2}\ln^{1/2} n}{m} \leqslant 2m^{-1/2}\ln^{1/2} n$$

by property 2) of the system $\{\bar{y}_i\}_1^n$.

Thus

$$\left\|\bar{l}_i - \left(\sum_{k=1}^{m} y_{ki}\bar{z}_k\right) \Big/ m\right\|_{l_\infty^n} \leqslant 2m^{-1/2}\ln^{1/2} n, \quad i=1,2,\ldots,n,$$

which proves the theorem.

Remark. It is easy to show that $d_m(l_1^{2m}, l_\infty^{2m}) \sim m^{-1/2}$, from which it follows that the estimate obtained in the Theorem cannot be improved in the "power scale".

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Translated by W. L. BYNUM